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WEIGHTED VARIABLE HARDY SPACES ASSOCIATED WITH OPERATORS SATISFYING DAVIES-GAFFNEY ESTIMATES

Abstract. We introduce the weighted variable Hardy space $H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$ associated with the operator L, which has a bounded holomorphic functional calculus and fulfills the Davies-Gaffney estimates. More precisely, we establish the molecular characterization of $H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$ and we show that the new weighted variable bounded mean oscillation-type space $BMO^{p(\cdot),M}_{L^*,w}$ represents the dual space of $H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$, where L^* denotes the adjoint operator of L on $L^2(\mathbb{R}^n)$.

Key words: *weighted Hardy spaces, variable exponent, Davies-Gaffney estimates, molecular decomposition, maximal function, dual space*

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1. Introduction. The theory of Hardy spaces in $\mathbb{R}^n$ was first introduced by Stein and Weiss [23] and was originally tied closely to the theory of harmonic functions. On the other hand, the real-variable methods were introduced by Fefferman and Stein [7]. It is well-known that the classical Hardy space $H^p(\mathbb{R}^n)$ is a suitable substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ for any $p \in (0,1]$; for example, when $p \in (0,1]$, various well-known operators from harmonic analysis, such as Hilbert and Riesz transforms, are bounded on $H^p(\mathbb{R}^n)$, but not on the classical Lebesgue spaces $L^p(\mathbb{R}^n)$. As a generalization of the classical Hardy spaces, Nakai and Sawano [20] introduced and studied the atomic characterization of the Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent. Independently, Cruz-Uribe and Wang [4] studied the variable Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ with $p(\cdot)$ satisfying some conditions slightly weaker than those used in [20]. Recently, Zhuo et al. [27] investigated the intrinsic square function characterizations of the variable Hardy spaces, then Saibi [21] extended the results of [27] to the variable Hardy-Lorentz spaces.

The weighted variable Lebesgue space is a natural generalization for the classical weighted Lebesgue space and the variable exponent Lebesgue space. This space has been considered in a series of papers, see for example [3], [17]. Regarding the theory of the Hardy spaces, Ho [12], presented the atomic characterization for the variable weighted Hardy spaces. Additionally, Melkemi et al. [19] explored the weighted Hardy spaces with variable exponents on a proper open subset Ω of $\mathbb{R}^n$.

In the last decade, the study of function spaces associated with different operators has been a very active area of research in harmonic analysis and has attracted the attention of many researchers. In particular, Yang and Zhuo [26] introduced the variable Hardy space $H_L^{p(\cdot)}(\mathbb{R}^n)$ associated with the operator L, where $p(\cdot)\colon \mathbb{R}^n \to (0,1]$ is a measurable function satisfying the globally log-Hölder continuous condition and L is a linear operator on $L^2(\mathbb{R}^n)$, which generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ with kernels having pointwise upper bounds. As a generalization of these results, Yang et al. [24] considered the variable Hardy spaces $H_L^{p(\cdot)}(\mathbb{R}^n)$ associated with the operator L, which obeys the Davies-Gaffney estimates. More generally, Zuo et al. [28] investigated the variable Hardy-Lorentz spaces associated with operators satisfying Davies-Gaffney estimates. We point out that the notion of the Davies-Gaffney estimates (or the so-called L^2 off-diagonal estimates) of the semigroup $\{e^{-tL}\}_{t>0}$ was first introduced by Gaffney [8] and Davies [5], which is considered as a generalization of the Gaussian upper bound of the associated heat kernel.

The main purpose of this paper is to introduce and study the weighted variable Hardy space $H_{L,w}^{p(\cdot)}(\mathbb{R}^n)$ associated with the operator L, which satisfies Davies-Gaffney estimates. We establish its molecular characterization by means of the atomic decomposition of the weighted variable tent space. Furthermore, using this molecular characterization, we formulate the dual space of the variable weighted Hardy space $H_{L,w}^{p(\cdot)}(\mathbb{R}^n)$. The rest of this paper is arranged as follows: in Section 2, we describe the Assumption(A) and Assumption(B) imposed on the operator L and we recall some definitions and basic properties of the weighted Lebesgue spaces with variable exponent. In Section 3, we introduce the weighted variable tent space $T_w^{p(\cdot)}(\mathbb{R}_+^{n+1})$, establish its atomic characterization, and give the definition of the weighted Hardy space with variable exponents associated to the operator L in terms of the square function of the heat semi-group generated by L. Our main results on the molecular characterization of $H_{L,w}^{p(\cdot)}(\mathbb{R}^n)$ is given in this section (see theorem 3 below). In Section 4, we introduce the weighted BMO space with variable exponent $BMO_{L^*,w}^{p(\cdot),M}(\mathbb{R}^n)$, where $M \in \mathbb{N}$ and L^* denotes the adjoint operator of L on $L^2(\mathbb{R}^n)$, and we establish the duality between $H_{L,w}^{p(\cdot)}(\mathbb{R}^n)$ and $BMO_{L^*,w}^{p(\cdot),M}(\mathbb{R}^n)$.

We end this introduction by describing the basic notation. We denote by $\mathbb{N}$ the set $\{1, 2, \dots\}$ and by $\mathbb{Z}_+$ the set $\mathbb{N} \cup \{0\}$. The square function S_L associated with L is defined by setting, for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$S_L(f)(x) := \left[\int_0^\infty \int_{B(x,t)} \left| t^2 L e^{-t^2 L} (f)(y) \right|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}.$$

The symbols $A \lesssim B$ and $A \approx B$ stand for the inequalities $A \leqslant CB$ and $A \lesssim B \lesssim A$, respectively, and C denotes a positive constant independent of the parameters, which can vary from line to line. Finally, for a measurable subset $\Omega \subset \mathbb{R}^n$ we denote by $|\Omega|$ and χ_Ω the Lebesgue measure of Ω and the characteristic function of Ω, respectively.

2. Preliminaries. We first give some notions, notation, and useful definitions; we also describe the assumptions required for the operator L considered in this paper.

Let $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$. For any $\alpha \in (0, \infty)$ and $x \in \mathbb{R}^n$, define

$$\Gamma_\alpha(x) := \{(y,t) \in \mathbb{R}_+^{n+1} \colon |y - x| < \alpha t\}.$$

If $\alpha = 1$, for the sake of simplicity, we write $\Gamma(x)$ instead of $\Gamma_\alpha(x)$. For any ball $B := B(x_B, r_B) \subset \mathbb{R}^n$ with $x_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$, $\lambda \in (0, \infty)$ and $j \in \mathbb{N}$, let $\lambda B := B(x_B, \lambda r_B)$,

$$\widehat{B} = \{(y,t) \in \mathbb{R}_+^{n+1},\ \operatorname{dist}(y, B^c) \geqslant t\}.$$

Now, we recall some notions of bounded holomorphic calculi which was introduced by McIntosh [18]. Let $0 \leqslant \eta < \pi$. The closed sector in the complex plane $\mathbb{C}$ is defined as follows:

$$S_\eta = \{z \in \mathbb{C} \colon |\arg z| \leqslant \eta\} \cup \{0\}$$

and its interior denoted by S_η^0 is defined by

$$S_\eta^0 = \{z \in \mathbb{C} \backslash \{0\} \colon |\arg z| < \eta\}.$$

Denote the set of all holomorphic functions on S^0_η by $H(S^0_\eta)$ and for any $b \in H(S^0_\eta)$ we define $\|b\|_\infty$ by

$$\|b\|_\infty = \sup\{|b(z)|\colon z \in S^0_\eta\}.$$

The set of all $b \in H(S^0_\eta)$ satisfying $\|b\|_\infty < \infty$ is denoted by $H_\infty(S^0_\eta)$ and define the set $\Psi(S^0_\eta)$ by

$$\Psi(S^0_\eta) = \left\{\psi \in H_\infty(S^0_\eta)\colon \exists\, \nu, C > 0\colon |\psi(z)| \leqslant \frac{c|z|^\nu}{1+|z|^{2\nu}}, \forall z \in S^0_\eta\right\}.$$

Let $\eta \in [0,\pi)$ and denote the spectrum of L by $\sigma(L)$. Then, we say that the closed operator L on $L^2(\mathbb{R}^n)$ is of type η if

1) $\sigma(L)$ is a subset of S_η,
2) for any $\upsilon \in (\eta, \pi)$, there exists a positive constant C_υ, such that for all $\lambda \notin S_\upsilon$:
$$\|(L - \lambda I)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leqslant C_\upsilon |\lambda|^{-1},$$
where $\mathcal{L}(L^2(\mathbb{R}^n))$ denotes the set of all linear continuous operators from $L^2(\mathbb{R}^n)$ to itself and for any operator $T \in \mathcal{L}(L^2(\mathbb{R}^n))$, its norm is denoted by $\|T\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$.

Let $\eta \in [0, \pi)$, L be a one-to-one operator of type η in $L^2(\mathbb{R}^n)$, $\upsilon \in (\eta, \pi)$ and $\psi \in \Psi(S^0_\upsilon)$. The operator $\psi(L)$ is defined as follows:

$$\psi(L) = \frac{1}{2\pi i} \int\limits_\Theta \psi(\lambda)(\lambda I - L)^{-1} d\lambda, \tag{1}$$

where $\Theta := \{re^{i\nu}\colon r \in (0, \infty)\} \cup \{re^{-i\nu}\colon r \in (0, \infty)\}$, $\nu \in (\eta, \upsilon)$ is the curve consisting of two rays parameterized anti-clockwise. It is well-known that the integral in (1) is absolutely convergent in $L^2(\mathbb{R}^n)$ (see [9], [18] for more details) and $\psi(L)$ does not depend on the choice of ν (see, for instance, [1, Lecture 2]). By a limiting procedure, we can extend the above holomorphic functional calculus on $\Psi(S^0_\upsilon)$ to $H_\infty(S^0_\upsilon)$ (the reader is referred to [18] for more details). Let $0 < \upsilon < \pi$; we say that the operator L has a bounded $H_\infty(S^0_\upsilon)$-calculus in $L^2(\mathbb{R}^n)$ if there exists a positive constant C, such that for all $\psi \in H_\infty(S^0_\upsilon)$,

$$\|\psi(L)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leqslant C\|\psi\|_{L^\infty(S^0_\upsilon)}.$$

We assume that L is an operator satisfying the following assumptions:
Assumption (A). L is one-to-one operator of type η in $L^2(\mathbb{R}^n)$ with $\eta \in [0, \frac{\pi}{2})$ and has a bounded holomorphic functional calculus.
Assumption (B). The semigroup $\{e^{-tL}\}_{t>0}$ generated by L satisfies the Davies-Gaffney estimates, namely, there exist positive constants c_1 and c_2, such that, for any function f in $L^2(\mathbb{R}^n)$ and closed sets E and F of $\mathbb{R}^n$ with $\operatorname{supp} f \subset E$,

$$\|e^{-tL}(f)\|_{L^2(F)} \leqslant c_1 e^{-c_2 \frac{[\operatorname{dist}(E,F)]^2}{t}} \|f\|_{L^2(E)},$$

where $\operatorname{dist}(E,F) := \inf\{|x-y| \colon x \in E, y \in F\}$.

A measurable function $p(\cdot) \colon \mathbb{R}^n \to (0,\infty]$ is called a variable exponent. We set

$$p_- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p_+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x),$$

and

$$\mathcal{P}(\mathbb{R}^n) := \Big\{ p(\cdot) \text{ variable exponent} \colon 0 < p_- \leqslant p_+ < \infty \Big\}.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions $f \colon \mathbb{R}^n \to \mathbb{C}$, such that $\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty$, equipped with the Luxemburg quasi-norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \Big\{ \lambda > 0 : \int_{\mathbb{R}^n} \Big[\frac{|f(x)|}{\lambda}\Big]^{p(x)} dx \leqslant 1 \Big\}.$$

We recall that for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined for all $x \in \mathbb{R}^n$ by setting

$$M(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

Let $w \colon \mathbb{R}^n \to (0,\infty)$ be a locally integrable function. The weighted Lebesgue space with variable exponent $L^{p(\cdot)}_w(\mathbb{R}^n)$ is defined as

$$L^{p(\cdot)}_w(\mathbb{R}^n) := \Big\{ f : \mathbb{R}^n \to \mathbb{C} \colon \|f\|_{L^{p(\cdot)}_w(\mathbb{R}^n)} = \|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty \Big\}.$$

Note that if $w = 1$, then $L^{p(\cdot)}_w(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$ and if $p(\cdot)$ is a constant, then $L^{p(\cdot)}_w(\mathbb{R}^n)$ is the classical weighted Lebesgue space $L^p_w(\mathbb{R}^n)$.

We recall in the following lemma the Hölder inequality, for the proof see [6, Lemma 3.2.20].

Lemma 1. *Let $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ and $w : \mathbb{R}^n \to (0,\infty)$ be two Lebesgue measurable functions. Then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant 2\|f\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}_{w^{-1}}(\mathbb{R}^n)},$$

where, $p'(\cdot)$ denotes the conjugate function of $p(\cdot)$, that is: $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Let $\underline{p} = \min\{p_-, 1\}$.

Definition 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w\colon \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function. We denote by $W_{p(\cdot)}$ the set of all Lebesgue measurable functions w, such that*

- $\|\chi_B\|_{L^{p(\cdot)/\underline{p}}_{w^{\underline{p}}}(\mathbb{R}^n)} < \infty$ *and* $\|\chi_B\|_{L^{(p(\cdot)/\underline{p})'}_{w^{-\underline{p}}}(\mathbb{R}^n)} < \infty$, *for any ball $B \subset \mathbb{R}^n$;*
- *there exist $k > 1$ and $s > 1$, such that the Hardy-Littlewood maximal operator is bounded on $L^{(sp(\cdot))'/k}_{w^{-k/s}}(\mathbb{R}^n)$.*

Remark 1. *It is easy to see that $L^{sp(\cdot)}_{w^{1/s}}(\mathbb{R}^n)$ is the s-convexification of $L^{p(\cdot)}_w(\mathbb{R}^n)$.*

For any $w \in W_{p(\cdot)}$, set

$$s_w = \inf\{s \geqslant 1\colon M \text{ is bounded on } L^{(sp(\cdot))'}_{w^{-1/s}}(\mathbb{R}^n)\}$$

and

$$\mathbb{S}_w = \{s \geqslant 1\colon M \text{ is bounded on } L^{(sp(\cdot))'/k}_{w^{-k/s}}(\mathbb{R}^n), \text{for some } k > 1\}.$$

For any fixed $s \in \mathbb{S}_w$, we define

$$k^s_w = \sup\{k > 1\colon M \text{ is bounded on } L^{(sp(\cdot))'/k}_{w^{-k/s}}(\mathbb{R}^n)\}.$$

The following theorem is the Fefferman-Stein vector-valued maximal inequalities on $L^{p(\cdot)}_w(\mathbb{R}^n)$. For the proof, we refer to [12, Theorem 3.1].

Theorem 1. *Let $p(\cdot)\colon \mathbb{R}^n \to (0,\infty)$ be a Lebesgue-measurable function with $0 < p_- \leqslant p_+ < \infty$ and $q \in (1,\infty)$. If $w \in W_{p(\cdot)}$, then, for any $r > s_w$, we have*

$$\left\| \left(\sum_{i\in\mathbb{N}} (Mf_i)^q \right)^{1/q} \right\|_{L^{rp(\cdot)}_{w^{1/r}}(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_{i\in\mathbb{N}} |f_i|^q \right)^{1/q} \right\|_{L^{rp(\cdot)}_{w^{1/r}}(\mathbb{R}^n)}.$$

The following lemma plays a key role in the proofs of the main results of this paper; we refer the reader to [12, Lemma 5.4].

Lemma 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $w \in W_{p(\cdot)}$, $r \in (0,1]$, such that $\frac{1}{r} \in \mathbb{S}_w$ and $q \in \big(r(k_w^{1/r})', \infty\big)$. Then there exists a positive constant C, such that for any sequence $\{B_j\}_{j\in\mathbb{N}}$ of balls in $\mathbb{R}^n$, $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and functions $\{a_j\}_{j\in\mathbb{N}}$ satisfying the condition that for any $j \in \mathbb{N}$, Supp $a_j \subset B_j$ and $\|a_j\|_{L^q(\mathbb{R}^n)} \leqslant |B_j|^{1/q}$,*

$$\left\| \left(\sum_{j=1}^{\infty} |\lambda_j a_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}_w(\mathbb{R}^n)} \leqslant C \left\| \left(\sum_{j=1}^{\infty} |\lambda_j \chi_{B_j}|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}.$$

3. Weighted variable Hardy spaces. In this section, we introduce the weighted variable Hardy space associated to the operator L, and we establish its molecular characterization via the atomic decomposition of the weighted variable tent spaces given in this section. We begin by recalling some notation.

For all measurable functions f on $\mathbb{R}^{n+1}_+$ and for any $x \in \mathbb{R}^n$, define the operator $\mathcal{A}$ by

$$\mathcal{A}(f)(x) := \left(\iint\limits_{\Gamma(x)} |f(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$

Let $p(\cdot) \in (0,\infty)$. The tent space $T^p(\mathbb{R}^{n+1}_+)$ is the space of all measurable functions f on $\mathbb{R}^{n+1}_+$, such that

$$\|f\|_{T^p_2(\mathbb{R}^{n+1}_+)} = \|\mathcal{A}(f)\|_{L^p(\mathbb{R}^n)} < \infty.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w\colon \mathbb{R}^n \to (0,\infty)$. The weighted variable tent space $T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)$ is defined to be the space of all measurable functions f on $\mathbb{R}^{n+1}_+$, such that $\mathcal{A}(f) \in L^{p(\cdot)}_w(\mathbb{R}^n)$. For any $f \in T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)$, define

$$\|f\|_{T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)} = \|\mathcal{A}(f)\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}.$$

Let F be a closed set in $\mathbb{R}^n$ and $O \equiv F^{\complement}$; we denote by $\widehat{O}$ the tent over O, which is the set

$$\widehat{O} := \{(x,t) \in \mathbb{R}^{n+1}_+ \colon \operatorname{dist}(x,F) \geqslant t\}.$$

Next, we give the definition of $(p(\cdot), w, \infty)$-atoms:

Definition 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $w\colon \mathbb{R}^n \to (0,\infty)$ be a Lebesgue-measurable function and $r \in (1,\infty)$. A function a on $\mathbb{R}^{n+1}_+$ is called a $(p(\cdot), w, \infty)$-atom, if*

(i) *there exists a ball $B \subset \mathbb{R}^n$, such that* $\operatorname{supp} a \subset \widehat{B}$;

(ii) $\|a\|_{T^r(\mathbb{R}^{n+1}_+)} \leqslant |B|^{1/r} \|\chi_B\|^{-1}_{L^{p(\cdot)}_w(\mathbb{R}^n)}$.

The theorem presents the atomic characterization of $T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)$:

Theorem 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $w \in W_{p(\cdot)}$. Then, for any $f \in T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)$, there exist $(p(\cdot), w, \infty)$-atoms $\{a_i\}_{i\in\mathbb{N}}$ associated with the balls $\{B_i\}_{i\in\mathbb{N}}$, respectively, and numbers $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$, such that for almost every $(x,t) \in \mathbb{R}^{n+1}_+$,*

$$f(x,t) = \sum_{i\in\mathbb{N}} \lambda_i a_i(x,t). \tag{2}$$

Moreover, there exists a constant $C > 0$, such that for all $f \in T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)$:

$$\Lambda(\{\lambda_i\}_{i\in\mathbb{N}}, \{B_i\}_{i\in\mathbb{N}}) \leqslant C\|f\|_{T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)}, \tag{3}$$

where for any sequence of numbers $\{\lambda_i\}_{i\in\mathbb{N}} \in \mathbb{C}$ and sequence of balls $\{B_i\}_{i\in\mathbb{N}}$

$$\Lambda(\{\lambda_i\}_{i\in\mathbb{N}},\{B_i\}_{i\in\mathbb{N}}) := \left\| \left(\sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{B_i}}{\|\chi_{B_i}\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}} \right]^{\theta} \right)^{\frac{1}{\theta}} \right\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}. \tag{4}$$

Here and hereafter $\theta := \sup\{s^{-1} \colon s \in \mathbb{S}_w\}$.

Proof. Let $f \in T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)$. Let $\Omega_i = \{x \in \mathbb{R}^n : \mathcal{A}(f)(x) > 2^i\}$ for any $i \in \mathbb{Z}$. Since $f \in T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)$, it is easy to check that Ω_i is a proper open set and $|\Omega_i| < \infty$ for each $i \in \mathbb{Z}$. By a similar argument used in the proof of [15, Theorem 3.2], we can show that $\operatorname{supp} f \subset \left[\left(\cup_{i\in\mathbb{Z}} \widehat{\Omega^*_i} \right) \cup E \right]$, where $E \subset \mathbb{R}^{n+1}_+$ satisfying $\int_E \frac{dydt}{t} = 0$. Thus, for each $i \in \mathbb{Z}$, by applying the Whitney decomposition (see [22, p. 167]) to Ω^*_i, we get a sequence $\{Q_{i,j}\}_{j\in\mathbb{N}}$ of disjoint cubes, such that

1) $\bigcup_{j\in\mathbb{N}} Q_{i,j} = \Omega_i^*$ and $\{Q_{i,j}\}_{j\in\mathbb{N}}$ have disjoint interiors,

2) for all $j \in \mathbb{N}$,

$$c_1\sqrt{n}l_{Q_{i,j}} \leqslant \operatorname{dist}(Q_{i,j}, (\Omega_i^*)^{\complement}) \leqslant c_2\sqrt{n}l_{Q_{i,j}},$$

where $l_{Q_{i,j}}$ denotes the side-length of the cube $Q_{i,j}$ with $\operatorname{dist}(Q_{i,j},(\Omega_i^*)^{\complement}) := \inf\{|x-y| \colon x \in Q_{i,j}, y \in (\Omega_i^*)^{\complement}\}$.

For each $j \in \mathbb{N}$, choose a ball $B_{i,j}$ with the same center as $Q_{i,j}$ and with radius $\frac{11}{2}\sqrt{n}l(Q_{i,j})$. Define

$$A_{i,j} = \widehat{B}_{i,j} \cap (Q_{i,j} \times (0,\infty)) \cap (\widehat{\Omega_i^*} \setminus \widehat{\Omega_{i+1}^*}),$$

$$a_{i,j} = 2^{-i}\|\chi_{B_{i,j}}\|^{-1}_{L_w^{p(\cdot)}(\mathbb{R}^n)} f\chi_{A_{i,j}} \quad \text{and} \quad \lambda_{i,j} = 2^i\|\chi_{B_{i,j}}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}.$$

Note that $\{(Q_{i,j}\times(0,\infty))\cap(\widehat{\Omega_i^*}\setminus\widehat{\Omega_{i+1}^*})\} \subset \widehat{B}_{i,j}$. Following the proof used in [27, Theorem 2.16], we can show that $a_{i,j}$ is a $(p(\cdot), w, \infty)$-atom associated to the ball $B_{i,j}$ for any $i \in \mathbb{Z}$ and $j \in \mathbb{N}$. We see that $f = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{N}} \lambda_{i,j}a_{i,j}$ almost everywhere. Then remains to show that

$$\Lambda(\{\lambda_i\}_{i\in\mathbb{N}}, \{B_i\}_{i\in\mathbb{N}}) \leqslant C\|f\|_{T_w^{p(\cdot)}(\mathbb{R}_+^{n+1})}.$$

Indeed, by the definition of $\lambda_{i,j}$, we have

$$\begin{aligned}\Lambda(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}) &= \sum_{i\in\mathbb{Z}}\left\|\left(\sum_{j\in\mathbb{N}}\left(2^i\chi_{B_{i,j}}\right)^{\theta}\right)^{\frac{1}{\theta}}\right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \lesssim \\ &\lesssim \sum_{i\in\mathbb{Z}} 2^i\left\|\left(\sum_{j\in\mathbb{N}}\left(\chi_{B_{i,j}}\right)^{\theta}\right)^{\frac{1}{\theta}}\right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \lesssim \\ &\lesssim \sum_{i\in\mathbb{Z}} 2^i\left\|\chi_{\Omega_i^*}\right\|^{1/\theta}_{L_{w^\theta}^{p(\cdot)/\theta}(\mathbb{R}^n)}.\end{aligned}$$

Since $\frac{1}{\theta} \in \mathbb{S}_w$ and $\frac{1}{\theta} > s_w$. Hence, by Theorem 1 and the fact that $\chi_{\Omega_i^*} \lesssim \left[\mathcal{M}(\chi_{\Omega_i})\right]$, we find

$$\begin{aligned}\Lambda(\{\lambda_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}, \{B_{i,j}\}_{i\in\mathbb{Z},j\in\mathbb{N}}) &\lesssim \sum_{i\in\mathbb{Z}} 2^i\|\mathcal{M}(\chi_{\Omega_i})\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \lesssim \\ &\lesssim \|\mathcal{A}(f)\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} = \|f\|_{T_w^{p(\cdot)}(\mathbb{R}_+^{n+1})}.\end{aligned}$$

Now the proof is completed. □

We denote by $T^{p(\cdot)}_{c,w}(\mathbb{R}^{n+1}_+)$ and $T^2_c(\mathbb{R}^{n+1}_+)$ respectively, the set of all functions in $T^{p(\cdot)}_w(\mathbb{R}^{n+1}_+)$ and $T^2(\mathbb{R}^{n+1}_+)$ which have compact supports.

Proposition 1. *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $w\in W_{p(\cdot)}$. Then $T^{p(\cdot)}_{c,w}(\mathbb{R}^{n+1}_+)\subset T^2_c(\mathbb{R}^{n+1}_+)$ in the meaning of sets.*

Proof. By [16, Lemma 3.3(i)], we know that $T^q_c(\mathbb{R}^{n+1}_+)\subset T^2_c(\mathbb{R}^{n+1}_+)$ for any $q\in(0,\infty)$. Then it is sufficient to show that $T^{p(\cdot)}_{c,w}(\mathbb{R}^{n+1}_+)\subset T^{q'}_c(\mathbb{R}^{n+1}_+)$, for some $q'\in(0,\infty)$. Indeed, let $f\in T^{p(\cdot)}_{c,w}(\mathbb{R}^{n+1}_+)$ be such that $\operatorname{supp} f\subset E$, where E is a compact set of $\mathbb{R}^{n+1}_+$. Let B be a ball of $\mathbb{R}^{n+1}_+$ such that $E\subset\widehat{B}$. Then $\operatorname{supp}\mathcal{A}f\subset B$ and by Lemma 1 we have:

$$\int\limits_{\mathbb{R}^n}[\mathcal{A}f(x)]^{\underline{p}}\,dx=\int\limits_{\{x\in B:\mathcal{A}f(x)\leqslant 1\}}[\mathcal{A}f(x)]^{\underline{p}}\,dx+\int\limits_{\{x\in B:\mathcal{A}f(x)>1\}}[\mathcal{A}f(x)]^{\underline{p}}\,dx\lesssim$$
$$\lesssim|B|+\|(\mathcal{A}f)^{\underline{p}}\|_{L^{p(\cdot)/\underline{p}}_{w^{\underline{p}}}}\|\chi_B\|_{L^{(p(\cdot)/\underline{p})'}_{w^{-\underline{p}}}}\lesssim|B|+\|\mathcal{A}f\|^{\underline{p}}_{L^{p(\cdot)}_w}.$$

□

Next, we establish the molecular characterization of the weighted variable Hardy spaces associated with operators satisfying the Davies-Gaffney estimates. These kind of spaces are denoted by $H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$. We begin with some definitions.

Definition 3. *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $w:\mathbb{R}^n\to(0,\infty)$ be a Lebesgue measurable function. Let L be an operator satisfying Assumption (A) and Assumption (B). The weighted variable Hardy space $H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$ is defined as the completion of the space $\widetilde{H}^{p(\cdot)}_{L,w}(\mathbb{R}^n)$,*

$$\widetilde{H}^{p(\cdot)}_{L,w}(\mathbb{R}^n):=\{f\in L^2(\mathbb{R}^n)\colon\|S_L(f)\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}<\infty\},$$

with respect to the quasi-norm

$$\|f\|_{H^{p(\cdot)}_{L,w}(\mathbb{R}^n)}:=\|S_L(f)\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}=\inf\left\{\lambda>0\colon\rho_{p(\cdot),w}\Big(\frac{S_L(f)}{\lambda}\Big)\leqslant 1\right\},$$

where $\rho_{p(\cdot),w}\big(\frac{S_L(f)}{\lambda}\big):=\int\limits_{\mathbb{R}^n}\big[\frac{|S_L(f)(x)w(x)|}{\lambda}\big]^{p(x)}dx.$

To introduce the molecular weighted variable Hardy spaces $H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$, we give the definition of a $(p(\cdot),w,M,\epsilon)_L$-molecule.

Definition 4. *Let L be an operator satisfying Assumption (A) and Assumption (B) and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w\colon \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function. Assume that $M \in \mathbb{N}$ and $\epsilon \in (0,\infty)$. A function $m \in L^2(\mathbb{R}^n)$ is called a $(p(\cdot), w, M, \epsilon)_L$-molecule, if $m \in R(L^M)$ (the range of L^M) and there exists a ball $B := B(x_B, r_B) \subset \mathbb{R}^n$, where $x_B \in \mathbb{R}^n$ and $r_B > 0$ is such that for every $k = 0, \dots, M$ and $j \in \mathbb{Z}_+$:*

$$\left\|(r_B^{-2}L^{-1})^k m\right\|_{L^2(U_j(B))} \leqslant 2^{-\epsilon j}|2^j B|^{1/2}\|\chi_B\|^{-1}_{L_w^{p(\cdot)}(\mathbb{R}^n)},$$

where for $j \in \mathbb{N}$:

$$U_j(B) := B(x_B, 2^j r_B)\backslash B(x_B, 2^{j-1}r_B),$$

and for $j = 0$:

$$U_0(B) := B(x_B, r_B).$$

Definition 5. *Let L be an operator satisfying Assumption (A) and Assumption (B). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in W_{p(\cdot)}$. Assume that $M \in \mathbb{N}$ and $\epsilon \in (0,\infty)$. For a measurable function f on $\mathbb{R}^n$, $f = \sum_{j=1}^{\infty} \lambda_j m_j$ is called a molecular $(p(\cdot), w, M, \epsilon)$-representation of f, if $\{m_j\}_{j\in\mathbb{N}}$ is a family of $(p(\cdot), w, M, \epsilon)_L$-molecules, the sum converges in $L^2(\mathbb{R}^n)$ and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ satisfies*

$$\Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}\right) < \infty,$$

where $\Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}\right)$ is as in (4) and for any $j \in \mathbb{N}, B_j$ is the ball associated with m_j. The space $\widetilde{H}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^2(\mathbb{R}^n)$, which has a molecular $(p(\cdot), w, M, \epsilon)$-representation. The molecular weighted variable Hardy spaces $H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$ is the completion of $\widetilde{H}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$ with respect to the quasi-norm

$$\|f\|_{H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)} := \inf\Bigg\{\Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}\right) : f = \sum_{j=1}^{\infty} \lambda_j m_j \text{ is a molecular } (p(\cdot), w, M, \epsilon) - \text{representation}\Bigg\}.$$

To establish the molecular characterization of $H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$, we need the following technical lemmas. Let L be an operator satisfying Assumptions (A) and (B) and $M \in \mathbb{N}$. The next lemma can be proved by using an argument similar to that used in the proof of [24, Proposition 3.10].

Lemma 3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n), w \in W_{p(\cdot)}, M \in \left(\frac{n}{2}\left[\frac{1}{\theta} - \frac{1}{2}\right], \infty\right) \cap \mathbb{N}$ and $\epsilon > \frac{n}{\theta}$. There exist a constant C and $\sigma \in (\frac{n}{\theta}, \infty)$, such that, for any $j \in \mathbb{Z}_+$ and $(p(\cdot), w, M, \epsilon)_L$-molecule m, associated with the ball $B := B(x_B, r_B) \subset \mathbb{R}^n$, where $x_B \in \mathbb{R}^n$ and $r_B > 0$,*

$$\|S_L(m)\|_{L^2(U_j(B))} \leqslant C2^{-j\sigma}|2^j B|^{1/2}\|\chi_B\|^{-1}_{L^{p(\cdot)}_w(\mathbb{R}^n)}.$$

Proposition 2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n), w \in W_{p(\cdot)}$. Let $M \in \mathbb{N}$ and $\epsilon \in (0, \infty)$. Then the set of all finite linear combinations of $(p(\cdot), w, M, \epsilon)_L$-molecule denoted by $H^{p(\cdot),M,\epsilon}_{L,w,fin}(\mathbb{R}^n)$ is dense in $H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)}$.*

Proof. Let $g \in H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$. Then, by the definition of $H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$, we know that for any $\delta \in (0, \infty)$ there exists a function $f \in \widetilde{H}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$, such that

$$\|g - f\|_{H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)} \leqslant \frac{\delta}{2}.$$

By the definition of $\widetilde{H}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$, we conclude that there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and a family $\{m_j\}_{j\in\mathbb{N}}$ of $(p(\cdot), w, M, \epsilon)_L$-molecules, associated with the balls $\{B_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$, such that

$$f = \sum_{j=1}^{\infty} \lambda_j m_j \text{ in } L^2(\mathbb{R}^n) \text{ and } \Lambda(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}) < \infty.$$

Let $f_N = \sum_{j=1}^{N} \lambda_j m_j$ for any $N \in \mathbb{N}$; then we have

$$\begin{aligned}
\|f - f_N\|_{H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)} &= \left\| \sum_{j=N+1}^{\infty} \lambda_j m_j \right\|_{H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)} \leqslant \\
&\leqslant \Lambda\left(\{\lambda_j\}_{j=N+1}^{\infty}, \{B_j\}_{j=N+1}^{\infty}\right) = \\
&= \left\| \left\{ \sum_{j=N+1}^{\infty} \left[\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}} \right]^{\theta} \right\}^{1/\theta} \right\|_{L^{p(\cdot)}_w(\mathbb{R}^n)} = \\
&= \left\| \sum_{j=N+1}^{\infty} \left[\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}} \right]^{\theta} \right\|^{1/\theta}_{L^{p(\cdot)/\theta}_{w^\theta}(\mathbb{R}^n)}.
\end{aligned}$$

On the other hand, since

$$\Lambda\left(\{\lambda_j\}_{j=N+1}^{\infty},\{B_j\}_{j=N+1}^{\infty}\right)=\left\|\sum_{j=N+1}^{\infty}\left[\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\theta}\right\|_{L_{w^\theta}^{p(\cdot)/\theta}(\mathbb{R}^n)}^{1/\theta}<\infty,$$

it follows that for almost every $x\in\mathbb{R}^n$,

$$\lim_{N\to\infty}\sum_{j=N+1}^{\infty}\left[\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\theta}=0.$$

Combining this and the dominated convergence theorem (see, for example, [6, Lemma 3.2.8]), we obtain

$$\lim_{N\to\infty}\left\|\sum_{j=N+1}^{\infty}\left[\frac{|\lambda_j|\chi_{B_j}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\theta}\right\|_{L_{w^\theta}^{p(\cdot)/\theta}(\mathbb{R}^n)}^{1/\theta}=0.$$

Thus, we conclude that

$$\lim_{N\to\infty}\|f-f_N\|_{H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)}=0.$$

Hence, we find that for any $\delta\in(0,\infty)$ there exists some $N_0\in\mathbb{N}$, such that for any $N>N_0$:

$$\|f-f_N\|_{H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)}<\frac{\delta}{2}.$$

Obviously, for any $N\in\mathbb{N}$, $f_N\in H_{L,\omega,fin}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$. Then, for any $\delta\in(0,\infty)$, when $N>N_0$:

$$\|g-f_N\|_{H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)}\lesssim\|g-f\|_{H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)}+\|f-f_N\|_{H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)}\lesssim\delta.$$

Then we conclude that $H_{L,w,fin}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$ is dense in $H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)}$. □

The following theorem deals with the molecular characterization of $H_{L,w}^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 3. *Let L be an operator satisfying Assumption (A) and Assumption (B). Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $w\in W_{p(\cdot)}$. Let $M\in(\frac{n}{2}[\frac{1}{\theta}-\frac{1}{2}],\infty)\cap\mathbb{N}$*

and let $\epsilon \in (\frac{n}{\theta}, \infty)$*. Then* $H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$ *and* $H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$ *coincide with the equivalent quasi-norms.*

Let first show that $\widetilde{H}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n) \subset \Big[H^{p(\cdot)}_{L,w}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\Big]$.

Proposition 3. *Let* L *be an operator satisfying Assumption (A) and (B). Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n), w \in W_{p(\cdot)}$*. Let* $M \in (\frac{n}{2}[\frac{1}{\theta} - \frac{1}{2}], \infty) \cap \mathbb{N}$ *and let* $\epsilon \in (\frac{n}{\theta}, \infty)$*. Then there exists a positive constant* C*, such that for any* $f \in \widetilde{H}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$*,*

$$\|f\|_{H^{p(\cdot)}_{L,w}(\mathbb{R}^n)} \leqslant C\|f\|_{\widetilde{H}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)}.$$

Proof. Let $f \in \widetilde{H}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$. Then, by Definition 5, we know that there exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ and a family $\{m_j\}_{j\in\mathbb{N}}$ of $(p(\cdot), w, M, \epsilon)_L$-molecules associated with the balls $\{B_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$, such that $f = \sum\limits_{j=1}^{\infty} \lambda_j m_j$ in $L^2(\mathbb{R}^n)$ and

$$\|f\|_{H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)} \sim \Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}\right). \tag{5}$$

Since the operator S_L is bounded on $L^2(\mathbb{R}^n)$, we find that

$$\left\|S_L(f) - S_L\Big(\sum_{j=1}^{N} \lambda_j m_j\Big)\right\|_{L^2(\mathbb{R}^n)} \xrightarrow[N\to\infty]{} 0.$$

Hence, there exists a subsequence $\left\{S_L\Big(\sum_{j=1}^{N_k} \lambda_j m_j\Big)\right\}_{k\in\mathbb{N}}$, such that for almost every $x \in \mathbb{R}^n$:

$$\lim_{k\to\infty} S_L\left(\sum_{j=1}^{N_k} \lambda_j m_j\right)(x) = S_L(f)(x).$$

Thus, for almost every $x \in \mathbb{R}^n$ we have

$$S_L(f)(x) \leqslant \sum_{j=1}^{\infty}\sum_{i=0}^{\infty} |\lambda_j| S_L(m_j)(x)\chi_{U_i(B_j)}(x).$$

Then

$$\|S_L(f)\|^{\theta}_{L^{p(\cdot)}_{\omega}(\mathbb{R}^n)} = \left\|[S_L(f)]^{\theta}\right\|_{L^{p(\cdot)/\theta}_{w^{\theta}}(\mathbb{R}^n)} \leqslant$$

$$\leqslant \sum_{i=0}^{\infty} \left\| \sum_{j=1}^{\infty} |\lambda_j|^\theta \left[S_L(m_j)\chi_{U_i(B_j)} \right]^\theta \right\|_{L_{w^\theta}^{p(\cdot)/\theta}(\mathbb{R}^n)} \leqslant$$

$$\leqslant \sum_{i=0}^{\infty} \left\| \left[\sum_{j=1}^{\infty} |\lambda_j|^\theta \left[S_L(m_j)\chi_{U_i(B_j)} \right]^\theta \right]^{1/\theta} \right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^\theta . \quad (6)$$

By Lemma 3, we find that for any $j \in \mathbb{N}$ and $i \in \mathbb{Z}_+$,

$$\left\| S_L(m_j) \right\|_{L^2(U_i(B_j))} = \left\| S_L(m_j)\chi_{U_i(B_j)} \right\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-i\sigma} |2^i B_j|^{1/2} \|\chi_{B_j}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^{-1}, \quad (7)$$

where $\sigma \in (\frac{n}{\theta}, \infty)$. Multiplying (7) by $2^{i\sigma} \|\chi_{B_j}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}$, we obtain

$$\left\| 2^{i\sigma} \|\chi_{B_j}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} S_L(m_j)\chi_{U_i(B_j)} \right\|_{L^2(\mathbb{R}^n)} \lesssim |2^i B_j|^{1/2}.$$

We apply Lemma 2 for $a_j := 2^{i\sigma} \|\chi_{B_j}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} S_L(m_j)\chi_{U_i(B_j)}$, to conclude that

$$\left\| \left\{ \sum_{j=1}^{\infty} |\lambda_j|^\theta \left[S_L(m_j)\chi_{U_i(B_j)} \right]^\theta \right\}^{1/\theta} \right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left[2^{-i\sigma} \|\chi_{B_j}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^{-1} |\lambda_j| \chi_{2^i B_j} \right]^\theta \right\}^{1/\theta} \right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} .$$

From the fact that $\chi_{2^i B_j}(x) \leqslant 2^{in} M(\chi_{B_j})(x)$, we deduce that

$$\left\| \left\{ \sum_{j=1}^{\infty} |\lambda_j|^\theta \left[S_L(m_j)\chi_{U_i(B_j)} \right]^\theta \right\}^{1/\theta} \right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left[2^{-i\sigma} \|\chi_{B_j}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^{-1} |\lambda_j| 2^{in} M(\chi_{B_j}) \right]^\theta \right\}^{1/\theta} \right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} .$$

We choose $r \in (0, \theta)$ such that $\sigma > nr^{-1}$. Then, by Remark 1 and Theorem 1, we have

$$\left\| \left\{ \sum_{j=1}^{\infty} |\lambda_j|^\theta \left[S_L(m_j)\chi_{U_i(B_j)} \right]^\theta \right\}^{1/\theta} \right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \left\| \left\{ \sum_{j=1}^{\infty} \left[(2^{-i\sigma})^r (2^{in})^r \|\chi_{B_j}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}^{-r} |\lambda_j|^r M(\chi_{B_j}) \right]^{\theta/r} \right\}^{1/\theta} \right\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \lesssim$$

$$\lesssim 2^{-i(\sigma-\frac{n}{r})}\left\|\left\{\sum_{j=1}^{\infty}\left[M\Big(\frac{|\lambda_j|^r}{\|\chi_{B_j}\|^r_{L^{p(\cdot)}_w(\mathbb{R}^n)}}\chi_{B_j}\Big)\right]^{\theta/r}\right\}^{r/\theta}\right\|^{1/r}_{L^{p(\cdot)/r}_{w^r}(\mathbb{R}^n)}\lesssim$$

$$\lesssim 2^{-i(\sigma-\frac{n}{r})}\left\|\left\{\sum_{j=1}^{\infty}\left[\frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}}\chi_{B_j}\right]^{\theta}\right\}^{1/\theta}\right\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}\lesssim$$

$$\lesssim 2^{-i(\sigma-\frac{n}{r})}\Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}}\right)\sim 2^{-i(\sigma-\frac{n}{r})}\|f\|_{H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)}.$$

From this, (5) and (6), we infer that for any $f\in\widetilde{H}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$,

$$\|f\|_{H^{p(\cdot)}_{L,w}(\mathbb{R}^n)}=\|S_L(f)\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}\lesssim\left\{\sum_{i=0}^{\infty}2^{-i(\sigma-\frac{n}{r})}\right\}^{1/\theta}\|f\|_{H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)}\sim$$
$$\sim\|f\|_{H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)},$$

which is the desired result. □

The following proposition shows that $\left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right]$ is a subset of the space $\widetilde{H}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$.

Proposition 4. *Let L be an operator satisfying Assumptions (A) and (B). Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $w\in W_{p(\cdot)}$. Let $M\in\mathbb{N}$ and let $\epsilon\in(0,\infty)$. Then for any $f\in\left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\right]$ there exist $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$ and a family $\{m_j\}_{j\in\mathbb{N}}$ of $(p(\cdot),w,M,\epsilon)_L$-molecules, associated with the balls $\{B_j\}_{j\in\mathbb{N}}$ of $\mathbb{R}^n$, such that $f=\sum_{j=1}^{\infty}\lambda_j m_j$ in $L^2(\mathbb{R}^n)$,*

$$\Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}}\right)\leqslant C\|f\|_{H^{p(\cdot)}_{L,w}(\mathbb{R}^n)}.$$

Proof. Let $f\in H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and $F(x,t):=t^2Le^{-t^2L}f(x)$, for all $(x,t)\in\mathbb{R}^{n+1}_+$. Then $F\in T^{p(\cdot)}_w(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Hence, by Theorem 2, there exist $(p(\cdot),w,\infty)$-atoms $\{a_j\}_{j\in\mathbb{N}}$, associated with the balls $\{B_j\}_{j\in\mathbb{N}}$, respectively, and numbers $\{\lambda_j\}_{j\in\mathbb{N}}\subset\mathbb{C}$, such that for almost $(x,t)\in\mathbb{R}^{n+1}_+$:

$$F(x,t)=\sum_{j\in\mathbb{N}}\lambda_j a_j(x,t),\ \text{ in } T^{p(\cdot)}_w(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$$

and

$$\Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}},\{B_j\}_{j\in\mathbb{N}}\right)\leqslant C\|F\|_{T^{p(\cdot)}_w(\mathbb{R}^n)}\sim\|f\|_{H^{p(\cdot)}_{L,w}(\mathbb{R}^n)}.$$

By the $H_\infty-$calculi of L, we know that

$$f = C_M \int_0^\infty (t^2L)^{M+1}e^{-t^2L}\big(t^2Le^{-t^2L}(f)\big)\frac{dt}{t} = \pi_{M,L}(F), \text{ in } L^2(\mathbb{R}^n),$$

where $C_M \int_0^\infty t^{2(M+2)}e^{-t^2}\frac{dt}{t} = 1$. From the fact that $\pi_{M,L}$ is a bounded map from $T^2(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^n)$, it follows that

$$f = C_M \times \pi_{M,L}\Big(\sum_{j\in\mathbb{N}}\lambda_j a_j\Big) = C_M\Big(\sum_{j\in\mathbb{N}}\lambda_j\pi_{M,L}(a_j)\Big), \text{ in } L^2(\mathbb{R}^n).$$

Following the argument used for [24, Lemma 3.11], we can prove that $m_j = \pi_{M,L}(a_j)$ is a multiple of a $(p(\cdot), w, M, \epsilon)_L$-molecule adopted to B_j; this implies the desired result. □

Next, we give the proof of Theorem 3.

Proof. By Proposition 3, Proposition 4, and the density argument, we have for $M \in (\frac{n}{2}[\frac{1}{\theta} - \frac{1}{2}], \infty) \cap \mathbb{N}$ and each $\epsilon \in (\frac{n}{\theta}, \infty)$: the spaces $H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$ and $H^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$ coincide with equivalent quasi-norms. □

4. Dual space. In this section, we study the duality of $H^{p(\cdot)}_{L,\omega}(\mathbb{R}^n)$. Here and hereafter, we denote by L^* the adjoint operator of L in $L^2(\mathbb{R}^n)$. Let us first recall some basic notions and definitions.

Definition 6. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w\colon \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function. Let L be an operator satisfying Assumption (A) and Assumption (B). Then for any $M \in \mathbb{N}$ and $\epsilon \in (0,\infty)$ define*

$$\mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n) := \Big\{ f = L^M(g) \in L^2(\mathbb{R}^n)\colon g \in \mathcal{D}(L^M), \|f\|_{\mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)} < \infty \Big\},$$

where $\mathcal{D}(L^M)$ denotes the domain of the operator L^M and

$$\|f\|_{\mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)} := \sup_{j\in\mathbb{Z}_+}\Big\{ 2^{j(\epsilon-\frac{n}{2})}\|\chi_{B(0,1)}\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}\sum_{k=0}^{M}\|L^{-k}(f)\|_{L^2(U_j(B(0,1)))}\Big\}.$$

The dual space of $\mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$ denoted by $\Big[\mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)\Big]^*$ is defined as the set of all bounded linear functionals on $\mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$. Then, *for any* $f \in \left[\mathcal{M}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)\right]^*$ *and* $g \in \mathcal{M}_{L,}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$*, the duality between* $\left[\mathcal{M}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)\right]^*$ *and* $\mathcal{M}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$ *is denoted by* $\langle f, g\rangle_{\mathcal{M}}$*. Let* $\mathcal{M}_{L,w}^{p(\cdot),M,*}(\mathbb{R}^n) = \cap_{\epsilon\in(0,\infty)}\left[\mathcal{M}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)\right]^*$.

Definition 7. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $w\colon \mathbb{R}^n \to (0,\infty)$ *be a Lebesgue measurable function. Let* $M \in \mathbb{N}$ *and* L *be an operator satisfying Assumption (A) and Assumption (B). We say that an element* $f \in \mathcal{M}_{L,w}^{p(\cdot),M,*}(\mathbb{R}^n)$ *is in* $BMO_{L^*,w}^{p(\cdot),M}(\mathbb{R}^n)$ *if*

$$\|f\|_{BMO_{L^*,w}^{p(\cdot),M}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \frac{|B|^{1/2}}{\|\chi_B\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}\left[\int_B |(I - e^{r_B^2 L^*})^M(f)(x)|^2 dx\right]^{1/2} < \infty,$$

where the supremum is taken over all balls of $\mathbb{R}^n$.

The following result can be seen as an extension of [24, Proposition 4.3] to the weighted $\mathcal{M}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$.

Proposition 5. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $w \in W_{p(\cdot)}$*. Let* $M \in \mathbb{N}$ *and* $\epsilon \in (0,\infty)$*. If* $f \in \mathcal{M}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$*, then* f *is a harmless positive constant multiple of a* $(p(\cdot), w, M, \epsilon)_L$*-molecule associated with the ball* $B(0,1)$*. Conversely, if* m *is a* $(p(\cdot), w, M, \epsilon)_L$*-molecule associated with the ball* $B \subset \mathbb{R}^n$*, then* $m \in \mathcal{M}_{L,w}^{p(\cdot),M,\epsilon}(\mathbb{R}^n)$.

The following three estimates play important roles in the proofs of our main results in this section. The proof of the next lemma can be done with similar arguments as in [24, Lemma 9].

Lemma. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $w \in W_{p(\cdot)}$, $M \in \mathbb{N}$*. Then* $f \in BMO_{L,w}^{p(\cdot),M}(\mathbb{R}^n)$ *is equivalent to*

$$\|f\|_{BMO_{L,w}^{p(\cdot),M,res}(\mathbb{R}^n)} :=$$
$$:= \sup_{B\subset\mathbb{R}^n} \frac{|B|^{1/2}}{\|\chi_B\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}\left[\int_B \left|\left(I - (I + r_B^2 L)^{-1}\right)^M(f)(x)\right|^2 dx\right]^{1/2} < \infty,$$

where the supremum is taken over all balls of $\mathbb{R}^n$*. Moreover, there exists a positive constant* C*, such that for any* $f \in BMO_{L,w}^{p(\cdot),M}(\mathbb{R}^n)$ *we have*

$$C^{-1}\|f\|_{BMO_{L,w}^{p(\cdot),M}(\mathbb{R}^n)} \leqslant \|f\|_{BMO_{L,w}^{p(\cdot),M,res}(\mathbb{R}^n)} \leqslant C\|f\|_{BMO_{L,w}^{p(\cdot),M}(\mathbb{R}^n)}.$$

Similarly to [24, Lemma 4.5], we have the next lemma. The proof is left to the reader.

Lemma 4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in W_{p(\cdot)}$. Let $\epsilon, \widetilde{\epsilon} \in (0, \infty)$ and $M \in \mathbb{N}$ and $\widetilde{M} > M + \widetilde{\epsilon} + \frac{n}{4}$. Suppose that $f \in \mathcal{M}_{L,w}^{p(\cdot),M,*}(\mathbb{R}^n)$ satisfies*

$$\int\limits_{\mathbb{R}^n} \frac{\left|\left[I - (I + L^*)^{-1}\right]^M (f)(x)\right|^2}{1 + |x|^{n+\widetilde{\epsilon}}} dx < \infty. \tag{8}$$

Then, for any $(p(\cdot), w, \widetilde{M}, \epsilon)_L$-molecule m, it is true that

$$\langle f, m\rangle_{\mathcal{M}} = C_M \iint\limits_{\mathbb{R}^{n+1}_+} (t^2L^*)^M e^{-t^2L^*}(f)(x) \overline{t^2 L e^{-t^2 L}(m)(x)} \frac{dx dt}{t},$$

where C_M is a positive constant, depending on M, which satisfies

$$C_M \int\limits_0^\infty t^{M+1} e^{2t^2} \frac{dt}{t} = 1.$$

The proof of the following lemma is similar to that of [14, Lemma 8.3] and [24, Lemma 4.7]:

Lemma 5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in W_{p(\cdot)}$ and $M \in \mathbb{N}$. Then there exists a positive constant C, such that for any $f \in BMO_{L,w}^{p(\cdot),M}(\mathbb{R}^n)$:*

$$\sup_{B \subset \mathbb{R}^n} \frac{|B|^{1/2}}{\|\chi_B\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}} \left[\iint\limits_{\widehat{B}} |(t^2L)^M e^{-t^2L}(f)(x)|^2 \frac{dx dt}{t}\right]^{1/2} \leqslant C\|f\|_{BMO_{L,w}^{p(\cdot),M}(\mathbb{R}^n)},$$

where the supremum is taken over all balls of $\mathbb{R}^n$.

Proposition 6. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in W_{p(\cdot)}$. For any $\widetilde{\epsilon} \in (\frac{2n}{\theta}, \infty)$, $M \in \mathbb{N}$ and $f \in BMO_{L^*,w}^{p(\cdot),M}(\mathbb{R}^n)$. Then f satisfies (8).*

Proof. We can check that f satisfies (8) by following the argument used for [25, Remark 4.8], with $\mathcal{X} = \mathbb{R}^n$ and p_- replaced by θ. □

In the following result, we prove the duality of the space $H_{L^*,w}^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $w \in W_{p(\cdot)}$ such that $p_+ \in (0, 1]$. Let $M \in (\frac{n}{2}[\frac{1}{\theta} - \frac{1}{2}], \infty) \cap \mathbb{N}$, $\widetilde{M} > M + \frac{2n}{\theta} + \frac{n}{4}$ and $\epsilon \in (\frac{n}{\theta}, \infty)$. Then, $\left[H_{L,w}^{p(\cdot)}(\mathbb{R}^n)\right]^*$ coincides with $BMO_{L^*,w}^{p(\cdot),M}(\mathbb{R}^n)$ in the following sense:*

(i) *Let* $g \in \left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\right]^*$. *Then* $g \in BMO^{p(\cdot),M}_{L^*,w}(\mathbb{R}^n)$ *and for any* $f \in H^{p(\cdot),M,\epsilon}_{L,w,fin}(\mathbb{R}^n)$ *the following holds true:* $g(f) = \langle g, f\rangle_{\mathcal{M}}$, *and there exists a positive constant* C, *such that for any* $g \in \left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\right]^*$:

$$\|g\|_{BMO^{p(\cdot),M}_{L^*,w}(\mathbb{R}^n)} \leqslant C\|g\|_{\left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\right]^*}.$$

(ii) *Conversely, let* $g \in BMO^{p(\cdot),M}_{L^*,w}(\mathbb{R}^n)$. *Then, for any* $f \in H^{p(\cdot),\widetilde{M},\epsilon}_{L,w,fin}(\mathbb{R}^n)$, *the linear functional* ℓ_g *defined by* $\ell_g(f) = \langle g, f\rangle_{\mathcal{M}}$ *has a unique bounded extension to* $H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$ *and there exists a positive* C, *such that for any* $g \in BMO^{p(\cdot),M}_{L^*,w}(\mathbb{R}^n)$:

$$\|\ell_g\|_{\left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\right]^*} \leqslant C\|g\|_{BMO^{p(\cdot),M}_{L^*,w}(\mathbb{R}^n)}.$$

Proof. First we show (i). Let $g \in \left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\right]^*$. For any $f \in H^{p(\cdot)}_{L,w}(\mathbb{R}^n)$, we have:

$$|g(f)| \leqslant \|g\|_{\left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\right]^*}\|f\|_{H^{p(\cdot)}_{L,w}(\mathbb{R}^n)}.$$

We also know that for any $(p(\cdot), w, M, \epsilon)_L$-molecule m $\|m\|_{H^{p(\cdot)}_{L,w}(\mathbb{R}^n)} \lesssim 1$. Thus,

$$|g(m)| \lesssim \|g\|_{\left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\right]^*}. \tag{9}$$

On the other hand, by Proposition 5, we find that for any $h \in \mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$ with $\|h\|_{\mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)} = 1$: h is multiple of a $(p(\cdot), w, M, \epsilon)_L$-molecule up to harmless positive constant associated with the ball $B(0, 1)$. From (9), we know that for any $\epsilon \in (0, \infty)$, $g \in \left[\mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)\right]^*$. Hence, $g \in \mathcal{M}^{p(\cdot),M,*}_{L,w}(\mathbb{R}^n)$, and for any $h \in \mathcal{M}^{p(\cdot),M,\epsilon}_{L,w}(\mathbb{R}^n)$:

$$\langle g, h\rangle_{\mathcal{M}} = g(h).$$

Next, we show that

$$\|g\|_{BMO^{p(\cdot),M}_{L^*,w}(\mathbb{R}^n)} \leqslant C\|g\|_{\left[H^{p(\cdot)}_{L,w}(\mathbb{R}^n)\right]^*}.$$

We take a ball $B \subset \mathbb{R}^n, h \in L^2(B)$ with $\|h\|_{L^2(B)} = 1$. Following the argument used in [25], we learn that $\frac{|B|^{1/2}}{\|\chi_B\|_{L^{p(\cdot)}_w(\mathbb{R}^n)}}\left(I - e^{r_B^2 L}\right)^M(h)$ is a harmless positive constant multiple of a $(p(\cdot), w, M, \epsilon)_L$-molecule. Therefore,

$$\left|\frac{|B|^{1/2}}{\|\chi_B\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}\int\limits_B \left(I - e^{r_B^2 L^*}\right)^M (g)(x)h(x)dx\right| =$$

$$= \left|\left\langle g, \frac{|B|^{1/2}}{\|\chi_B\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}\left(I - r^{r_B^2 L}\right)^M (h)\right\rangle_{\mathcal{M}}\right| \lesssim \|g\|_{[H_{L,w}^{p(\cdot)}(\mathbb{R}^n)]^*},$$

which implies that for any ball $B \subset \mathbb{R}^n$:

$$\frac{|B|^{1/2}}{\|\chi_B\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}\left\{\int\limits_B \left|(I - e^{r_B^2 L^*})^M (g)(x)\right|^2 dx\right\}^{1/2} \lesssim \|g\|_{[H_{L,w}^{p(\cdot)}(\mathbb{R}^n)]^*}.$$

Hence, we get the desired result.

Now we turn to prove (ii). Let $g \in BMO_{L^*,w}^{p(\cdot),M}(\mathbb{R}^n)$. We define

$$\ell_g(f) := \int\limits_{\mathbb{R}^n} f(x)g(x)dx$$

for $f \in H_{L,w,fin}^{p(\cdot),\widetilde{M},\epsilon}(\mathbb{R}^n)$. Since $f \in H_{L,w,fin}^{p(\cdot),\widetilde{M},\epsilon}(\mathbb{R}^n) \subset H_{L,w}^{p(\cdot)}(\mathbb{R}^n)$, we have $t^2 L e^{-t^2 L} f \in T_w^{p(\cdot)}(\mathbb{R}_+^{n+1})$. Then, by Theorem 2, $t^2 L e^{-t^2 L} f = \sum\limits_{j\in\mathbb{N}} \lambda_j a_j$, where $\{a_j\}_{j\in\mathbb{N}}$ is a sequence of $(p(\cdot), w, \infty)$-atoms supported by $\{B_j\}_{j\in\mathbb{N}}$. By Proposition 6, we know that g satisfies inequality (8) for $\widetilde{\epsilon} > \frac{2n}{\theta}$. Thus, it follows from Lemma 4, the Hölder inequality, and Lemma 5 that:

$$\ell_g(f) = \left|C_M \iint\limits_{\mathbb{R}_+^{n+1}} (t^2 L^*)^M e^{-t^2 L^*}(g)(x)\overline{t^2 L e^{-t^2 L}(f)(x)}\frac{dxdt}{t}\right| \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} |\lambda_j| \iint\limits_{\mathbb{R}_+^{n+1}} \left|(t^2 L^*)^M e^{-t^2 L^*}(g)(x)\right| |a_j(x,t)| \frac{dxdt}{t} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} |\lambda_j| \left[\iint\limits_{\widehat{B}_j} \left|(t^2 L^*)^M e^{-t^2 L^*}(g)(x)\right|^2 \frac{dxdt}{t}\right]^{1/2} \left[\iint\limits_{\widehat{B}_j} \left|a_j(x,t)\right|^2 \frac{dxdt}{t}\right]^{1/2} \lesssim$$

$$\lesssim \sum_{j=1}^{\infty} |\lambda_j| \|g\|_{BMO_{L^*,w}^{p(\cdot),M}(\mathbb{R}^n)} \lesssim$$

$$\lesssim \Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}\right) \|g\|_{BMO^{p(\cdot),M}_{L^*,w}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}_{L,w}(\mathbb{R}^n)} \|g\|_{BMO^{p(\cdot),M}_{L^*,w}(\mathbb{R}^n)},$$

where in the third inequality uses the fact that

$$D = \sum_{j=1}^{\infty} |\lambda_j| \leqslant \Lambda\left(\{\lambda_j\}_{j\in\mathbb{N}}, \{B_j\}_{j\in\mathbb{N}}\right).$$

Indeed,

$$\int_{\mathbb{R}^n} \left(\sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{B_i}}{D\|\chi_{B_i}\|} \right]^{\theta} \right)^{\frac{p(x)}{\theta}} w(x)^{\theta} dx \geqslant \int_{\mathbb{R}^n} \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{B_i}}{D\|\chi_{B_i}\|} \right]^{p(x)} w(x)^{\theta p(x)} dx \geqslant$$
$$\geqslant \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^n} \frac{|\lambda_i|}{D} \left[\frac{\chi_{B_i}}{\|\chi_{B_i}\|} \right]^{p(x)} w(x)^{\theta p(x)} dx \geqslant 1.$$

Hence, the proof of Theorem 4 is finished. □

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