Probl. Anal. Issues Anal. Vol. 11 (29), No 3, 2022, pp. 45–55
DOI: 10.15393/j3.art.2022.11390

UDC 517.18

F. N. Garif'yanov, E. V. Strezhneva

ON A CARLEMAN PROBLEM IN THE CASE OF A DOUBLY PERIODIC GROUP

Abstract. Let Γ be a doubly periodic group whose fundamental region D is a rectangle, in which the ratio of the largest side to the shortest one does not exceed 3. The generating transformations of the group and their inverse transformations induce, on the boundary, an involutive inverse shift, discontinuous at the vertices. We consider a particular case of the Carleman problem for functions that are analytic in D (the so-called *jump problem*). We show that the regularization of the unknown function suggested by Torsten Carleman leads to an equivalent regularization of the problem. For this, we rely on the contraction mapping principle for Banach spaces and use the theory of Weierstrass elliptic functions. The integral representation was first introduced by Carleman during his talk at the International Congress of Mathematicians in Zürich in 1932. However, he did not investigate the Fredholm integral equation obtained by regularizing the jump problem. In particular, the question of equivalence of the jump problem and the corresponding Fredholm equation obtained through the given representation remained open.

Key words: *Carleman problem, regularization method, contraction mapping principle*

2020 Mathematical Subject Classification: *30F10;45B05*

1. Introduction and the problem statement. Let D be the interior of the fundamental polygon of a properly discontinuous group Γ (i. e., a specific realization of a closed Riemannian surface of finite genus ρ on the plane). Assume that $S_j(z)$ are such transformations of this group that $S_j(\overline{D}) \cap \overline{D} \neq \emptyset$, $S_j(z) \neq z$. The generating transformations of Γ and their inverse transformations induce, on each side, a shift

$\alpha(t)\colon \partial^+ D \to \partial^- D$ that has discontinuities of the first kind at the vertices t_k. In 1932, Torsten Carleman [3] first directed attention to studying the Fredholm integral equation

$$(T\phi)(t) \equiv \phi(t) + (4\pi i)^{-1} \int_L K(t,\tau)\phi(\tau)\,d\tau = 2^{-1} g(t), \tag{1}$$

where $L = \partial D$, and

$$K(t,\tau) = A(t,\tau) - \alpha'(\tau) A(\alpha(t), \alpha(\tau)), \tag{2}$$

with

$$A(z,\tau) = (\tau - z)^{-1} + \sum_j (\tau - S_j(z))^{-1}, \tag{3}$$

$$g(t) + g(\alpha(t)) = 0. \tag{4}$$

For details on the history of this issue and, in particular, the emergence of a kernel with such a structure, see [6].

The main difficulty with Equation (1) is the following. The solutions to the union equation

$$T'\psi = 0 \tag{5}$$

are first-order automorphic forms holomorphic in D (automorphic covariants). This assertion can be proven by straightforward substitution. But apart from these solutions, Equation (5) may have complementary solutions, that is, solutions that satisfy the condition

$$\psi(t) = \alpha'(t)\psi(\alpha(t)) \tag{6}$$

and, moreover, are not boundary values of functions that are analytic in D. If there exist complementary solutions, Equation (1) has more solvability conditions than the corresponding jump problem

$$F^+(t) - F^+[\alpha(t)] = g(t), \qquad t \in L \setminus \{t_k\}, \tag{7}$$

where $F^+(t) = \lim_{z \to t} F(z), z \in D$. Asserting the equivalence of Equation (1) and problem (7) is what the Carleman problem is; in other words, the problem consists in proving that Equation (5) has no complementary solutions.

It is proven in [1] that one can always obtain an equivalent regularization by introducing additional terms in kernel (3).

T. Carleman suggested that the solution of problem (7) be expressed in the form

$$F(z) = \frac{1}{2\pi i} \int\limits_L \varphi(\tau) A(z,\tau)\, d\tau + C, \tag{8}$$

where the density of the integral satisfies condition (4). Carleman's idea turned out to be relevant not only to the theory of boundary-value problems for functions analytic on Riemannian surfaces, but also to such branches of mathematics as the theory of entire functions and convolution operators (see, for instance, [4], [5], [7]). It should be noted that the used groups have the property that each vertex of the fundamental polygon is common to an even number of congruent fundamental polygons, which meet at this point. In this case, the regularization of problem (7) can be achieved using kernels of type (3), containing only some of the terms indicated by Carleman [2]. In particular, the Cauchy kernel can be excluded from the number of terms.

Let us state the main result of the paper.

Theorem 1. *Assume that the fundamental region of the doubly periodic group Γ is a rectangle in which the ratio of the largest side to the shortest does not exceed* 3. *Then the adjoint equation* (5) *does not have complementary solutions.*

The novelty of this result is that it establishes the equivalence of integral equation (1) and jump problem (7).

In the first part of the paper, we consider the properties of Equation (1) that hold for every doubly periodic group. In the second part, we obtain the main result (Theorem 1). For this, we rely on the contraction mapping principle for Banach spaces, as in the previous study (see [1]).

2. Properties of the Carleman equation. Assume that the fundamental region of the doubly periodic group Γ is a parallelogram with vertices t_k ($t_j = -t_{j+2}$, $j = 1, 2$) and sides $l_k, k = 1, 2, 3, 4$, listed in the same order as they occur on the boundary L when it is traversed in the positive sense. Moreover, $\mathrm{Im}((t_3 - t_2)(t_2 - t_1)^{-1}) > 0$ and $\{t_1, t_2\} \subset \overline{l_1}$. The following results are from [1]:

1. The kernel (2) is skew-symmetric; that is, $K(t, \tau) = -K(\tau, t)$.
2. One of the solutions of Equation (5) is a constant.
3. The fundamental system of solutions of the adjoint equation or the Carleman equation

$$T\phi = 0 \tag{9}$$

can be constructed in such a way that some of the functions belonging to the system satisfy the condition (4) and the others satisfy the opposite condition, namely,

$$g(t) = g(\alpha(t)). \tag{10}$$

Note that condition (10) is condition (6) in the considered case of a doubly periodic group.

4. If homogeneous equation (1) is solvable, then it has a solution that satisfies the condition (4).

An analog of the Sokhotski–Plamelj formula holds for the integral (8); namely, $F^+(t) = 2^{-1}(\phi(t) - \phi(\alpha(t))) + F(t)$, where the last term on the right-hand side follows from (8) by formal substitution $z \in D$ with $t \in L$ and should be understood in the sense of the Cauchy principal value.

Lemma 1. *Assume that the fundamental system of solutions of equation (9) contains a function $\phi(t)$ with property (10). Then the function*

$$\psi(t) = \phi(t) + F^+(t) \tag{11}$$

is a nontrivial solution of the adjoint equation (5) with property (4). Conversely, assume that the fundamental system of solutions of equation (5) contains a function $\psi(t)$ with property (4). Then the function $\phi(t) = 2\psi(t) - \Phi^+(t)$, where $\Phi(z)$ is the integral in (8) ($C = 0$) with density $\psi(t)$, is a nontrivial solution of Equation (9) with property (10).

Proof. The functions $\phi(t)$ satisfy equation (9), that is,

$$2\phi(t) + F^+(t) + F^+(\alpha(t)) = 0.$$

Moreover, the function given in (11) satisfies the condition (10). Furthermore, it follows from (11) that $2\psi(t) = F^+(t) - F^+(\alpha(t))$. The solutions of the adjoint equation having the property (4) satisfy Equation (5), which, in turn, implies that $4\psi(t) = \Phi^+(t) - \Phi^+(\alpha(t))$. The first part of the lemma can now be proven by direct substitution, taking into account that kernel (3) is a generating one. Next, assume that $\psi \equiv 0$. Then $\phi(t) = \phi^+(t)$ is the boundary value of a function analytic in D. By (11), we have $\phi(t) \equiv \text{const}$. The constant is a solution of the adjoint equation (5), but not of Equation (9). This finishes the proof of the lemma. The second part of the lemma can be proven according to a similar scheme. □

Corollary. *The fundamental systems of solutions of Equations* (5) *or* (9) *can be constructed in such a way that the one for Equation* (5) *contains as many solutions with property* (10) *as the one for Equation* (9) *contains functions with the property* (4). *Furthermore, the fundamental system of solutions of Equation* (9) *contains as many functions with property* (4) *as the one for the adjoint equation contains functions with the property* (10).

The second part of the corollary follows from the first one by the Fredholm alternative, that is, the fundamental system of solutions of Equation (9) contains at least one function having the property (4). In what follows, we assume that the function $\phi(t)$ has property (4) and the function $\psi(t)$ has property (10). For such a solution $\phi(t)$ of Equation (9), it follows that $F^+(t) - F^+(\alpha(t)) = 0$, which means that $F(z) = c$, $z \in D$. The solutions can be of three kinds.

Definition 1. *If*

$$\frac{1}{2\pi i}\int\limits_L A(z,\tau)\phi(\tau)\,d\tau = c, \qquad z \in D \tag{12}$$

with $c \neq 0$, then we say that $\phi(t)$ is a solution of the first kind.

Lemma 2. *If $\phi(\tau)$ is a solution of the first kind, then*

$$\int\limits_L \phi(\tau)\,d\tau = 0. \tag{13}$$

Proof. The function $\phi^2(t)$ satisfies condition (10), that is,

$$\int\limits_L \phi^2(\tau)\,d\tau = 0.$$

Pass in (12) to the limit as $z \to t \in L$. As a result, we have

$$\phi(t) + \frac{1}{2\pi i}\int\limits_L A(t,\tau)\phi(\tau)\,d\tau = c. \tag{14}$$

Multiply this equality by $\phi(t)$ and integrate over L, then change the order of integration in the double integral. Taking into account that the kernel (3) is skew-symmetric, we have

$$-c\int\limits_L \phi(t)dt = c\int\limits_L \phi(t)\,dt,$$

thus finishing the proof of the lemma. □

Definition 2. *If the solution $\phi(t)$ satisfies condition (12) with $c = 0$, but condition (13) does not hold, then we say that $\phi(t)$ is a solution of the second kind.*

Lemma 3. *Equation (9) cannot have both a solution $\phi(t)$ of the first kind and a solution $\phi_1(t)$ of the second kind.*

Proof. Assume the opposite. Multiply (14) by $\phi_1(t)$ and integrate the result over L to get

$$0 = c \int_L \phi_1(t)\, dt.$$

This finishes the proof. □

Definition 3. *If $\phi(t)$ satisfies both (12) with $c = 0$ and (13), then we say that $\phi(t)$ is a solution of the third kind.*

Solutions of the third kind are closely related to possible complementary solutions of the adjoint equation. Let $\zeta(z) = \zeta(z; \omega_1, \omega_2)$ be a quasiperiodic Weierstrass zeta function with primitive periods $\omega_1 = t_2 - t_1$ and $\omega_2 = t_3 - t_2$. Kernel (3) is the sum of the first nine terms in the expansion of the function $\zeta(\tau - z)$ in a series of partial fractions.

Lemma 4. *Assume that Equation (9) has a solution $\phi(t)$ of the third kind. Then the integral*

$$\psi(t) = \frac{1}{2\pi i} \int_L \zeta(\tau - t)\phi(\tau)\, d\tau, \tag{15}$$

understood in the sense of the Cauchy principal value, is a complementary solution. Conversely, if $\psi(t)$ is a complementary solution, then the function

$$\phi(t) = \psi(t) - \Phi^+(t) \tag{16}$$

is a solution of the third kind.

Proof. It follows from (13) and the skew-symmetry of the kernel that function (15) satisfies condition (10). For a solution of the adjoint equation with property (10), we have

$$2\psi(t) = \Phi^+(t) + \Phi^+(\alpha(t)). \tag{17}$$

Consider the integral

$$\psi(z) = \frac{1}{2\pi i} \int\limits_L \zeta(\tau - z)\phi(\tau)\, d\tau.$$

An analog of the Sokhotski–Plamelj formula holds: $\psi^+(t) = \phi(t) + \psi(t)$. By substituting (15) into (17), we see that the first part of the lemma is indeed true. If we suppose that $\psi(t) \equiv 0$, then we obtain a contradiction, since, in that case, $\phi(t)$ is the boundary value of a function analytic in D and, by virtue of (5), $\phi \equiv 0$. To prove the second part of the lemma, it is enough to multiply (16) by the kernel $A(t, z)$ and integrate over L with $z \in D$. This concludes the proof.

Note that the shift $\alpha(t)$ is an odd function. Let us replace the variables τ and t in Equation (9) with $-\tau$ and $-t$, respectively. Then $(T\phi)(t) = 0 \Leftrightarrow (T\phi)(-t) = 0$. □

Lemma 5. *The fundamental system of solutions of Equations* (5) *and* (9) *can be constructed in such a manner that some functions in it are even and the rest are odd.*

Lemma 6. *Assume that $h(t) \in C(L)$ is either an even function with property* (4) *or an odd function with property* (9)*. Then the equality*

$$\int\limits_{l_j} h(\tau)\, d\tau = 0$$

holds for any j.

This result is also valid for functions having discontinuities of the first kind (jump discontinuities) at the vertices. Let us define a new kernel:

$$H(z, \tau) = \zeta(\tau - z) - A(z, \tau). \tag{18}$$

Consider the integral

$$\Omega(z) = \frac{1}{2\pi i} \int\limits_L \zeta(\tau - z)\psi(\tau)\, d\tau. \tag{19}$$

Lemma 7. *Integral* (19) *is a constant.*

Proof. Replace in (19) the variable of integration τ with $\alpha(\tau)$. As a result, we obtain $\Omega(z) = -\Omega(z) + c_\psi$, where

$$c_\psi = \frac{-1}{2\pi i} \int\limits_L \psi(\tau)\eta_\tau \, d\tau. \tag{20}$$

In this expression, $\eta_\tau = \zeta(\alpha(\tau)) - \zeta(\tau)$ is an odd piecewise-constant function, discontinuous at the vertices. The values of this function are related by the well-known Legendre's relation with periods ω_1 and ω_2. This concludes the proof. $\square$

We can now rewrite (17) in the form

$$2\psi(t) = -\Omega_1(t) - \Omega_1(\alpha(t)) + 2c_\psi, \tag{21}$$

where

$$\Omega_1(t) = \frac{1}{2\pi i} \int\limits_L H(t,\tau)\psi(\tau)\, d\tau. \tag{22}$$

Therefore, $\Omega_1(z)$ is analytic in the closure of the region D.

Lemma 8. *Assume that $\psi(t)$ is an even complementary solution. Then the derivative $\psi'(t)$ is also a complementary solution.*

Proof. Since the integral in (22) is an even function, we have $\psi(t) \in C(L)$; that is, an even piecewise-constant function discontinuous at the vertices cannot be a complementary solution. Let us differentiate relation (21) at points that are not vertices:

$$2\psi'(t) = -\Omega_1'(t) - \Omega_1'(\alpha(t)). \tag{23}$$

Integrate by parts each integral on the right-hand side of (23) taking into account that $\partial H/\partial t = -\partial H/\partial \tau$. At the same time, the nonintegral terms vanish because they are the increment of a continuous function over a closed path. As a result, we have $2\psi(t) = -\Omega_2(t) - \Omega_2(\alpha(t))$. Here

$$Q_2(t) = \frac{1}{2\pi i} \int\limits_L H(t,\tau)\psi'(\tau) d\tau.$$

Replace now kernel (18) in the last relation with kernel $A(t,\tau)$. We finish the proof by noting that integral (20) of an odd function $\psi(\tau)$ is zero. $\square$

3. The proof of Theorem 1. Let Γ be a doubly periodic group whose fundamental region is a rectangle with a ratio of the largest side to the shortest one not exceeding 3. We can assume, without loss of generality, that the vertices of the rectangle coincide with the points $t_1 = -a - i$, $t_2 = a - i$, $t_3 = -t_1$, $t_4 = -t_2$, where $a \in [3], [1]$. Assume that the adjoint equation (5) has an odd complementary solution $\psi(t)$, which can have discontinuities of the first kind at the vertices. Furthermore, assume that

$$M = \max |\psi(t)|, \qquad t \in \Gamma. \tag{24}$$

Let us estimate the absolute value of the integral term in (5) by decomposing it into integrals over separate sides. We proceed as follows. Assume that if $\tau \in l_j$ and $t \in l_k$, we have

$$K(t,\tau) = f(\beta) i^k,\ k = 0,1, \quad \beta = \beta(t,\tau), \quad \mathrm{Im}(f(\beta)) = 0, \quad \mathrm{Im}(\beta) = 0,$$

that is, we have either a real or a purely imaginary function. The specific kind of the function $f(\beta)$ and the limits of β change depend on j and k and are given below. By Lemma 6, we can use the estimate

$$B_j(t) = \left| \int\limits_{l_j} K(t,\tau)\psi(\tau)\, d\tau \right| \leqslant 2^{-1}\mu_j M(\max f(\beta) - \min f(\beta)), \tag{25}$$

where μ_j is the length of side l_j. For the sake of brevity, we introduce the following notations:

$$u = \tau - t, \qquad L_1 = l_2 \cup l_4, \qquad L_2 = l_1 \cup l_3,$$

$$G_j(t) = \left| \int\limits_{L_j} K(t,\tau)\psi(\tau)\, d\tau \right|, \qquad j = 1, 2,$$

$$Q_j(t) = \left| \int\limits_{l_j} K_1(t,\tau)\psi(\tau)\, d\tau \right|, \qquad j = 1, 2, 3, 4.$$

Note that the kernel K_1 will be defined below.

Assume that there exists an odd nontrivial complemetary solution $\psi(t)$. The Lemma 6 is satisfied for it. Let us estimate from above the modulus of the integral term in the equation (5) using the skew-symmetry of its kernel and the Lemma 6.

By virtue of the symmetry, it is enough to consider only two cases.

I. Equality (24) is attained for $t \in l_1$. In this case, we have three possible subcases.

a) $\tau \in l_1$, that is, $K(t,\tau) = 0$.

b) $\tau \in l_3$. Then $u = \beta + 2i$, $|\beta| \leqslant 2a$, $\operatorname{Im}\beta = 0$. We obtain $K(t,\tau) = -if(\beta)$, where

$$f(\beta) = 4((\beta^2+4)^{-1} + ((\beta+2a)^2+4)^{-1} + ((\beta-2a)^2+4)^{-1} + \\ + 2(\beta^2+16)^{-1} + 2((\beta+2a)^2+16)^{-1} + 2((\beta-2a)^2+16)^{-1}).$$

By virtue of estimate (25), we have $B_3(t) \leqslant 6.97M$.

c) $\tau \in L_1$. Then, taking into account the property (10), we have

$$G_1(t) = Q_4(t),$$

where

$$K_1(t,\tau) = 2((u+4a)^{-1} + (u+4a-2i)^{-1} - (u-2a)^{-1} - (u-2a-2i)^{-1}) + \\ + (u+4a-4i)^{-1} + (u+4a+2i)^{-1} - (u-2a+2i)^{-1} - (u-2a-4i)^{-1}.$$

It follows from this that $|K_1(t,\tau)| \leqslant 2.4$.
Since $2 \cdot 2.4 + 6.97 < 4\pi$, we have $\psi \equiv 0$.

II. Equality (24) is attained for $t \in l_2$. In this case, there are three possible subcases.

a) $\tau \in l_2$, that is, $K(t,\tau) = 0$.

b) $\tau \in l_4$. Then $u = \beta i - 2a$, $|\beta| \leqslant 2|$. Since

$$K(t,\tau) = 4a((\beta^2+4a^2)^{-1} + ((\beta+2)^2+4a^2)^{-1} + ((\beta-2)^2+4a^2)^{-1} + \\ + 2(\beta^2+16a^2)^{-1} + 2((\beta+2)^2+16a^2)^{-1} + 2((\beta-2)^2+16a^2)^{-1}),$$

taking into account the estimate (25), we obtain $B_4(t) \leqslant 0.1M$.

c) $\tau \in L_2$. By virtue of property (10), we have $G_2(t) = Q_1(t)$. Here

$$K_1(t,\tau) = 2((u-2i)^{-1} + (u+2a-2i)^{-1} - (u+4i+2a)^{-1} - (u+4i)^{-1}) = \\ = (u-2a-2i)^{-1} + (u+4a-2i)^{-1} - (u+4a+4i)^{-1} - (u+4i-2a)^{-1}.$$

So, we have $|K_1(t,\tau)| \leqslant 1.93$.
Since $6 \cdot 1.93 + 0.1 < 4\pi$, it follows that $\psi \equiv 0$.

Therefore, the complementary solution must be an even function, which is impossible by Lemma 8. This finishes the proof of the theorem.

As for the result we have obtained, note that the restriction $a \leqslant 3$ is necessary since the accuracy of the estimates of the absolute values of the integrals required to prove Theorem 1 would degrade with increasing a. The best estimates are obtained for $a = 1$, that is, when the rectangle D is a square.

References

[1] Aksent'eva E. P., Garif'yanov F. N. *On the Analysis of an Integral Equation with Carleman Kernel*, Soviet Mathematics, 1983, vol. 27:4, pp. 53 – 63.

[2] Aksent'eva E. P., Garif'yanov F. N. *On Lacunary Analogs of the Poincare Theta-Series and Their Applications*, Siberian Math. J., 2002, vol. 43:5, pp. 787 – 794. DOI: https://doi.org/10.1023/A:1020191219922

[3] Carleman T., *Sur la theorie des equations integrales et ses applications*, Verh. Int. Math. Kongress. Zurich. Bd I, 1932, pp. 138 – 151.

[4] Garif'yanov F. N. *Stieltjes moments of entire functions of exponential type*, Math. Notes, 2000, vol. 67:5, pp. 572 – 576.

[5] Garif'yanov F. N. *On a Certain Difference Equation and Its Application to the Moment Problem*, Math. Notes, 2003, vol. 73:6, pp. 777 – 782.

[6] Garif'yanov F. N., Modina S. A. *The Carleman kernel and its applications*, Siberian Math. J., 2012, vol. 53:6, pp. 1011 – 1020.

[7] Garif'yanov F. N., Strezhneva E. V. *On moment problem for entire function generated by doubly periodic group*, Ufa Math. J., 2020, vol. 12:2, pp. 21 – 27. DOI: https://doi.org/10.13108/2020-12-2-21

Received January 31, 2022.
In revised form, July 24, 2022.
Accepted July 29, 2022.
Published online September 19, 2022.

F. N. Garif'yanov
Kazan State Power Engineering University
51 Krasnosel'skaya street, Kazan 420066, Russia
E-mail: f.garifyanov@mail.ru

E. V. Strezhneva
Kazan National Research Technical University named after A. N. Tupolev
10 K. Marx street, Kazan, 42011, Russia
E-mail: strezh@yandex.ru