DOI: 10.15393/j3.art.2022.11770

UDC 517.521

Kuldip Raj, Kavita Saini, M. Mursaleen

APPLICATIONS OF THE FRACTIONAL DIFFERENCE OPERATOR FOR STUDYING EULER STATISTICAL CONVERGENCE OF SEQUENCES OF FUZZY REAL NUMBERS AND ASSOCIATED KOROVKIN-TYPE THEOREMS

Abstract. The present work focuses on the statistical Euler summability, Euler statistical convergence, and Euler summability of sequences of fuzzy real numbers via the generalized fractional difference operator. We make an effort to establish some relations between different sorts of Euler convergence. Further, we discuss the fuzzy continuity and demonstrate a fuzzy Korovkin-type approximation theorem. Finally, we study fuzzy rate of the convergence of approximating fuzzy positive linear operators through the modulus of continuity.

Key words: *Euler mean, sequences of fuzzy real numbers, statistical convergence, rate of convergence, approximation theorem*

2020 Mathematical Subject Classification: *40A05, 40A30, 46S40, 47S40*

1. Introduction and Preliminaries. The theory of statistical convergence was initially presented by Fast [7] and Steinhaus [21]. It has been further designed by Connor [5], Fridy [8], Miller and Orhan [13]. For advanced developments in the field of statistical convergence and the neighbour topics, see [15], [16]. Mursaleen and Alotaibi [17] also proved an approximation theorem for a function of two variables by means of statistical $A-$summability. For a detailed study on summability theory and approximation results, see [1], [11], [14], [18], and many others.
In 1965, Zadeh [27] introduced the concept of fuzzy numbers. Savaş [22] studied statistical convergence for a sequence of fuzzy numbers. Later, Aytar et al. [2] expanded the concept of statistical superior limit and inferior limit to statistically bounded sequences of fuzzy real numbers. Talo and

Başar ([23], [24]) studied certain classes of sequences of fuzzy numbers; for further study, see [6], [25], [26].

A fuzzy set $\hat{u}\colon \mathbb{R} \to [0,1]$ is called a fuzzy number if it satisfies the following criteria:

(i) $\hat{u}$ is normal, i. e., there exists an $x_0 \in \mathbb{R}$, such that $\hat{u}(x_0) = 1$;

(ii) $\hat{u}$ is convex, i. e., for $x_0, x \in \mathbb{R}$ and $0 \leqslant \tau \leqslant 1$:

$$\hat{u}(\tau x_0 + (1-\tau)x) \geqslant \min\{\hat{u}(x_0), \hat{u}(x)\};$$

(iii) $\hat{u}$ is upper semi-continuous;

(iv) $\operatorname{supp}(\hat{u}) = cl\{x \in \mathbb{R}\colon \hat{u}(x) > 0\}$ is compact and it is denoted by $[\hat{u}]^0$.

Throughout the paper, $\mathbb{R}_{\mathbb{F}}$ denotes the space of all fuzzy numbers. Suppose $[\hat{u}]^0 = \overline{\{x \in \mathbb{R}\colon \hat{u}(x) > 0\}}$ and the i-level set is $[\hat{u}]^i = \{x \in \mathbb{R}\colon \hat{u}x) \geqslant i\}$, $(0 < i \leqslant 1)$. For any $\hat{u}, \hat{v} \in \mathbb{R}_{\mathbb{F}}$ and $\lambda \in \mathbb{R}$, it is positive to define uniquely the sum $\hat{u} \oplus \hat{v}$ and the scalar multiplication to $\lambda \in \mathbb{R}$ of $\hat{u}$ as

$$[\hat{u} \oplus \hat{v}]^i = [\hat{u}]^i + [\hat{v}]^i \ \text{ and } \ [\lambda \odot \hat{u}]^i = \lambda[\hat{u}]^i.$$

Now, the interval $[\hat{u}]^i$ is denoted by $[\hat{u}_-^{(i)}, \hat{u}_+^{(i)}]$, where $\hat{u}_-^{(i)} \leqslant \hat{u}_+^{(i)}$ and $u_-^{(i)}, \hat{u}_+^{(i)} \in \mathbb{R}$, $i \in [0,1]$. Then, for $\hat{u}, \hat{v} \in \mathbb{R}_{\mathbb{F}}$ define

$$\hat{u} \leqslant \hat{v} \Leftrightarrow \hat{u}_-^{(i)} \leqslant \hat{v}_-^{(i)} \text{ and } \hat{u}_+^{(i)} \leqslant \hat{v}_+^{(i)},\ \forall\, 0 \leqslant i \leqslant 1.$$

Now, $d\colon \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}$ is given by

$$d(\hat{u}, \hat{v}) = \sup_{i \in [0,1]} \max\left\{\left|\hat{u}_-^{(i)} - \hat{v}_-^{(i)}\right|, \left|\hat{u}_+^{(i)} - \hat{v}_+^{(i)}\right|\right\}.$$

Here, $(\mathbb{R}_{\mathbb{F}}, d)$ is a complete metric space [20]. Let $g, h\colon [a,b] \to \mathbb{R}_{\mathbb{F}}$ be fuzzy-valued functions. Then the distance between g and h is given by

$$d^*(g,h) = \sup_{u \in [a,b]} \sup_{i \in [0,1]} \max\left\{\left|g_-^{(i)} - h_-^{(i)}\right|,\ \left|g_+^{(i)} - h_+^{(i)}\right|\right\}.$$

Let $y = (y_\vartheta)$ be a sequence of fuzzy real numbers. Then (y_ϑ) is called statistically convergent to a fuzzy number L, if, for every $\epsilon > 0$:

$$\lim_{\eta} \frac{1}{\eta}|\{\vartheta \leqslant \eta : d(y_\vartheta, L) \geqslant \epsilon\}| = 0.$$

By Ω, we mean the space of real-valued sequences. Let $\hat{y} = (\hat{y}_\vartheta)$ be any sequence in Ω and h be a constant. Recently, Baliarsingh [3], [4] introduced a new version of difference sequence space of fractional order, given by

$$(\Delta_h^{r,s,t}\hat{y})_\vartheta = \sum_{j=0}^{\infty} \frac{(-r)_j(-s)_j}{j!(-t)_j h^{r+s-t}} \hat{y}_{\vartheta-j},\ \forall \vartheta \in \mathbb{N} \tag{1}$$

where r, s, t are real numbers and $(\tau)_\vartheta$ is the Pochhammer symbol of a real number τ, which is defined as

$$(\tau)_\vartheta = \begin{cases} 1, & (\vartheta = 0) \\ \Gamma(\tau) = [\tau][\tau+1][\tau+2]\cdots[\tau+\vartheta-1], & (\vartheta \in \mathbb{N}). \end{cases}$$

Here the series (1) is convergent for all $t > r + s$ (see [9]).

Definition 1. [19] *A sequence $\hat{y} = (\hat{y}_\vartheta)$ is said to be Euler statistically convergent to l, if for each $\varepsilon > 0$*

$$B_\varepsilon = \{\vartheta \leqslant (1+\mu)^\eta \colon \mu^{\eta-\vartheta}|\hat{y}_\vartheta - l| \geqslant \varepsilon\}$$

has zero natural density, i. e.,

$$\lim_{\eta\to\infty} \frac{|B_\varepsilon|}{(1+\mu)^\eta} = 0.$$

Definition 2. *A sequence $y = (y_\vartheta)$ of fuzzy real numbers is said to be statistically $\Delta_h^{r,s,t}$ Euler summable $(st - \Delta_h^{r,s,t} S_E^F)$ to a fuzzy number L, if for every $\varepsilon > 0$*

$$\lim_{\eta\to\infty} \frac{1}{\eta}\left|\left\{\vartheta \leqslant \eta \colon d\left(\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta}\Delta_h^{r,s,t} y_\vartheta,\ L\right) \geqslant \varepsilon\right\}\right| = 0.$$

Definition 3. *A sequence $y = (y_\vartheta)$ of fuzzy real numbers is said to be $\Delta_h^{r,s,t}$ Euler statistically $(\Delta_h^{r,s,t} St_E^F)$ convergent to a fuzzy number L, if for each $\varepsilon > 0$*

$$\{\vartheta \leqslant (1+\mu)^\eta \colon \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta,\ L) \geqslant \varepsilon\}$$

has zero natural density, i. e.,

$$\lim_{\eta\to\infty} \frac{1}{(1+\mu)^\eta} |\{\vartheta \leqslant (1+\mu)^\eta : \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta,\ L) \geqslant \varepsilon\}| = 0.$$

Definition 4. *A sequence $y = (y_\vartheta)$ of fuzzy real numbers is said to be $\Delta_h^{r,s,t}$ Euler summable to a fuzzy number L, if*

$$\lim_{\eta\to\infty} \frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} \Delta_h^{r,s,t} y_\vartheta = L \ \text{ as } \eta \to \infty.$$

Also, $y = (y_\vartheta)$ is said to be strongly $\Delta_h^{r,s,t}$ Euler summable $(\Delta_h^{r,s,t} S_E^F)$ to L, if

$$\lim_{\eta\to\infty} \frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) = 0.$$

Definition 5. *A sequence $y = (y_\vartheta)$ of fuzzy real numbers is said to be strongly $\Delta_h^{r,s,t}$ Euler summable with order α $(\Delta_h^{r,s,t} S_E^F)_\alpha$ $(0 < \alpha < \infty)$ to L, if*

$$\lim_{\eta\to\infty} \frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L)^\alpha = 0.$$

2. Relation between different convergence concepts of sequences of fuzzy real numbers.

Theorem 1. *Suppose $\mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta - L) \leqslant M$, $\forall\, \eta, \vartheta \in \mathbb{N}$. If a sequence $y = (y_\vartheta)$ is $\Delta_h^{r,s,t}$ Euler statistically $(\Delta_h^{r,s,t} St_E^F)$ convergent to L, then it is strongly $\Delta_h^{r,s,t}$ Euler summable $(\Delta_h^{r,s,t} S_E^F)$ to L.*

Proof. Suppose $\mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) \leqslant M$, $\forall\, \eta, \vartheta \in \mathbb{N}$. By the given condition, we have

$$\lim_{\eta\to\infty} \frac{1}{(1+\mu)^\eta} |\{\vartheta \leqslant (1+\mu)^\eta \colon \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) \geqslant \varepsilon\}| = 0.$$

Consider

$$G(\varepsilon) = \{\vartheta \leqslant (1+\mu)^\eta \colon \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) \geqslant \varepsilon\}$$

and

$$G^c(\varepsilon) = \{\vartheta \leqslant (1+\mu)^\eta \colon \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) < \varepsilon\}.$$

Then

$$\frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) =$$

$$= \frac{1}{(1+\mu)^\eta} \sum_{\substack{\vartheta=1 \\ \vartheta\in G(\varepsilon)}}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta - L) + \frac{1}{(1+\mu)^\eta} \sum_{\substack{\vartheta=1 \\ \vartheta\in G^c(\varepsilon)}}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) \leqslant$$

$$\leqslant \frac{1}{(1+\mu)^\eta}\left(\sup_\vartheta \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L)\right)|G(\varepsilon)| + \varepsilon \leqslant$$
$$\leqslant \frac{1}{(1+\mu)^\eta} M|G(\varepsilon)| + \varepsilon \to 0 + \varepsilon = \varepsilon$$

as $\eta \to \infty$, which implies $y = (y_\vartheta)$ is strongly $\Delta_h^{r,s,t}$ Euler summable $(\Delta_h^{r,s,t} S_E^F)$ to L. □

Theorem 2.

(a) A sequence of fuzzy numbers $y = (y_\vartheta)$ be $(\Delta_h^{r,s,t} S_E^F)_\alpha$ to L. If

$$0 < \alpha < 1 \text{ and } 0 \leqslant d(\Delta_h^{r,s,t} y_\vartheta, L) < 1$$

or

$$1 \leqslant \alpha < \infty \text{ and } 1 \leqslant d(\Delta_h^{r,s,t} y_\vartheta, L) < \infty,$$

then (y_ϑ) is $(\Delta_h^{r,s,t} St_E^F)$ convergent to L.

(b) A sequence of fuzzy numbers $y = (y_\vartheta)$ is $(\Delta_h^{r,s,t} St_E^F)$ convergent to L and

$$\mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) \leqslant M, \text{ for all } \vartheta \in \mathbb{N}.$$

If $0 < \alpha < 1$ and $1 \leqslant M < \infty$ or $1 \leqslant \alpha < \infty$ and $0 \leqslant M < 1$, then (y_ϑ) is $(\Delta_h^{r,s,t} S_E^F)_\alpha$ to L.

Proof. (a) Let $y = (y_\vartheta)$ be $(\Delta_h^{r,s,t} S_E^F)_\alpha$ to L, i. e.,

$$\lim_{\eta\to\infty} \frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L)^\alpha = 0.$$

Consider

$$G(\varepsilon) = \{\vartheta \leqslant (1+\mu)^\eta \colon \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) \geqslant \varepsilon\}$$

and

$$G^c(\varepsilon) = \{\vartheta \leqslant (1+\mu)^\eta \colon \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) < \varepsilon\}.$$

By the given conditions, we have

$$d(\Delta_h^{r,s,t} y_\vartheta - L) \leqslant d(\Delta_h^{r,s,t} y_\vartheta - L)^\alpha$$

and

$$\mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta - L) \leqslant \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta - L)^\alpha.$$

Then

$$\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta-L)^\alpha \geqslant$$

$$\geqslant \frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)=$$

$$=\frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G(\varepsilon)}}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)+\frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G^c(\varepsilon)}}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)\geqslant$$

$$\geqslant \frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G(\varepsilon)}}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)\geqslant \frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G(\varepsilon)}}^{\eta}\varepsilon=$$

$$=\varepsilon\frac{|G(\varepsilon)|}{(1+\mu)^\eta}\to 0 \ as\ \eta\to\infty,$$

which implies (y_ϑ) is $(\Delta_h^{r,s,t}St_E^F)$ converge to L.
(b) Let $y{=}y_\vartheta$ be $(\Delta_h^{r,s,t}St_E^F)$ convergent to L and $\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)\leqslant M$, for all $\vartheta\in\mathbb{N}$. Then

$$\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta-L)^\alpha =$$

$$=\frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G(\varepsilon)}}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)^\alpha+\frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G^c(\varepsilon)}}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)^\alpha=$$

$$=V_1(\eta)+V_2(\eta),$$

where

$$V_1(\eta)=\frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G(\varepsilon)}}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)^\alpha,$$

$$V_2(\eta)=\frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G^c(\varepsilon)}}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)^\alpha.$$

If $\vartheta\in G(\varepsilon)$, then

$$V_1(\eta)=\frac{1}{(1+\mu)^\eta}\sum_{\substack{\vartheta=1\\ \vartheta\in G(\varepsilon)}}^{\eta}\mu^{\eta-\vartheta}d(\Delta_h^{r,s,t}y_\vartheta,L)^\alpha\leqslant$$

$$\leqslant \frac{1}{(1+\mu)^\eta} \sum_{\substack{\vartheta=1 \\ \vartheta \in G(\varepsilon)}}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) \leqslant$$

$$\leqslant \frac{1}{(1+\mu)^\eta} \left(\sup_\vartheta r^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) |G(\varepsilon)| \right) \leqslant M \frac{|G(\varepsilon)|}{(1+\mu)^\eta} \to 0 \text{ as } \eta \to \infty.$$

If $\vartheta \in G^c(\varepsilon)$, then

$$\begin{aligned} V_2(\eta) &= \frac{1}{(1+\mu)^\eta} \sum_{\substack{\vartheta=1 \\ \vartheta \in G^c(\varepsilon)}}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L)^\alpha \leqslant \\ &\leqslant \frac{1}{(1+\mu)^\eta} \sum_{\substack{\vartheta=1 \\ \vartheta \in G^c(\varepsilon)}}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta, L) \frac{\varepsilon |G(\varepsilon)|}{(1+\mu)^\eta} = \varepsilon \text{ as } \eta \to \infty. \end{aligned}$$

Thus, $\frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} d(\Delta_h^{r,s,t} y_\vartheta - L)^\alpha \to 0$ as $\eta \to \infty$. Hence, (y_ν) is $(\Delta_h^{r,s,t} S_E^F)_\alpha$ to L. □

3. Korovkin-type theorem and rates of equi-statistical convergence. In this section, we use the concept of statistical $\Delta_h^{r,s,t}-$ Euler summability method $(st - \Delta_h^{r,s,t} S_E^F)$ to prove a Korovkin-type approximation theorem. A fuzzy number valued function $g : [a, b] \to \mathbb{R}_F$ is said to be fuzzy continuous at $y_0 \in [a, b]$, iff $y_\nu \to y_0$; then $d(y_\vartheta, y_0) \to 0$ as $\vartheta \to \infty$. In other words, we can say that on an interval $[a, b]$ g is fuzzy continuous if it is fuzzy continuous for any $u \in [a, b]$, and we denote the space of all fuzzy continuous functions on the interval $[a, b]$ by $C_F[a, b]$. In this case, $C_F[a, b]$ is just a cone, not a vector space. Now let $\xi : C_F[a, b] \to C_F[a, b]$ be an operator. We say that ξ is fuzzy linear, if for every $\zeta_1, \zeta_2 \in \mathbb{R}$, $g_1, g_2 \in C_F[a, b]$, and $u \in [a, b]$:

$$\xi(\zeta_1 \odot g_1 \oplus \zeta_2 \odot g_2; u) = \zeta_1 \odot \xi(g_1; u) \oplus \zeta_2 \odot \xi(g_2; u).$$

Also, ξ is called fuzzy positive linear operator, if it is fuzzy linear and

$$\xi(g_1; u) \leqslant \xi(g_2; u)$$

for any $g_1, g_2 \in C_{F[a,b]}$ and for any $u \in [a, b]$ with

$$g_1(u) \leqslant g_2(u).$$

In this paper, we use the test function e_j, which is given by $e_j(u) = u^j$; here $j = 0, 1, 2$.

Theorem 3. *Consider the fuzzy sequence $\{\xi_m\}$ of positive linear operators from $C_F[a,b]$ into itself. Suppose that there exists a corresponding sequence $\{\bar{\xi}_m\}$ of positive linear operators from $C[a,b]$ into itself s.t.*

$$\{\xi_m(g;u)\}_{\pm}^{(i)} = \bar{\xi}_m(g_{\pm}^{(i)};u), \tag{2}$$

for all $u \in [a,b]$, $g \in C_F[a,b]$ and $m \in \mathbb{N}$. Suppose also that

$$st - \lim_{\eta\to\infty} \left\| \frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} \Delta_h^{r,s,t} \bar{\xi}_\vartheta(e_j) - e_j \right\| = 0 \quad (j = 0, 1, 2). \tag{3}$$

Then, for all $g \in C_F[a,b]$,

$$st - \lim_{\eta\to\infty} d^*\Big(\frac{1}{(1+\mu)^n} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} \Delta_h^{r,s,t} \xi_\vartheta(g), g \Big) = 0. \tag{4}$$

Proof. Suppose $g \in C_F[a,b]$, $u \in [a,b]$ and $i \in [0,1]$. Since $g_{\pm}^{(i)} \in C[a,b]$, for every $\varepsilon > 0$, there exists a number $\rho > 0$, such that $|g_{\pm}^{(i)}(y) - g_{\pm}^{(i)}(x)| < \varepsilon$ whenever $|v - u| < \rho$. Since g is fuzzy bounded, we get $|g_{\pm}^{(i)}(u)| \leqslant R_{\pm}^{(i)}$. Then, for all $v \in [a,b]$, we have

$$|g_{\pm}^{(i)}(v) - g_{\pm}^{(i)}(u)| \leqslant \varepsilon + 2R_{\pm}^{(i)} \frac{(v-u)^2}{\rho^2},$$

which implies

$$-\varepsilon - \frac{2R_{\pm}^{i}}{\rho^2}(v-u)^2 < \Big(g_{\pm}^{(i)}(v) - g_{\pm}^{(i)}(u) \Big) < \varepsilon + \frac{2R_{\pm}^{i}}{\rho^2}(v-u)^2.$$

Using the positivity and linearity of the operators $\bar{\xi}_m$, we have

$$\frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} \Delta_h^{r,s,t} \bar{\xi}_\vartheta(1,u) \Big(-\varepsilon - \frac{2R_{\pm}^{i}}{\rho^2}(v-u)^2 \Big) <$$

$$< \frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} \Delta_h^{r,s,t} \bar{\xi}_\vartheta(1,u) \Big(g_{\pm}^{(i)}(v) - g_{\pm}^{(i)}(u) \Big) <$$

$$< \frac{1}{(1+\mu)^\eta} \sum_{\vartheta=1}^{\eta} \mu^{\eta-\vartheta} \Delta_h^{r,s,t} \bar{\xi}_\vartheta(1,u) \Big(\varepsilon + \frac{2R_{\pm}^{i}}{\rho^2}(v-u)^2 \Big).$$

Suppose u is fixed; then $g_{\pm}^{(i)}(u)$ is a constant number and we have:

$$-\varepsilon\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(1,u)-$$
$$-\frac{2R_{\pm}^{i}}{\rho^2}\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}((v-u)^2,u)<$$
$$<\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(g_{\pm}^{(i)}(v),u)-$$
$$-g_{\pm}^{(i)}(u)\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(1,u)<$$
$$<\varepsilon\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(1,u)+$$
$$+\frac{2R_{\pm}^{i}}{\rho^2}\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}((v-u)^2,u).$$

Also,

$$\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(g_{\pm}^{(i)}(v),u)-g_{\pm}^{(i)}(u)=$$
$$=\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(g_{\pm}^{(i)}(v),u)-$$
$$-g_{\pm}^{(i)}(u)\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(1,u)+$$
$$+g_{\pm}^{(i)}(u)\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}((1,u)-1),$$

which gives:

$$\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(g_{\pm}^{(i)}(v),u)-g_{\pm}^{(i)}(u)<$$
$$<\varepsilon\frac{1}{(1+\mu)^{\eta}}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_{\vartheta}(1,u)+$$

$$+\frac{2R^i_{\pm}}{\rho^2}\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta((v-u)^2,u)+$$
$$+g^{(i)}_{\pm}(u)\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta((1,u)-1). \quad (5)$$

Next, consider the second part of the above inequality:

$$\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta((v-u)^2,u)=$$
$$=\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v^2+u^2-2uv,u)=$$
$$=u^2\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)-2u\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v,u)+$$
$$+\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v^2,u)=$$
$$=\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v^2,u)-u^2\Big]-$$
$$-2u\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v,u)-u\Big]+$$
$$+u^2\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)-1\Big].$$

Using the above equality with (5), we have

$$\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(g^{(i)}_{\pm}(v),u)-g^{(i)}_{\pm}(u)\leqslant$$
$$\leqslant\varepsilon\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)+$$
$$+\frac{2R^i_{\pm}}{\rho^2}\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v^2,u)-u^2\Big]-$$
$$-2u\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v,u)-u\Big]+$$

$$+ u^2\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)-1\Big]+$$
$$+ g_{\pm}^{(i)}(u)\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)-1\Big]=$$
$$=\varepsilon\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)-1\Big]+\varepsilon+$$
$$+\frac{2R_{\pm}^i}{\rho^2}\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v^2,u)-u^2\Big]-$$
$$-2u\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v,u)-u\Big]+$$
$$+ u^2\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)-1\Big]+$$
$$+ g_{\pm}^{(i)}(u)\Big[\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)-1\Big].$$

Now,

$$\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(g_{\pm}^{(i)}(v),u)-g_{\pm}^{(i)}(u)\Big|<$$
$$<\varepsilon+\Big(\varepsilon+\frac{2R_{\pm}^i c^2}{\rho^2}+R_{\pm}^i\Big)\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(1,u)-1\Big|+$$
$$+\frac{4R_{\pm}^i c}{\rho^2}\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v,u)-u\Big|+$$
$$+\frac{2R_{\pm}^i}{\rho^2}\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(v^2,u)-u^2\Big|,$$

where $c=\max\{|a|,|b|\}$. Let $N_{\pm}^i(\varepsilon)=\max\Big(\varepsilon+\frac{2R_{\pm}^i c^2}{\rho^2}+R_{\pm}^i,\frac{4R_{\pm}^i c}{\rho^2},\frac{2R_{\pm}^i}{\rho^2}\Big)$. Taking supermum over $u\in[a,b]$, we transform the above inequality to

$$\Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(g_{\pm}^{(i)},u)-g_{\pm}^{(i)}(u)\Big\|\leqslant$$

$$\leqslant \varepsilon + N^i_{\pm}(\varepsilon)\Big\{\Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0,u)-e_0\Big\|+ \\ +\Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_1,u)-e_1\Big\|+ \\ +\Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_2,u)-e_2\Big\|\Big\}. \quad (6)$$

Then, from (2), we have

$$d^*\Big(\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\xi_\vartheta(g),g\Big)= \\ = \sup_{u\in[a,b]} d\Big(\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\xi_\vartheta(g;u),g\Big)= \\ = \sup_{u\in[a,b]}\sup_{i\in[0,1]}\max\Big\{\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(g^i_-(v))-g^i_-(u)\Big|, \\ \Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(g^i_+(v))-g^i_+(u)\Big|\Big\}= \\ = \sup_{i\in[0,1]}\max\Big\{\Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(g^i_-(v))-g^i_-(u)\Big\|, \\ \Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(g^i_+(v))-g^i_+(u)\Big\|\Big\}. \quad (7)$$

From (6) and (7), we get

$$d^*\Big(\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\xi_\vartheta(g),g\Big)\leqslant \\ \leqslant \varepsilon + N(\varepsilon)\Big\{\Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0,u)-e_0\Big\|+ \\ +\Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_1,u)-e_1\Big\|+ \\ +\Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_2,u)-e_2\Big\|\Big\},$$

where $N(\varepsilon) = \sup_{i\in[0,1]} \max\Big\{|N_-^i(\varepsilon)|, |N_+^i(\varepsilon)|\Big\}$. For a given $\varepsilon_1 > 0$, take a number $\varepsilon > 0$, such that $\varepsilon < \varepsilon_1$. Consider

$$S = \Big\{n \in \mathbb{N}\colon d^*\Big(\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\xi_\vartheta(g), g\Big) \geqslant \varepsilon_1\Big\},$$

$$S_j = \Big\{n \in \mathbb{N}\colon \Big\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_j; u) - e_j\Big\| \geqslant \frac{\varepsilon_1-\varepsilon}{3N(\varepsilon)}\Big\},$$

where $j = 0, 1, 2$. Therefore, $S \subset \sum_{j=0}^{2} S_j$. Thus, $\delta(S) \leqslant \sum_{j=0}^{2}\delta(S_j)$. Hence, using (3), we obtain (4). □

Definition 6. *Consider the positive non-increasing sequence (s_η) of real numbers. A sequence (y_ϑ) of fuzzy number is said to be statistically $\Delta_h^{r,s,t}$ Euler summable $(st - \Delta_h^{r,s,t}S_E^F)$ convergent to L with the fuzzy rate $o(s_\eta)$ for every $\varepsilon > 0$, if we have*

$$\lim_{\eta\to\infty}\frac{1}{s_\eta}\Big|\Big\{\vartheta \leqslant \eta\colon d\Big(\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}y_\vartheta, L\Big) \geqslant \varepsilon\Big\}\Big| = 0$$

and, therefore, we can write it as

$$y_\vartheta - L = (st - \Delta_h^{r,s,t}S_E^F)\ o(s_\eta).$$

Now, the modulus of continuity of $g \in C_F[a,b]$ is given by

$$z(g;\varphi) = \sup_{u,v\in[a,b],|u-v|\leqslant\varphi} d(g(u), g(v))$$

for any $0 < \varphi \leqslant b - a$ that satisfies

$$z(g, |u-v|) \leqslant \Big(1 + \frac{|u-v|}{\varphi}\Big)z(g;\varphi).$$

Next, we have the following result:

Theorem 4. *Consider the fuzzy sequence $\{\xi_m\}$ of positive linear operators from $C_F[a,b]$ into itself. Suppose that there exists a corresponding sequence $\{\bar{\xi}_m\}$ of positive linear operators from $C[a,b]$ into itself, such that $\{\xi_m(g;u)\}_\pm^{(i)} = \bar{\xi}_m(g_\pm^{(i)};u)$, for all $u \in [a,b]$, $g \in C_F[a,b]$ and $m \in \mathbb{N}$. Consider two positive non-increasing sequences (s_η) and (p_η) of real numbers. Further, suppose that*

(i) $\|\bar{\xi}_m(e_0) - e_0\| = st - \Delta_h^{r,s,t} S_E^F \ o(s_\eta)$,
(ii) $z(g, \sqrt{\|\bar{L}_m((v-u)^2; x)\|}) = st - \Delta_h^{r,s,t} S_E^F \ o(p_\eta)$.
Then we have

$$d^*(\xi_m(g), g) = st - \Delta_h^{r,s,t} S_E^F \ o(\alpha_\eta),$$

where $\alpha = \max\{s_\eta, p_\eta\}$.

Proof. Consider $g \in C_F[a, b]$ and $u \in [a, b]$. Then, using positivity and linearity of the operator $\bar{\xi}_m$ with the continuity of fuzzy modulus, we have

$$\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\xi_\vartheta(g^i_\pm; u) - g^i_\pm(u)\Big| \leqslant$$

$$\leqslant |g^i_\pm(u)|\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0; u) - e_0\Big| +$$

$$+ \frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(|g^i_\pm(v) - g^i_\pm(u)|; u) \leqslant$$

$$\leqslant R^i_\pm\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0; u) - e_0\Big| +$$

$$+ \frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta\Big(\Big(1 + \frac{|v-u|}{\varphi}\Big)z(g^i_\pm, \varphi); u\Big) \leqslant$$

$$\leqslant R^i_\pm\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0; u) - e_0\Big| +$$

$$+ \frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta\Big(\Big(1 + \frac{|v-u|^2}{\varphi^2}\Big)z(g^i_\pm, \varphi); u\Big) \leqslant$$

$$\leqslant R^i_\pm\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0; u) - e_0\Big| +$$

$$+ z(g^i_\pm, \varphi)\Big|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0; u) - e_0\Big| +$$

$$+ z(g^i_\pm, \varphi) + \frac{z(g^i_\pm, \varphi)}{\varphi^2}\Big(\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta((v-u)^2; u)\Big),$$

where $R^i_\pm = \|g^i_\pm\|$. Taking supermum norm of both sides of the above inequality, we have

$$\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\xi_\vartheta(g_\pm^i;u)-g_\pm^i(u)\right\| \leqslant$$

$$\leqslant R_\pm^i\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0;u)-e_0\right\|+$$

$$+z(g_\pm^i,\varphi)\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0;u)-e_0\right\|+$$

$$+z(g_\pm^i,\varphi)+\frac{z(g_\pm^i,\varphi)}{\varphi^2}\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta((v-u)^2;u)\right\|.$$

Now, take $\varphi=\mu^\eta$; then we have

$$\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\xi_\vartheta(g_\pm^i;u)-g_\pm^i(u)\right\| \leqslant$$

$$\leqslant R_\pm^i\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0;u)-e_0\right\|+$$

$$+z(g_\pm^i,\mu_\vartheta^\eta)\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0;u)-e_0\right\|+2z(g_\pm^i,\mu_\vartheta^\eta).$$

Therefore, we have

$$d^*\Big(\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\xi_\vartheta(g),g\Big) \leqslant$$

$$\leqslant R_\pm^i\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0;u)-e_0\right\|+$$

$$+z(g_\pm^i,\mu_\vartheta^\eta)\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0;u)-e_0\right\|+2z(g_\pm^i,\mu_\vartheta^\eta)\leqslant$$

$$\leqslant H\Big\{\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0;u)-e_0\right\|+z(g_\pm^i,\mu_\vartheta^\eta)+$$

$$+z(g_\pm^i,\mu_\vartheta^\eta)\left\|\frac{1}{(1+\mu)^\eta}\sum_{\vartheta=1}^{\eta}\mu^{\eta-\vartheta}\Delta_h^{r,s,t}\bar{\xi}_\vartheta(e_0;u)-e_0\right\|\Big\},$$

where $H=\max\{R_\pm^i,2\}$. Using Definition 6 and conditions of Theorem 4, we have desired result. □

Acknowledgment. The author deeply appreciates the suggestions of the reviewers and the editor that improved the presentation of the paper.

References

[1] Ayman-Mursaleen M., Serra-Capizzano S. *Statistical Convergence via q-Calculus and a Korovkin's Type Approximation Theorem.* Axioms. 2022, vol. 11, 70. DOI: https://doi.org/10.3390/axioms11020070

[2] Aytar S., Mammadov M. A., Pehlivan S. *Statistical limit inferior and limit superior for sequences of fuzzy numbers.* Fuzzy Sets Syst., 2006, vol. 157, pp. 976 – 985.

[3] Baliarsingh P. *On a fractional difference operator.* Alex. Eng. J., 2016, vol. 55, pp. 1811 — 1816.

[4] Baliarsingh P., Kadak U., Mursaleen M. *On statistical convergence of difference sequences of fractional order and related Korovkin type approximation theorems.* Quaest. Math., 2018, vol. 41, pp. 1117 – 1133.

[5] Connor J. *Two valued measures and summability.* Analysis, 1990, vol. 10, pp. 373 – 386.

[6] Çolak R., Altın Y., Mursaleen M. *On some sets of difference sequences of fuzzy numbers.* Soft Comput., 2010, vol. 15, pp. 787 – 793.

[7] Fast H. *Sur le convergence.* Colloq. Math., 1951, vol. 2, pp. 241 – 244.

[8] Fridy J. A. *On statistical convergent.* Analysis, 1985, vol. 5, pp. 301 – 314.

[9] Gasper G., Rahman M. *Basic Hypergeometric Series.* Camb. Univ. Press, 2004.

[10] Hardy G. H. *Divergent Series.* Oxford, at the Clarendon Press, Oxford, 1949.

[11] Kadak U., Mursaleen M., Mohiuddine S. A. *Statistical weighted matrix summability of fuzzy mappings and associated approximation results.* J. Intel. Fuzzy Syst., 2019, vol. 36, pp. 3483 — 3494.

[12] Kadak U., Srivastava H. M., Mursaleen M. *Relatively uniform weighted summability based on fractional-order difference operator.* Bull. Malays. Math. Sci. Soc., 2019, vol. 42, pp. 2453 – 2480.

[13] Miller H. I., Orhan C. *On almost convergent and statistically convergent subsequences.* Acta Math. Hung., 2001, vol. 93, pp. 131 – 151.

[14] Mohiuddine S. A., Asiri A., Hazarika B. *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems.* Int. J. General Syst., 2019, vol. 48, pp. 492 – 506.

[15] Mursaleen M. *$\lambda-$statistical convergence.* Math. Slovaca, 2000, vol. 50, pp. 111 – 115.

[16] Mursaleen M., Alotaibi A. *Statistical summability and approximation by de la Vallée-Poussin mean.* Appl. Math. Lett., 2011, vol. 24, pp. 320 – 324.

[17] Mursaleen M., Alotaibi A. *Korovkin type approximation theorems for functions of two variables through statistical $A-$summability.* Adv. Difference Equ., 2012, vol. 2012, pp. 1 – 10.
DOI: https://doi.org/10.1186/1687-1847-2012-65

[18] Mursaleen M., Başar F. *Sequence Spaces: Topics in Modern Summability Theory.* CRC Press/Taylor & Francis Group, Series: Mathematics and Its Applications, Boca Raton · London · New York, 2020.

[19] Patro M., Paikray S. K., Dutta H. *Statistical Deferred Euler Summability Mean and Associated Korovokin-type Approximation Theorem.* Sci. Technol. Asia, 2020, vol. 25, pp. 31 – 37.

[20] Puri M. L., Ralescu D. A. *Differentials of fuzzy functions.* J. Math. Anal. Appl., 1983, vol. 91, pp. 552 – 558.

[21] Steinhaus H. *Sur la convergence ordinarie et la convergence asymptotique.* Colloq. Math., 1951, vol. 2, pp. 73 – 74.

[22] Savaş E. *On statistically convergent sequences of fuzzy numbers.* Inf. Sci., 2001, vol. 137, pp. 277 – 282.

[23] Talo Ö., Başar F. *Certain spaces of sequences of fuzzy numbers defined by a modulus function.* Demonstratio Math., 2010, vol. 43, pp. 139 – 149.

[24] Talo Ö., Başar F. *Quasilinearity of the classical sets of sequences of fuzzy numbers and some related results.* Taiwanese J. Math., 2010, vol. 14, pp. 1799 – 1819.

[25] Tripathy B. C., Baruah A. *New type of difference sequence spaces of fuzzy real numbers.* Math. Modell. Analysis, 2009, vol. 14, pp. 391 – 397.

[26] Tripathy B. C., Borgogain S. *Some classes of difference sequence spaces of fuzzy real numbers defined by Orlicz function.* Advances in Fuzzy Systems, 2011, Article ID216414, pp. 6.

[27] Zadeh L. A. *Fuzzy sets.* Inform. and Control, 1965, vol. 8, pp. 338 – 353.

Received April, 26, 2022.
In revised form, September 20, 2022.
Accepted September 23, 2022.
Published online October 14, 2022.

Kuldip Raj
School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India
E-mail: kuldipraj68@gmail.com

Kavita Saini
School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India
E-mail: kavitasainitg3@gmail.com

M. Mursaleen
Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail: mursaleenm@gmail.com