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A NOTE ON ALMOST UNIFORM CONTINUITY OF BOREL FUNCTIONS ON POLISH METRIC SPACES

Abstract. With a simple short proof, this article improves a classical approximation result of Lusin's type; specifically, it is shown that, on any given finite Borel measure space with the ambient space being a Polish metric space, every Borel real-valued function is almost a bounded, uniformly continuous function in the sense that for every $\varepsilon > 0$ there is some bounded, uniformly continuous function, such that the set of points at which they would not agree has measure less than ε. This result also complements the known result of almost uniform continuity of Borel real-valued functions on a finite Radon measure space whose ambient space is a locally compact metric space.

Key words: *almost uniform continuity; Borel functions; extension theorems; finite Borel measures; Lusin's theorem; Polish metric spaces*

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Let Ω be a metric space; let M be a finite Borel measure over Ω. It follows from the well-known version of Lusin's Theorem (e.g., Theorem 2.24 in Rudin [6]) that if Ω is locally compact, if M is Radon, and if $f\colon \Omega \to \mathbb{R}$ is Borel-measurable, then for every $\varepsilon > 0$ there is some bounded, uniformly continuous function $g\colon \Omega \to \mathbb{R}$ such that the set of points at which f and g possibly disagree has measure less than ε. For our purposes, we refer to a Borel function $f\colon \Omega \to \mathbb{R}$ satisfying the conclusion of the above proposition as (M)-*almost uniformly continuous*, where Ω is simply a metric space and M, with respect to which the meaning of "almost" is clearly assigned, is simply a finite Borel measure over Ω. Thus, boundedness is also a requirement of almost uniform continuity.

However, depending on one's purposes, local compactness is not always a "nice" property. For instance, there are metric spaces that are interesting and important in analysis but are not locally compact. It can be shown that the space $\mathbb{R}^{\mathbb{N}}$ of real sequences, equipped with the product metric (in terms of summation) of the (equivalent) Euclidean metric $(x, y) \mapsto \min(1, |x - y|)$ of $\mathbb{R}$, is not locally compact, and that the Banach space $C([0,1], \mathbb{R})$ is not locally compact with respect to the uniform metric; no closed ball in either space is compact. On the other hand, these spaces are indeed complete and separable with respect to the respective metrics, i.e, they are Polish metric spaces.

Polish spaces play a central role in the descriptive set theory. Moreover, as the Alexandroff (one-point) compactification of every given topological manifold is completely metrizable, every topological manifold is a Polish space. Every compact metric space is a Polish metric space. For another example, probability theory is very concerned with Polish metric spaces (e.g. Billingsley [1] or Villani [7]).

Given the importance of Polish spaces, it would be desirable to have an almost-uniform-continuity result, serving Polish metric spaces and requiring no local compactness, for Borel real-valued functions, such that the assumption on the underlying finite measure is hopefully mild.

Certainly, given an arbitrary metric space and a finite Borel measure over it, it is well-known that every Borel real-valued function is "almost" a continuous function (e.g., Theorem 17.12 in Kechris [4], or Section 2.3.6 in Federer [2]). Nevertheless, without further assumptions, the approximating continuous functions need not be bounded or uniformly continuous.

This article indicates an improvement for the result above by asserting boundedness and uniform continuity instead of continuity.

For reference, by a *Borel* measure over a metric space Ω we mean a measure defined on the Borel sigma-algebra $\mathscr{B}_\Omega$ of Ω generated by the metric topology of Ω.

We will stick to the standard measure-theoretic definitions of outer regularity (approximation by open sets from without) and inner regularity (approximation by compact sets from within) of measures (Rudin [6] or Folland [3], for concreteness). For our purposes, it would be convenient to introduce the following terminology. If M is a finite Borel measure over a metric space Ω, the measure M is called *co-outer regular*[1] if and only if

[1] In the literature of probability theory, co-outer regularity is sometimes also termed inner regularity, and is associated with Borel probability measures over a metric space.

$\mathsf{M}(B) = \sup\{\mathsf{M}(F) \,|\, F \subset B \text{ is closed}\}$ for every $B \in \mathscr{B}_\Omega$. The terminology reflects the fact that a closed set is the complement of some open set. Since $\sup\{\mathsf{M}(K) \,|\, K \subset B \text{ is compact}\} \leqslant \sup\{\mathsf{M}(F) \,|\, F \subset B \text{ is closed}\}$ whenever $B \in \mathscr{B}_\Omega$, the inner regularity (resp. co-outer regularity) of a finite Borel measure need not imply co-outer regularity (resp. inner regularity).

By a *Polish metric space* we mean a metric space that is complete and separable with respect to the given metric. If Ω is a metric space, and if M is a finite Borel measure over Ω, we denote by $L^0(\mathsf{M})$ the collection of all Borel functions $\Omega \to \mathbb{R}$, and by $C_{b,u}(\Omega)$ the collection of all bounded, uniformly continuous functions $\Omega \to \mathbb{R}$.

The desired improvement is the following:

Theorem 1. *If Ω is a Polish metric space, and if M is a finite Borel measure over Ω, then every Borel function $\Omega \to \mathbb{R}$ is M-almost uniformly continuous, i.e., then $f \in L^0(\mathsf{M})$ implies that for every $\varepsilon > 0$ there is some $g \in C_{b,u}(\Omega)$ such that*

$$\mathsf{M}(\{x \in \Omega \,|\, f(x) \neq g(x)\}) < \varepsilon.$$

Moreover, we may choose g such that $\sup_{x\in\Omega} |g(x)| \leqslant \sup_{x\in\Omega} |f(x)|$.

Proof. Let $f \in L^0(\mathsf{M})$; let $\varepsilon > 0$. Since every finite Borel measure over a metric space is both outer regular and co-outer regular (which may be obtained neatly from an apparent generalization of the simple proof of Theorem 1.1 in Billingsley [1]), by (the proof of) Theorem 17.12 in Kechris [4] we can choose some closed subset F_ε of Ω such that $\mathsf{M}(F_\varepsilon^c) < \varepsilon/2$ and $f|_{F_\varepsilon}$ is continuous.

Although it is well-known that every finite Borel measure over a Polish metric space is inner-regular, we give a quick justification. Since Ω is a Polish metric space, we have $\mathsf{M}(\Omega) = \sup\{\mathsf{M}(K) \,|\, K \subset \Omega \text{ is compact}\}$; this follows from the direct apparent application of the simple proof of Theorem 1.3 in Billingsley [1]. If $B \in \mathscr{B}_\Omega$, then, since M is co-outer regular, given any $\delta > 0$, we can choose some closed $F \subset B$ such that $\mathsf{M}(F) > \mathsf{M}(B) - \delta/2$; choose, in turn, some compact $K \subset \Omega$ such that $\mathsf{M}(K) > \mathsf{M}(\Omega) - \delta/2$. Then $K \cap F$ is compact and included in B, and

$$\mathsf{M}(K \cap F) = \mathsf{M}(F) - \mathsf{M}(F \setminus K) >$$

Since our arguments will involve both inner regularity in the standard measure-theoretic sense and co-outer regularity, and since every Borel probability measure over a metric space is a finite Borel measure, we choose to assign a new name to the property.

$$> \mathsf{M}(B) - \delta/2 - \mathsf{M}(\Omega \setminus K) >$$
$$> \mathsf{M}(B) - \delta;$$

the inner regularity of M follows.

Now we can choose a compact $K \subset F_\varepsilon$ such that $\mathsf{M}(F_\varepsilon \setminus K) < \varepsilon/2$, and so $f|_K$ is bounded and uniformly continuous. Then, by the McShane extension theorem (Corollary 2 in McShane [5]), we can choose some $g \in C_{b,u}(\Omega)$ such that $g|_K = f|_K$ and g preserves the bounds. Since $\{x \in \Omega \,|\, f(x) \neq g(x)\} \subset K^c$, we have

$$\mathsf{M}(\{x \in \Omega \mid f(x) \neq g(x)\}) \leqslant \mathsf{M}(K^c) \leqslant$$
$$\leqslant \mathsf{M}(F_\varepsilon \setminus K) + \mathsf{M}(F_\varepsilon^c) < \varepsilon.$$

Upon further choosing g such that $\sup\limits_{x\in\Omega} |g(x)| = \sup\limits_{x\in K} |f(x)|$, we complete the proof. □

Remark. *The existence of a compact $K \subset F_\varepsilon$ in the proof of Theorem 1 is asserted by Theorem 17.12 in Kechris [4]; in the proof of Theorem 1 we merely indicate a seemingly overlooked quicker way to see it.* □

For various application purposes, there are some convergence results, following immediately from Theorem 1, which might be worth indicating:

Proposition 1. *Let Ω be a Polish metric space; let M be a finite Borel measure over Ω. Then*

(i) for every $f \in L^0(\mathsf{M})$ there is some sequence $(g_n)_{n\in\mathbb{N}}$ in $C_{b,u}(\Omega)$ such that $g_n \to f$ in measure;

(ii) given any $1 \leqslant p < +\infty$, for every bounded $f \in L^0(\mathsf{M})$ there is some sequence $(g_n)_{n\in\mathbb{N}}$ in $C_{b,u}(\Omega)$ such that $|g_n - f|_{L^p} \to 0$.

Proof. Since

$$\{x \in \Omega \mid g(x) \neq h(x)\} = \{x \in \Omega \mid |g(x) - h(x)| > 0\} \supset$$
$$\supset \{x \in \Omega \mid |g(x) - h(x)| > \varepsilon\}$$

for all $\varepsilon > 0$ and all $g, h : \Omega \to \mathbb{R}$, the assertion (i) follows from Theorem 1.

To prove (ii), consider a bounded element f of $L^0(\mathsf{M})$. Given any $\varepsilon > 0$, we have, by Theorem 1, some $g \in C_{b,u}(\Omega)$ such that $\mathsf{M}(\{x \in \Omega \mid |f(x) - g(x)| > 0\}) < \varepsilon$ and $\sup_{x\in\Omega} |g(x)| \leqslant \sup_{x\in\Omega} |f(x)|$; then the desired L^p-convergence follows. □

References

[1] Billingsley P. *Convergence of Probability Measures*, second edition. Wiley, 1999.

[2] Federer H. *Geometric Measure Theory*, reprint of the first edition. Springer, 1996.

[3] Folland G. B. *Real Analysis: Modern Techniques and Their Applications*, second edition. Wiley, 2007.

[4] Kechris A. *Classical Descriptive Set Theory*. Springer, 1995.

[5] McShane E. J. *Extension of range of functions*. Bull. Amer. Math. Soc., 1934, vol. 40, pp. 837 − 842.

[6] Rudin W. *Real and Complex Analysis*, (international) third edition. McGraw-Hill, 1987.

[7] Villani C. *Optimal Transport: Old and New*. Springer, 2009.

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