DOI: 10.15393/j3.art.2022.11230

UDC 517.521.2, 517.988

Sejal V. Puvar, R. G. Vyas

ĆIRIĆ-TYPE RESULTS IN QUASI-METRIC SPACES AND G-METRIC SPACES USING SIMULATION FUNCTION

Abstract. In this paper, we establish existence of some common fixed-point theorems for admissible mappings via a simulation function along with $\mathcal{C}$-class functions in quasi-metric spaces. As a consequence, these results are extended to G-metric spaces and metric spaces.

Key words: *quasi-metric space, G-metric space, simulation function, common fixed point, admissible mappings*

2020 Mathematical Subject Classification: *47H10, 54H25*

1. Introduction. Jleli and Samet [4], Samet et al. [13] have shown that a G-metric space has a quasi-metric type structure. Then many results for such spaces follow from results for quasi-metric spaces.

Khojasteh [7] introduced the simulation function and proved fixed-point theorems in metric spaces. Later, Roldań et al. [10] modified the definition of the simulation function by removing the symmetry condition, and introduced a $(\mathcal{Z}, g)$-contraction. Roldań et al. [12] investigated the existence and uniqueness of coincidence points via simulation functions in the setting of quasi-metric spaces and deduced corresponding results in the framework of G-metric spaces.

Radenović and Chandok [9] proved common fixed-point theorems for a $(\mathcal{Z}_{\mathcal{G}}, g)$-contraction and a generalized $(\mathcal{Z}_{\mathcal{G}}, g)$-contraction. They also introduced a $(\mathcal{Z}_{\mathcal{G}}, g)$-quasi-contraction of Ćirić-Das-Naik type and posed an open problem regarding common fixed point theorems for a $(\mathcal{Z}_{\mathcal{G}}, g)$-quasi-contraction of Ćirić-Das-Naik type in metric spaces.

In this paper, we use the α-admissible mapping, introduce a $(\mathcal{Z}_{(\alpha,\mathcal{G})}, g)$-quasi-contraction of Ćirić type, prove common fixed-point theorems in quasi-metric spaces, and observe its consequences to G-metric spaces.

2. Preliminaries.

Definition 1. *[4] Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function, such that the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) \leqslant d(x, z) + d(z, y)$*, for any points* $x, y, z \in X$.

Then, d is called a quasi-metric on X and the pair (X, d) is called a quasi-metric space.

Definition 2. *Let $T, g : X \to X$ be self maps on X. A point $x \in X$ is called a:*

- *fixed point of the operator T, if $Tx = x$; we denote $x \in Fix(T)$;*
- *coincidence point of T and g, if $Tx = gx$; we denote $x \in C(T, g)$;*
- *common fixed point of T and g, if $Tx = gx = x$.*

Definition 3. *Let (X, d) be a quasi-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if*

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0. \tag{1}$$

The limit of a sequence in a quasi-metric space is unique.

Definition 4. *Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is*

- *left-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geqslant m > N$.*
- *right-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geqslant n > N$.*
- *Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.*

A sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 5. *Let (X, d) be a quasi-metric space. We say that (X, d) is complete if and only if each Cauchy sequence in X is convergent.*

Lemma 1. *[5] Let $\{x_n\}$ be a sequence in a quasi-metric space (X, d), such that* *(i)* $d(x_{n+1}, x_{n+2}) \leqslant \lambda d(x_n, x_{n+1}), n \geqslant 0,$

(ii) $d(x_{n+2}, x_{n+1}) \leqslant \lambda d(x_{n+1}, x_n), n \geqslant 0,$

for some $\lambda \in (0, 1)$. *Then* $\{x_n\}$ *is a Cauchy sequence in* X.

Definition 6. *[11] A subset E of a metric space (X, d) is said to be precomplete if every Cauchy sequence $\{u_n\}$ in E converges to a point of X.*

Similarly, precompleteness is defined for quasi-metric space.

Lemma 2. *[12] Let (X, d) be a quasi-metric space and $T\colon X \to X$ be a given mapping. Suppose that T is continuous at $u \in X$. Then, for each sequence $\{x_n\}$ in X, such that $x_n \to u$, we have $Tx_n \to Tu$; that is,*

$$\lim_{n\to\infty} d(Tx_n, Tu) = \lim_{n\to\infty} d(Tu, Tx_n) = 0.$$

Every quasi-metric induces a metric, that is, if (X, d) is a quasi-metric space, then the function $\delta\colon X \times X \to [0, \infty)$, defined by

$$\delta(x, y) = \max\{d(x, y), d(y, x)\}$$

is a metric on X (see [4]).

The following result is an immediate consequence of the above definition:

Theorem 1. *[4] Let (X, d) be a quasi-metric space, $\delta\colon X \times X \to [0, \infty)$ be the function defined by $\delta(x, y) = \max\{d(x, y), d(y, x)\}$. Then*

(1) (X, δ) *is a metric space;*

(2) $\{x_n\} \subset X$ *is convergent to* x *in* (X, d) *if and only if* $\{x_n\}$ *is convergent to* x *in* (X, δ);

(3) $\{x_n\} \subset X$ *is Cauchy in* (X, d) *if and only if* $\{x_n\}$ *is Cauchy in* (X, δ);

(4) (X, d) *is complete if and only if* (X, δ) *is complete.*

Definition 7. *[14] Let $T, g\colon X \to X$ and $\alpha\colon X \times X \to [0, \infty)$ be mappings. We say that T is α-admissible for g if*

$$\alpha(gx, gy) \geqslant 1 \implies \alpha(Tx, Ty) \geqslant 1 \quad for \quad all \quad x, y \in X.$$

For $g = i_X$ (identity mapping on X), T is an α-admissible mapping.

Definition 8. *Let $T, g\colon X \to X$ and $\alpha\colon X \times X \to [0, \infty)$ be mappings. We say that T is triangular α-admissible for g if T is α-admissible for g and*

$$\alpha(gx, gy) \geqslant 1 \text{ and } \alpha(gy, gz) \geqslant 1 \implies \alpha(gx, gz) \geqslant 1 \text{ for all } x, y, z \in X.$$

Definition 9. *[10] Let $T, g\colon X \to X$ be self-mappings on X. A sequence $\{x_n\}$ in X is said to be a Picard-Jungck sequence of the pair (T, g) (based on x_0) if $gx_{n+1} = Tx_n$, for all $n \geqslant 0$.*

If $T(X) \subseteq g(X)$, then there exists a Picard-Jungck sequence of (T, g) based on any point $x_0 \in X$.

Definition 10. *[12] Let $T, g\colon X \to X$ be mappings on a quasi-metric space (X, d). We say that T and g are compatible if and only if*

$$\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0 \text{ or } \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$$

for all sequences $\{x_n\} \subseteq X$ such that the sequences $\{gx_n\}$ and $\{Tx_n\}$ are convergent and have the same limit.

Ćirić [2] introduced the quasi-contraction and proved fixed point theorems for metric spaces.

Definition 11. *[2] Let (X, d) be a metric space and $T\colon X \to X$ be a self-mapping on X. A mapping T is said to be a quasi-contraction if and only if there exists a number λ, $0 \leqslant \lambda < 1$, such that*

$$d(Tx, Ty) \leqslant \lambda max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$.

Later, Das and Naik generalized the quasi-contraction of Ćirić for two mappings and established the following result:

Theorem 2. *[3] Let (X, d) be a complete metric space. Let T be a continuous self-mapping on X and g be any self-mapping on X that commutes with T. Further, $g(X) \subseteq T(X)$ and there exists a constant $\lambda \in (0, 1)$, such that, for every $x, y \in X$,*

$$d(gx, gy) \leqslant \lambda \max\{d(Tx, Ty), d(Tx, gx), d(Ty, gy), d(Tx, gy), d(Ty, gx)\}.$$

Then T and g have a unique fixed point.

Roldan et al. [10] modified the definition of simulation function by Khojasteh et al. [7] as follows:

Definition 12. *A simulation function is a function* $\zeta\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ *satisfying the following conditions:*

(ζ_1) $\zeta(0,0)=0$;

(ζ_2) $\zeta(t,s)<s-t$ *for all* $t,s>0$;

(ζ_3) *if* $\{t_n\}$ *and* $\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n>0$ *and* $t_n<s_n$, *then* $\limsup_{n\to\infty}\zeta(t_n,s_n)<0$.

Set of all simulation functions is denoted by $\mathcal{Z}$. It is clear that a simulation function must satisfy $\zeta(s,s)<0$ for all $s>0$.

Definition 13. *[10] Let* (X,d) *be a metric space,* T,$g:X\to X$ *be self mappings. Then* T *is called a* $(\mathcal{Z},g)$*-contraction if there exists* $\zeta\in\mathcal{Z}$, *such that*

$$\zeta(d(Tu,Tv),d(gu,gv))\geqslant 0 \ \textit{for all}\ u,v\in X\ \textit{and}\ gu\neq gv.$$

If g is the identity mapping on X, we say that T is a $\mathcal{Z}$-contraction for ζ.

Example 4. Let $\zeta_\lambda:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be the function defined by $\zeta_\lambda(t,s)=\lambda s-t$, where $\lambda\in(0,1)$. Then, $\zeta_\lambda\in\mathcal{Z}$.

Ansari [1] introduced $\mathcal{C}$-class functions as follows:

Definition 14. *A function* $\mathcal{G}:[0,\infty)^2\to\mathbb{R}$ *is called a* $\mathcal{C}$*-class function if it is continuous and satisfies the following conditions:*

(i) $\mathcal{G}(s,t)\leqslant s$;

(ii) $\mathcal{G}(s,t)=s$ *implies that either* $s=0$ *or* $t=0$ *for all* $s,t\geqslant 0$.

Definition 15. *[8] A function* $\mathcal{G}:[0,\infty)^2\to\mathbb{R}$ *has the property* $\mathcal{C}_\mathcal{G}$, *if there exists* $\mathcal{C}_\mathcal{G}\geqslant 0$, *such that*

($\mathcal{G}_1$) $\mathcal{G}(s,t)>\mathcal{C}_\mathcal{G}$ *implies* $s>t$;

($\mathcal{G}_2$) $\mathcal{G}(t,t)\leqslant\mathcal{C}_\mathcal{G}$ *for all* $t\geqslant 0$.

Example 5. $\mathcal{G}(s,t)=s-t$, $\mathcal{C}_\mathcal{G}=r$, $r\geqslant 0$ is a $\mathcal{C}$-class function that has property $\mathcal{C}_\mathcal{G}$.

Definition 16. *[8]* $\mathcal{C}_\mathcal{G}$ *simulation function is a function* $\zeta\colon[0,\infty)^2\to\mathbb{R}$ *satisfying the following conditions:*

(ζ_a) *$\zeta(t,s) < \mathcal{G}(s,t)$ for all t,$s > 0$, where $\mathcal{G}\colon [0,\infty)^2 \to \mathbb{R}$ is a $\mathcal{C}$-class function with the property $\mathcal{C}_{\mathcal{G}}$;*

(ζ_b) *if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$, such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < \mathcal{C}_{\mathcal{G}}$.*

Example 6. Let $k \in \mathbb{R}$ be such that $k \leqslant 1$ and $\zeta : [0,\infty)^2 \to \mathbb{R}$ be the function defined by $\zeta(t,s) = k\mathcal{G}(s,t)$. Here take $\mathcal{C}_{\mathcal{G}} = 1$. Then, ζ is a $\mathcal{C}_{\mathcal{G}}$ simulation function.

The family of all $\mathcal{C}_{\mathcal{G}}$ simulation functions is denoted by $\mathcal{Z}_{\mathcal{G}}$.

Radenovic et al. [8] generalized the simulation function using $\mathcal{C}$-class function for two operators, as follows:

Definition 17. *[8] Let (X,d) be a metric space and $T,g\colon X \to X$ be self-mappings. A mapping T is called a $(\mathcal{Z}_{\mathcal{G}}, g)$-contraction if there exists $\zeta \in \mathcal{Z}_{\mathcal{G}}$ such that*

$$\zeta(d(Tx,Ty),d(gx,gy)) \geqslant \mathcal{C}_{\mathcal{G}} \quad for \quad all \quad x,y \in X \quad and \quad gx \neq gy.$$

3. Main result. In this section, we use admissible mappings, simulation functions, and $\mathcal{C}$-class functions to consider quasi-contraction of Ćirić-type on quasi-metric spaces, and we establish related results on existence and uniqueness of the coincidence point.

Lemma 3. *Let (X,d) be a quasi-metric space and $T,g\colon X \to X$ be mappings. Let $\{x_n\}$ be a Picard-Jungck sequence of (T,g). If T is triangular α-admissible for g with $\alpha(gx_0,Tx_0) \geqslant 1$ and $\alpha(Tx_0,gx_0) \geqslant 1$, then $\alpha(gx_n,gx_m) \geqslant 1$ for $n \neq m$.*

Proof. *Let $\{x_n\}$ be a Picard sequence of (T,g) based at x_0, that is,*

$$Tx_n = gx_{n+1},$$

for all $n \geqslant 0$. Since T is α-admissible for g, we have

$$\alpha(gx_0,Tx_0) = \alpha(gx_0,gx_1) \geqslant 1 \implies \alpha(Tx_0,Tx_1) = \alpha(gx_1,gx_2) \geqslant 1.$$

By induction, we get

$$\alpha(gx_n,gx_{n+1}) \geqslant 1 \; for \; all \; n \geqslant 0.$$

Since T is triangular α-admissible for g, we have

$$\alpha(gx_0,gx_1) \geqslant 1 \text{ and } \alpha(gx_1,gx_2) \geqslant 1 \implies \alpha(gx_0,gx_2) \geqslant 1.$$

Continuing this way, we get

$$\alpha(gx_n, gx_m) \geqslant 1 \; for \; all \; m > n.$$

Similarly, for $\alpha(Tx_0, gx_0) \geqslant 1$, *we get*

$$\alpha(gx_n, gx_m) \geqslant 1 \; for \; all \; m < n.$$

□

Definition 18. *Let* (X, d) *be a metric space,* $\alpha\colon X \times X \to [0, \infty)$ *and* $T, g\colon X \to X$ *be given mappings. A mapping* T *is called a* $(\mathcal{Z}_{(\alpha,\mathcal{G})}, g)$*-quasi-contraction of Ćirić type if there exist* $\zeta \in \mathcal{Z}_{\mathcal{G}}$ *and* $\lambda \in (0, 1)$ *such that*

$$\zeta(\alpha(gx, gy)d(Tx, Ty), \lambda M(gx, gy)) \geqslant \mathcal{C}_{\mathcal{G}} \quad for \; all \; x, y \in X, \tag{2}$$

where

$$M(gx, gy) = \max\{d(gx, gy), d(gx, Tx), d(gy,Ty), d(gx,Ty), d(gy, Tx)\}.$$

Remark 1.

(i) *If we take* $\alpha(x, y) = 1$, *inequality (2) becomes a* $(\mathcal{Z}_{\mathcal{G}}, g)$*-quasi-contraction of Ćirić-Das-Naik type contraction.*

(ii) *For* $\alpha(x, y) = 1$, $g = i_X$ *and* $\mathcal{C}_{\mathcal{G}} = 0$, *we get a* $\mathcal{Z}$*-quasi-contraction of Ćirić type.*

(iii) *For* $\alpha(x, y) = 1$ *and* $\zeta(t, s) < \mathcal{G}(s, t) = s - t$, *inequality (2) becomes a Das-Naik type quasi-contraction.*

Theorem 3. *Let* (X, d) *be a quasi-metric space,* $T, g\colon X \to X$ *be mappings with* $T(X) \subset g(X)$. *If* T *is a* $(\mathcal{Z}_{(\alpha,\mathcal{G})}, g)$*-quasi-contraction of Ćirić type satisfying the following conditions:*

(i) T *is triangular* α*-admissible for* g;

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(gx_0, Tx_0) \geqslant 1$ *and* $\alpha(Tx_0, gx_0) \geqslant 1$;

(iii) *at least one of the following conditions holds:*

   (a) $T(X)$ *is precomplete in* $g(X)$.

   (b) (X, d) *is a complete quasi-metric space and* T *and* g *are continuous and compatible.*

Then, T and g have a point of coincidence.

Proof. Start with $x_0 \in X$; since $T(X) \subset g(X)$, we get a sequence $\{x_n\}$ in X with $Tx_n = gx_{n+1}$ for all n $\geqslant 0$. If $gx_n = gx_{n+1}$ for some n, then $Tx_n = gx_n$; that is, x_n is a coincidence point of T and g. Thus, we assume that $d(gx_{n+1}, gx_n) > 0$ and $d(gx_n, gx_{n+1}) > 0$ for all $n \geqslant 0$.

In view of condition (i), by Lemma 3, we get

$$\alpha(gx_n, gx_m) \geqslant 1 \text{ for all } n \neq m. \tag{3}$$

Now,

$$d(gx_n, gx_{n+1}) = d(Tx_{n-1}, Tx_n) \leqslant \alpha(gx_{n-1}, gx_n)d(Tx_{n-1}, Tx_n). \tag{4}$$

Since T is a $(\mathcal{Z}_{(\alpha,\mathcal{G})}, g)$-quasi-contraction of Ćirić type,

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} \leqslant \zeta(\alpha(gx_{n-1}, gx_n)d(Tx_{n-1}, Tx_n), \lambda M(gx_{n-1}, gx_n)) <\\ < \mathcal{G}(\lambda M(gx_{n-1}, gx_n), \alpha(gx_{n-1}, gx_n)d(Tx_{n-1}, Tx_n)).\end{aligned}$$

Using $(\mathcal{G}_1)$, we get

$$\alpha(gx_{n-1}, gx_n)d(Tx_{n-1}, Tx_n) \leqslant \lambda M(gx_{n-1}, gx_n). \tag{5}$$

From (4) and (5), we have

$$d(gx_n, gx_{n+1}) \leqslant \lambda M(gx_{n-1}, gx_n), \tag{6}$$

where

$$\begin{aligned}M(gx_{n-1}, gx_n) = \max\{&d(gx_{n-1}, gx_n), d(gx_{n-1}, Tx_{n-1}),\\ &d(gx_n,Tx_n), d(gx_{n-1}, Tx_n), d(gx_n, Tx_{n-1})\} =\\ = \max\{&d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_{n+1}))\} \leqslant\\ &\leqslant d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}).\end{aligned}$$

Hence,

$$d(gx_n, gx_{n+1}) \leqslant \lambda(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})),$$

$$d(gx_n, gx_{n+1}) \leqslant \frac{\lambda}{1-\lambda}d(gx_{n-1}, gx_n),$$

$$d(gx_n, gx_{n+1}) \leqslant kd(gx_{n-1}, gx_n), \tag{7}$$

where $k = \frac{\lambda}{1-\lambda} < 1$.

Similarly, we get

$$d(gx_{n+1}, gx_n) \leqslant kd(gx_n, gx_{n-1}) \text{ for } k < 1.$$

By Lemma 1, the sequence $\{gx_n\}$ is a Cauchy sequence. Now, let us consider independently cases (a)–(b). Let us prove that T and g have a coincidence point.

(a): Assume $T(X)$ is precomplete in $g(X)$. The precompleteness of $T(X)$ in $g(X)$ ensures the existence of some $v \in X$ with

$$\lim_{n\to\infty} gx_n = gv = \lim_{n\to\infty} Tx_{n-1}. \tag{8}$$

We claim that v is a coincidence point of T and g. On contrary, assume that $d(gv, Tv) > 0$ and $d(Tv, gv) > 0$.
We have

$$\lim_{n\to\infty} M(gx_n, gv) = \lim_{n\to\infty} \max\{d(gx_n, gv), d(gx_n, Tx_n), d(gv, Tv), d(gx_n, Tv), d(gv, Tx_n)\} = d(gv, Tv) > 0. \tag{9}$$

Using (2), we get

$$\begin{aligned} \mathcal{C}_{\mathcal{G}} &\leqslant \zeta(\alpha(gx_n, gv)d(Tx_n, Tv), \lambda M(gx_n, gv)) \\ &< \mathcal{G}(\lambda M(gx_n, gv), \alpha(gx_n, gv)d(Tx_n, Tv)). \end{aligned}$$

By $(\mathcal{G}_1)$, we have

$$\alpha(gx_n, gv)d(Tx_n, Tv) < \lambda M(gx_n, gv) \text{ for all } n \in \mathbb{N}$$

Letting $n \to \infty$ in above inequality and using (9), we get

$$\lim_{n\to\infty} \alpha(gx_n, gv)d(gx_n, Tv) < \lambda d(gv, Tv).$$

Hence, $d(gv, Tv) < \lambda d(gv, Tv)$ and so is a contradiction.
Therefore, $d(gv, Tv) = 0$. So, v is a coincidence point of T and g.

(b): Assume that (X, d) is complete and T and g are continuous and compatible. In this case, the sequence $\{gx_n\}$ is a Cauchy sequence in the complete quasi-metric space (X, d), hence, there exists $u \in X$ such that $\lim_{n\to\infty} gx_n = u$. That is,

$$\lim_{n\to\infty} d(gx_n, u) = \lim_{n\to\infty} d(u, gx_n) = 0.$$

Since $Tx_n = gx_{n+1}$ for all $n \geqslant 0$, we have

$$\lim_{n\to\infty} d(Tx_n, u) = \lim_{n\to\infty} d(u, Tx_n) = 0.$$

The continuity of T yields that

$$\lim_{n\to\infty} d(Tgx_n, Tu) = \lim_{n\to\infty} d(Tu, Tgx_n) = 0.$$

The continuity of g yields that

$$\lim_{n\to\infty} d(gTx_n, gu) = \lim_{n\to\infty} d(gu, gTx_n) = 0.$$

Moreover, as T and g are compatible and the sequences $\{Tx_n\}$ and $\{gx_n\}$ have the same limit, we deduce that

$$\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0 \text{ or } \lim_{n\to\infty} d(gTx_n, Tgx_n) = 0.$$

Now,

$$d(Tu, gu) \leqslant d(Tu, Tgx_n) + d(Tgx_n, gTx_n) + d(gTx_n, gu).$$

By taking limit $n \to \infty$ in above inequality, we get $d(Tu, gu) = 0$. Similarly, we can show that $d(gu, Tu) = 0$. In any case, $Tu = gu$ and we conclude that u is a coincidence point of T and g.

□

If $T(X) \subset g(X)$, then there exists a Picard-Jungck sequence of (T, g) based on any point $x_0 \in X$.

Corollary 1. *Let (X, d) be a quasi-metric space, $T, g\colon X \to X$ be mappings and let $\{x_n\}$ be a Picard-Jungck sequence of (T, g). Let T be a $(\mathcal{Z}_{(\alpha,\mathcal{G})}, g)$-quasi-contraction of Ćirić-type and satisfy the following conditions:*

(i) T is triangular α-admissible for g;

(ii) there exists $x_0 \in X$ such that $\alpha(gx_0, Tx_0) \geqslant 1$ and $\alpha(Tx_0, gx_0) \geqslant 1$;

(iii) at least one of the following conditions holds:

- *(a) $(g(X), d)$ is complete.*
- *(b) (X, d) is a complete quasi-metric space and T and g are continuous and compatible.*

Then, T and g have a point of coincidence.

For the uniqueness of a coincidence point and existence and uniqueness of a fixed point of a $(\mathcal{Z}_{(\alpha,\mathcal{G})}, g)$-quasi-contraction of Ćirić type, we propose the following conjecture.

Theorem 4. *In addition to the assumptions of Theorem 3, suppose that for all $u, v \in C(T, g)$, there exists $w \in X$, such that $\alpha(gu, gw) \geqslant 1$, $\alpha(gw, gu) \geqslant 1$, $\alpha(gw, gv) \geqslant 1$, and $\alpha(gv, gw) \geqslant 1$. Also, T, g commute at their coincidence points. Then, T and g have a unique common fixed point.*

Proof. We claim that if $u, v \in C(T, g)$, then $gu = gv$. By the assumption, there exists $w \in X$, such that

$$\alpha(gw, gu) \geqslant 1 \textit{ and } \alpha(gw, gv) \geqslant 1.$$

Let us define the Picard sequence $\{w_n\}$ in X by $gw_{n+1} = Tw_n$ for all $n \geqslant 0$ and $w_0 = w$. Reasoning as in the proof of Theorem 3, we obtain that the sequence $\{gw_n\}$ converges to gz.

By condition (i) in Theorem 3, we have

$$\alpha(gw_n, gu) \geqslant 1 \text{ and } \alpha(gw_n, gv) \geqslant 1 \text{ for all } n \geqslant 1. \tag{10}$$

Using (2), we have

$$\begin{aligned}\mathcal{C}_{\mathcal{G}} \leqslant \zeta(\alpha(gw_n, gu)d(Tw_n, Tu), \lambda M(gw_n, gu)) <\\ < \mathcal{G}(\lambda M(gw_n, gu), \alpha(gw_n, gu)d(Tw_n, Tu)) =\\ = \mathcal{G}(\lambda M(gw_n, gu), \alpha(gw_n, gu)d(gw_{n+1}, gu)).\end{aligned} \tag{11}$$

By $(\mathcal{G}_1)$ and (10), we have

$$d(gw_{n+1}, gu) \leqslant \alpha(gw_n, gu)d(gw_{n+1}, gu) < \lambda M(gw_n, gu) \text{ for all } n \geqslant 1, \tag{12}$$

where

$$\begin{aligned}&M(gw_n, gu) =\\ &= \max\{d(gw_n, gu), d(gw_n, Tw_n), d(gu, Tu), d(gw_n, Tu), d(gu, Tw_n)\} =\\ &= \max\{d(gw_n, gu), d(gw_n, gw_{n+1}), d(gu, gw_{n+1})\}.\end{aligned}$$

Passing to the limit $n \to \infty$, we get

$$\lim_{n\to\infty} M(gw_n, gu) = \max\{d(gz, gu), d(gu, gz)\}.$$

Similarly, we get

$$d(gu, gw_{n+1}) < \lambda M(gu, gw_n) \quad \text{for} \quad \text{all } n \geqslant 1, \tag{13}$$

where

$$M(gu, gw_n) = \max\{d(gu, gw_n), d(gw_n, gw_{n+1}), d(gw_n, gu)\}.$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} M(gw_n, gu) = \max\{d(gu, gz), d(gz, gu)\}.$$

If $gu \neq gz$ and we take the limit $n \to \infty$ in (12) and (13), we get

$$d(gz, gu) < \lambda \max\{d(gz, gu), d(gu, gz)\},$$

$$d(gu, gz) < \lambda \max\{d(gz, gu), d(gu, gz)\}.$$

If $d(gz, gu) < \lambda d(gz, gu)$ or $d(gz, gu) < \lambda d(gu, gz) < \lambda^2 d(gz, gu)$, we get a contradiction. Thus, $d(gz, gu) = 0$. Therefore, $gu = gz$. Similarly, $gv = gz$ implies $gu = gv$. Hence, u is a unique coincidence point of T and g.

Existence of a common fixed point: Let $u \in C(T, g)$, that is, $Tu = gu$. Due to commutativity of T and g at their coincidence points, we get

$$ggu = gTu = Tgu.$$

Denote $gu = z^*$. Then $gz^* = Tz^*$, and, thus, z^* is a coincidence point of T and g. By uniqueness of the coincidence point, we have $z^* = gu = gz^* = Tz^*$. Then, z^* is a common fixed point of T and g.

Uniqueness: Assume that w^* is another common fixed point of T and g. Then $w^* \in C(T, g)$. Thus, we have $w^* = gw^* = gz^* = z^*$. This completes the proof. □

From Theorem 1 we see that the result above is valid also for metric spaces.

Corollary 1. *Let (X, d) be a metric space, $T, g \colon X \to X$ be mappings, and let $\{x_n\}$ be a Picard-Jungck sequence of (T, g). Assume that T is a $(\mathcal{Z}_{(\alpha,\mathcal{G})}, g)$-quasi-contraction of Ćirić type satisfying the following conditions:*

(i) T is triangular α-admissible for g;

(ii) *there exists* $x_0 \in X$*, such that* $\alpha(gx_0, Tx_0) \geqslant 1$*;*

(iii) *for all* $u, v \in C(T, g)$*, there exists* $w \in X$*, such that* $\alpha(gu, gw) \geqslant 1$*,* $\alpha(gv, gw) \geqslant 1$ *and* T, g *commute at their coincidence points.*

(iv) *at least one of the following conditions holds:*

  (a) $T(X)$ *is precomplete in* $g(X)$*.*

  (b) (X, d) *is a complete metric space and* T *and* g *are continuous and compatible.*

Then, T *and* g *have a unique common fixed point.*

The following result is a solution to an open problem posed by Radenovic and Chandok [9].

Corollary 2. *[9] Let* (X, d) *be a metric space,* $T, g\colon X \to X$ *be mappings, and let* $\{x_n\}$ *be a Picard-Jungck sequence of* (T, g)*. Let* T *be a* $(\mathcal{Z}_{\mathcal{G}}, g)$*-quasi-contraction of Ćirić-Das-Naik type. Assume that at least one of the following conditions holds:*

(a) $(g(X), d)$ *is complete.*

(b) (X, d) *is a complete metric space and* T *and* g *are continuous and compatible.*

Then, T *and* g *have a unique point of coincidence. Moreover, if* T *and* g *commute at their coincidence point, then they have a unique common fixed point in* X*.*

Proof. The result follows from Corollary 1 and Theorem 4, if we consider $\alpha(x, y) = 1$ and (X, d) is a metric space with metric d as defined in Theorem 1. □

Now, if we take $\mathcal{G}(s, t) = s - t$, $\mathcal{C}_{\mathcal{G}} = 0$, we get the following result:

Corollary 3. *Let* (X, d) *be a metric space,* $\alpha\colon X \times X \to [0, \infty)$*, and* $T, g\colon X \to X$ *be mappings. Let* $\{x_n\}$ *be a Picard-Jungck sequence of* (T, g) *and* $\lambda \in (0, 1)$*, such that*

$$\alpha(gx, gy)d(Tx, Ty) \leqslant \lambda M(gx, gy) \quad \text{for all} \quad x, y \in X,$$

$M(gx, gy) = \max\{d(gx, gy), d(gx, Tx), d(gy, Ty), d(gx, Ty), d(gy, Tx)\}$.

Assume that

(i) T *is triangular* α*-admissible for* g*;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(gx_0, Tx_0) \geqslant 1$*;*

(iii) for all $u, v \in C(T, g)$, there exists $w \in X$, such that $\alpha(gu, gw) \geqslant 1$, $\alpha(gv, gw) \geqslant 1$ and T, g commute at their coincidence points;

(iv) at least one of the following conditions holds:

(a) $(g(X), d)$ is complete.

(b) (X, d) is a complete metric space and T and g are continuous and compatible.

Then T and g have a unique common fixed point.

4. Consequences: Common fixed point results in the context of G-metric spaces. In this section, we give some consequences of our main results. For this purpose, we first recollect the basic concepts on G-metric spaces.

Definition 19. *[15] Let X be a nonempty set. Let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:*

(G_1) *$G(x, y, z) = 0$, if $x = y = z$,*

(G_2) *$G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,*

(G_3) *$G(x, x, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,*

(G_4) *$G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables),*

(G_5) *$G(x, y, z) \leqslant G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$(rectangle inequality).*

The function G is called G-metric on X and the pair (X, G) is called a G-metric space.

Definition 20. *A G-metric space (X, G) is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.*

The function defined by $d_{G'}(x, y) = G(x, y, y) + G(y, x, x)$ for all $x, y \in X$, is a metric on X. Furthermore, (X, G) is $G-$ complete if and only if $(X, d_{G'})$ is complete.

Recently, Jleli and Samet [4] obtained the following results.

Theorem 5. *Let (X, G) be a $G-$metric space. Let $d_G\colon X \times X \to [0, \infty)$ be the function defined by $d_G(x, y) = G(x, y, y)$. Then,*

(1) (X, d_G) is a quasi-metric space;

(2) $\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d_G);

(3) $\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, d_G);

(4) (X, G) is G-complete if and only if (X, d_G) is complete.

Definition 21. *A subset E of a G-metric space (X, G) is said to be precomplete if every Cauchy sequence $\{u_n\}$ in E converges to a point of X.*

Furthermore, a subset E of X is precomplete in (X, G) if and only if it is precomplete in (X, d_G).

Definition 22. *For a nonempty set X, let $T, g\colon X \to X$ and $\alpha_G\colon X^3 \to [0, \infty)$ be mappings. We say that T is α_G-admissible for g, if for all $x, y \in X$ we have*

$$\alpha_G(gx, gy, gy) \geqslant 1 \implies \alpha_G(Tx, Ty, Ty) \geqslant 1.$$

Definition 23. *For a nonempty set X, let $\alpha_G\colon X^3 \to [0, \infty)$ and $T\colon X \to X$ be mappings. We say that T is triangular α_G-admissible for g, if T is α_G-admissible for g and for all $x, y \in X$, we have*

$$\alpha_G(gx, gy, gy) \geqslant 1 \text{ and } \alpha_G(gy, gz, gz) \geqslant 1 \implies \alpha_G(gx, gz, gz) \geqslant 1.$$

By using the above definition, we get the following corollary:

Corollary 1. *Let X be a non-empty set. The mapping $T\colon X \to X$ is triangular α_G-admissible for g if and only if T is triangular α-admissible for g.*

Proof. It is obvious by taking $\alpha(x, y) = \alpha_G(x, y, y)$. □

Now, we present Theorem 3 and Theorem 4 in the context of G-metric spaces, using the quasi-metric d_G as defined in Theorem 5.

Corollary 2. *Let (X, G) be a G-metric space, $\alpha_G\colon X \times X \times X \to [0, \infty)$, and $T, g\colon X \to X$ be mappings with $T(X) \subset g(X)$. Let $\zeta \in \mathcal{Z}_{\mathcal{G}}$, and $\lambda \in (0, 1)$, such that*

$$\zeta(\alpha_G(gx, gy, gy)G(Tx, Ty, Ty), \lambda M(gx, gy, gy)) \geqslant \mathcal{C}_{\mathcal{G}} \tag{14}$$

for all $x, y \in X$, where

$$M(gx, gy, gy) = \\ = \max\{G(gx, gy, gy), G(gx, Tx, Tx), G(gy, Ty, Ty), G(gx, Ty, Ty), \\ G(gy, Tx, Tx)\}.$$

Suppose that

(i) T *is triangular* α_G*-admissible for* g;

(ii) *there exists* $x_0 \in X$, *such that* $\alpha_G(gx_0, Tx_0, Tx_0) \geqslant 1$ *and* $\alpha_G(Tx_0, gx_0, gx_0) \geqslant 1$;

(iii) *for all* $u, v \in C(T, g)$, *there exists* $w \in X$ *such that* $\alpha_G(gu, gw, gw) \geqslant 1$, $\alpha_G(gw, gu, gu) \geqslant 1$, $\alpha_G(gv, gw, gw) \geqslant 1$ $\alpha_G(gw, gv, gv) \geqslant 1$ *and* T, g *commute at their coincidence points;*

(iv) *at least one of the following conditions holds:*

  (a) $T(X)$ *is precomplete in* $g(X)$.

  (b) (X, G) *is a complete* G*-metric space and* T *and* g *are continuous and compatible.*

Then, T *and* g *have a unique common fixed point.*

Proof. It suffices to take $d_G(x, y) = G(x, y, y)$ *and* $\alpha(x, y) = \alpha_G(x, y, y)$. From (14), we get (2). Since (X, G) is complete, (X, d_G) is a complete quasi-metric space due to Theorem 5. Hence, the result follows from Lemma 1, Theorem 3, and Theorem 4. □

Corollary 3. *Let* (X, G) *be a* G*-metric space,* $\alpha_G\colon X \times X \times X \to [0, \infty)$, *and* $T, g\colon X \to X$ *be mappings. Let* $\{x_n\}$ *be a Picard-Jungck sequence of* (T, g), $\zeta \in \mathcal{Z}_\mathcal{G}$ *and* $\lambda \in (0, 1)$, *such that (1) is satisfied. Suppose that*

(i) T *is triangular* α_G*-admissible for* g;

(ii) *there exists* $x_0 \in X$ *such that* $\alpha_G(gx_0, Tx_0, Tx_0) \geqslant 1$ *and* $\alpha_G(Tx_0, gx_0, gx_0) \geqslant 1$;

(iii) *for all* $u, v \in C(T, g)$, *there exists* $w \in X$ *such that* $\alpha_G(gu, gw, gw) \geqslant 1$, $\alpha_G(gw, gu, gu) \geqslant 1$, $\alpha_G(gv, gw, gw) \geqslant 1$, $\alpha_G(gw, gv, gv) \geqslant 1$ *and* T, g *commute at their coincidence points;*

(iv) *at least one of the following conditions holds:*

  (a) $T(X)$ *is precomplete in* $g(X)$.

  (b) (X, G) *is a complete* G*-metric space and* T *and* g *are continuous and compatible.*

Then T *and* g *have a unique common fixed point.*

Corollary 4. *Let* (X, G) *be a* G*-metric space and* $T, g\colon X \to X$ *be mappings. Let* $\{x_n\}$ *be a Picard-Jungck sequence of* (T, g), $\zeta \in \mathcal{Z}_\mathcal{G}$, *and* $\lambda \in (0, 1)$ *such that*

$$\zeta(G(Tx, Ty, Ty), \lambda M(gx, gy, gy)) \geqslant \mathcal{C}_\mathcal{G} \tag{15}$$

for all $x, y \in X$, where

$$M(gx, gy, gy) = \max\{G(gx, gy, gy), G(gx, Tx, Tx), G(gy, Ty, Ty), \\ G(gx, Ty, Ty), G(gy, Tx, Tx)\}.$$

Also assume that at least one of the following conditions holds:

(a) *$T(X)$ is precomplete in $g(X)$.*

(b) *(X, G) is a complete G-metric space and T and g are continuous and compatible.*

Then T and g have unique point of coincidence. Moreover, if T, g commute at their coincidence points, then T and g have a unique common fixed point in X.

Proof. In (1), if we take $\alpha_G(x, y, y) = 1$, we get (2). □

Corollary 5. *Let (X, G) be a G-metric space, $\alpha_G\colon X \times X \times X \to [0, \infty)$, and $T, g\colon X \to X$ be mappings. Let $\{x_n\}$ be a Picard-Jungck sequence of (T, g) and $\lambda \in (0, 1)$, such that*

$$\alpha_G(gx, gy, gy)G(Tx, Ty, Ty) \leqslant \lambda M(gx, gy, gy)$$

for all $x, y \in X$, where

$$M(gx, gy, gy) = \max\{G(gx, gy, gy), G(gx, Tx, Tx), G(gy, Ty, Ty), \\ G(gx, Ty, Ty), G(gy, Tx, Tx)\}.$$

Suppose that

(i) *T is triangular α_G-admissible for g;*

(ii) *there exists $x_0 \in X$ such that $\alpha_G(gx_0, Tx_0, Tx_0) \geqslant 1$ and $\alpha_G(Tx_0, gx_0, gx_0) \geqslant 1$;*

(iii) *for all $u, v \in C(T, g)$, there exists $w \in X$ such that $\alpha_G(gu, gw, gw) \geqslant 1$, $\alpha_G(gw, gu, gu) \geqslant 1$, $\alpha_G(gv, gw, gw) \geqslant 1$, $\alpha_G(gw, gv, gv) \geqslant 1$ and T, g commute at their coincidence points;*

(iv) *at least one of the following conditions holds:*

   (a) *$T(X)$ is precomplete in $g(X)$.*

   (b) *(X, G) is a complete G-metric space and T and g are continuous and compatible.*

Then T and g have a unique common fixed point.

Acknowledgment. The first author acknowledges the financial support by SHODH-Scheme (Gujarat Government) with student reference number 202001720096.

References

[1] Ansari A. H. *Note on φ-ψ-contractive type mappings and related fixed point.* The 2nd Regional Conference on Math. Appl. PNU, Sept., 2014, pp. 377 – 380.

[2] Ćirić L. B. *A generalization of Banach's contraction principle.* Proc. Amer. Math. Soc., 1974, vol. 45, no. 2, pp. 267 – 273. DOI: https://doi.org/10.1090/S0002-9939-1974-0356011-2

[3] Das K. M., Naik K. V. *Common fixed point theorems for commuting maps on a metric space.* Proc. Amer. Math. Soc., 1979, vol. 77, no. 3, pp. 369 – 373. DOI: https://doi.org/10.2307/2042188

[4] Jleli M., Samet B. *Remarks on G-metric spaces and fixed point theorems.* Fixed Point Theory Appl., 2012, vol. 2012, no. 1, Paper No. 210, 7 pages. DOI: https://doi.org/10.1186/1687-1812-2012-210

[5] Jovanović M., Kadelburg K., and Radenović S. *Common fixed point results in metric-type spaces.* Fixed Point Theory Appl., 2010, vol. 2010, 978121, 15 pages. DOI: https://doi.org/10.1155/2010/978121

[6] Jungck G. *Compatible mappings and common fixed points (2).* Int. J. Math. Math. Sci., 1988, vol. 11, no. 2, pp. 285 – 288. DOI: https://doi.org/10.1155/S0161171288000341

[7] Khojasteh F., Shukla S., Radenović S. *A new approach to the study of fixed point theory for simulation functions.* Filomat, 2015, vol. 29, no. 6, pp. 1189 – 1194.

[8] Liu X. L., Ansari A. H., Chandok S. and Radenović S. *On some results in metric spaces using auxiliary simulation functions via new functions.* J. Comput. Anal. Appl., 2018, vol. 24, no. 6, pp. 1103 – 1114.

[9] Radenović S, Chandok S. *Simulation type functions and coincidence points.* Filomat, 2018, vol. 32, no. 1, pp. 141 – 147. DOI: https://doi.org/10.2298/FIL1801141R

[10] Roldán-López-de-Hierro A. F., Karapınar E. Roldán-López-de-Hierro C., and Martínez-Moreno J. *Coincidence point theorems on metric spaces via simulation functions.* J. Comput. Appl. Math., 2015, vol. 275, pp. 345 – 355. DOI: https://doi.org/10.1016/j.cam.2014.07.011

[11] Roldán-López-de-Hierro A. F., Shahzad N. *Common fixed point theorems under (R, S)-contractivity conditions.* Fixed Point Theory Appl., 2016, vol. 2016, Paper No. 55, 25 pages.
DOI: https://doi.org/10.1186/s13663-016-0532-5

[12] Roldán-López-de-Hierro A. F., Karapınar E., O'Regan D. *Coincidence point theorems on quasi-metric spaces via simulation functions and applications to G-metric spaces.* J. Fixed Point Theory Appl., 2018, vol. 20, no. 3, Paper No. 112, 21 pages.
DOI: https://doi.org/10.1007/s11784-018-0582-x

[13] Samet B., Vetro C., Vetro F. *Remarks on G-metric spaces.* Intern. J. Anal., 2013, vol. 2013, 917158, 6 pages.
DOI: https://doi.org/10.1155/2013/917158

[14] Shahi P., Kaur J., Bhatia S. S. *Coincidence and common fixed point results for generalized α-ψ contractive type mappings with applications.* Bull. Belg. Math. Soc. Simon Stevin, 2015, vol. 22, no. 2, pp. 299 – 318.
DOI: 10.36045/bbms/1432840866

[15] Zead M., Brailey S. *A new approach to generalized metric spaces.* J. Nonlinear Convex Anal., 2006, vol. 7, no. 2, pp. 289 – 297.

Received December 15, 2021.
In revised form, March 02, 2022.
Accepted March 05, 2022.
Published online March 28, 2022.

Department of Mathematics, Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara, Gujarat 390002, India

Sejal Puvar
E-mail: puvarsejal@gmail.com

R. G. Vyas
E-mail: drrgvyas@yahoo.com