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GENERALIZATIONS OF CERTAIN WELL-KNOWN INEQUALITIES FOR RATIONAL FUNCTIONS

Abstract. In this paper we generalize and refine a result of Wali and Shah concerning the estimate of the derivative of the maximum modulus of rational functions with prescribed poles and restricted zeros. The obtained results generalize and sharpen some well-known inequalities for the derivative of rational functions besides the refinement of some polynomial inequalities.

Key words: *polynomial, rational function, s-fold zeros, Bernstein-type inequality*

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1. Introduction. For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $p(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n over the field $\mathbb{C}$ of complex numbers. If $p \in \mathcal{P}_n$ and p' is its derivative, then we have the following:

Let $p \in \mathcal{P}_n$ and suppose $|p(z)| \leqslant M$ on $|z| = 1$. Then, for $|z| = 1$:

$$|p'(z)| \leqslant nM. \tag{1}$$

Inequality (1) is referred to as Bernsteins's inequality. Riesz [11] (see also [12, p. 557]) was probably the first to formulate this inequality in the present form. However, in a stronger version it was first proved by Smirnoff [13] (see also [7]). Equality holds in (1) if and only if p has all its zeros at the origin. However, if we impose restrictions on the location of zeros of p, then Erdös conjectured and latter Lax [8] proved the following:

If $p \in \mathcal{P}_n$ has all zeros in $|z| \geqslant 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)|.$$

On the other hand, if $p \in \mathcal{P}_n$ has all zeros in $|z| \leqslant 1$, then Turán [14] proved:

$$\max_{|z|=1} |p'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |p(z)|.$$

The above inequality of Turán was further refined by Dubinin [5], who obtained, under the same assumptions:

$$\max_{|z|=1} |p'(z)| \geqslant \frac{1}{2}\Big(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\Big) \max_{|z|=1} |p(z)|.$$

Recently, Kompaneets et al. [7] considered the problem for polynomials with zeros outside a disk and obtained some results related to classical inequalities of Bernstein and Smirnoff. These inequalities were further generalized in another paper of Kompaneets et.al [6], where it is assumed that all but one zero of $p(z)$ lie inside the disk.

We write

$$\mathbb{R}_{m,n} = \mathbb{R}_{m,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) := \Big\{\frac{p(z)}{w(z)} : p \in \mathcal{P}_m\Big\},\ m \leqslant n,$$

where

$$w(z) = \prod_{j=1}^{n} (z - \alpha_j),\ \ |\alpha_j| > 1, \quad j = 1, 2, \ldots, n.$$

Thus, $\mathbb{R}_{m,n}$ is the set of all rational functions with poles $\alpha_1, \alpha_2, \ldots, \alpha_n$ and with finite limit at ∞. Throughout this paper, we shall assume that all poles $\alpha_1, \alpha_2, \ldots, \alpha_n$ lie in $|z| > 1$. We also observe that the Blaschke product $B(z) \in \mathbb{R}_{n,n}$, where

$$B(z) := \prod_{j=1}^{n} \Big(\frac{1 - \overline{\alpha_j} z}{z - \alpha_j}\Big) = \frac{w^*(z)}{w(z)},$$

with $w^*(z) = z^n \overline{w\big(\frac{1}{\bar z}\big)} = \prod_{j=1}^{n} (1 - \overline{\alpha_j} z)$, satisfying $|B(z)| = 1$ for $|z| = 1$, and $\left|\frac{zB'(z)}{B(z)}\right| = |B'(z)|$. Li, Mohapatra, and Rodriguez [9] proved the following results for a rational function $r(z) \in \mathcal{R}_{n,n}$ with prescribed poles $\alpha_1, \alpha_2, \ldots, \alpha_n$, replacing z^n by $B(z)$:

Theorem 1. *If $r \in \mathbb{R}_{n,n}$ has all the n zeros in $|z| \geqslant 1$, then, for $|z| = 1$, we have:*

$$|r'(z)| \leqslant \frac{1}{2} |B'(z)||r(z)|. \tag{2}$$

The result is sharp and the equality holds for $r(z) = aB(z) + b$, with $|a| = |b| = 1$.

In the same paper, they proved the following:

Theorem 2. *If $r \in \mathbb{R}_{n,n}$ has all n zeros in $|z| \leqslant 1$, then, for $|z| = 1$, we have:*

$$|r'(z)| \geqslant \frac{1}{2}|B'(z)||r(z)|. \tag{3}$$

The result is sharp and the equality holds for $r(z) = aB(z) + b$, with $|a| = |b| = 1$.

In this paper, we relax the condition that all zeros of a rational function $r(z)$ lie in $|z| \leqslant 1$ and prove the following results, more general than those proved by Wali and Shah [15].

2. Main results.

Theorem 3. *If $r \in \mathbb{R}_{m,n}$ has a zero of order s at z_0 with $|z_0| > 1$, and the remaining $m - s$ zeros are in $|z| \leqslant 1$, then, for any $|z| = 1$:*

$$\max_{|z|=1}|r'(z)| \geqslant \frac{1}{2}\Big\{\Big(\frac{1-|z_0|}{1+|z_0|}\Big)^s \times \\ \times\Big(|B'(z)| - (n-m+s) + \frac{|c_{m-s}| - |c_0|}{|c_{m-s}| + |c_0|}\Big) - \frac{2s}{1+|z_0|}\Big\}\max_{|z|=1}|r(z)|. \tag{4}$$

For $s = 0$, (4) reduces to a result by Wali and Shah [15].

Corollary 1. *If we assume $r(z)$ has a pole of order n at $z = \alpha$, then $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$, where $p(z)$ is a polynomial of degree m. Then*

$$r'(z) = \Big(\frac{p(z)}{(z-\alpha)^n}\Big)' = -\frac{(n-m)p(z) + D_\alpha p(z)}{(z-\alpha)^{n+1}},$$

where $D_\alpha p(z) = mp(z) + (\alpha - z)p'(z)$ is the polar derivative of $p(z)$ with respect to the pole α.

Since $B(z) = \frac{w^*(z)}{w(z)} = \Big(\frac{1-\overline{\alpha}z}{z-\alpha}\Big)^n$, therefore, $B'(z) = \frac{n(|\alpha|^2-1)}{(z-\alpha)^2}\Big(\frac{1-\overline{\alpha}z}{z-\alpha}\Big)^{n-1}$.

*Also for $|z| = 1, |B'(z)| = \frac{n(|\alpha|^2-1)}{|z-\alpha|^2}$. Now for $|z| = 1$ and $m = n$, from Theorem **??** we get:*

$$\Big|\frac{D_\alpha p(z)}{(z-\alpha)^{n+1}}\Big| \geqslant \frac{1}{2}\Big\{\Big(\frac{1-|z_0|}{1+|z_0|}\Big)^s\Big(\frac{n(|\alpha|^2-1)}{|z-\alpha|^2} - s + \frac{|c_{n-s}| - |c_0|}{|c_{n-s}| + |c_0|}\Big) -$$

$$-\frac{2s}{1+|z_0|}\Big\}\Big|\frac{p(z)}{(z-\alpha)^n}\Big|. \tag{5}$$

Now, letting $|\alpha| \to \infty$, we get, from (5):

$$\max_{|z|=1}|p'(z)| \geqslant \frac{1}{2}\Big\{\Big(\frac{1-|z_0|}{1+|z_0|}\Big)^s\Big(n-s+\frac{|c_{n-s}|-|c_0|}{|c_{n-s}|+|c_0|}\Big)- \\ -\frac{2s}{1+|z_0|}\Big\}\max_{|z|=1}|p(z)|. \tag{6}$$

Remark. *If we put $s=0$ in (6), we get the result by Dubinin [5].*

Next, we obtain the following generalization of Theorem 3:

Theorem 4. *If $r \in \mathbb{R}_{m,n}$ has zeros at z_0 and z_1 of order s and d with $|z_0| > 1, |z_1| > 1$, and remaining $m-s-d$ zeros lie in $|z| \leqslant 1$, then, for $|z|=1$:*

$$\max_{|z|=1}|r'(z)| \geqslant \frac{1}{2}\Big\{\Big(\frac{1-|z_0|}{1+|z_0|}\Big)^s\Big(\frac{1-|z_1|}{1+|z_1|}\Big)^d \times \\ \times\Big(|B'(z)|-(n-m+s+d)+\frac{|c_{m-s-d}|-|c_0|}{|c_{m-s-d}|+|c_0|}\Big)- \\ -\frac{2s}{1+|z_0|}-\frac{2d}{1+|z_1|}\Big\}\max_{|z|=1}|r(z)|. \tag{7}$$

For $d=0$, Theorem 4 reduces to Theorem **??**; for $s=d=0$, (7) reduces to the result by Wali and Shah [15].

Again, if we consider $r(z)$ with a pole of order n at $z=\alpha$, then $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$. Now, using the same procedure as in Corollary 1 and letting $|\alpha| \to \infty$, we get:

Corollary 2. *If $p \in \mathcal{P}_n$ has zeros z_0 and z_1 of order s and d, respectively, with $|z_0| > 1, |z_1| > 1$, and remaining $n-s-d$ zeros lie in $|z| \leqslant 1$, then, for $|z|=1$:*

$$\max_{|z|=1}|p'(z)| \geqslant \frac{1}{2}\Big\{\Big(\frac{1-|z_0|}{1+|z_0|}\Big)^s\Big(\frac{1-|z_1|}{1+|z_1|}\Big)^d\Big(n-s-d+\frac{|c_{n-s-d}|-|c_0|}{|c_{n-s-d}|+|c_0|}\Big)- \\ -\frac{2s}{1+|z_0|}-\frac{2d}{1+|z_1|}\Big\}\max_{|z|=1}|p(z)|. \tag{8}$$

Using the similar technique, we also have:

Corollary 3. *Let $r(z) = (z - z_0)^{s_0}(z - z_1)^{s_1}...(z - z_t)^{s_t}s(z) \in \mathbb{R}_{m,n}$. Let $s(z)$ be a rational function of degree $m - \sum_{i=0}^{t} s_i = l$ (say), with each $|z_i| > 1, i = 0, 1, 2, \dots, t$ and $0 \leqslant \sum_{i=0}^{t} s_i \leqslant m - 1$, and let the remaining l zeros lie in $|z| \leqslant 1$. Then, for $|z| = 1$, we have:*

$$\max_{|z|=1}|r'(z)| \geqslant \frac{1}{2}\Big\{\Big(\frac{1-|z_0|}{1+|z_0|}\Big)^{s_0}\Big(\frac{1-|z_1|}{1+|z_1|}\Big)^{s_1}\dots\Big(\frac{1-|z_t|}{1+|z_t|}\Big)^{s_t}\times$$
$$\times\Big(|B'(z)| - (n-m+l) + \frac{|c_l|-|c_0|}{|c_l|+|c_0|}\Big)-$$
$$-\frac{2s_0}{1+|z_0|} - \frac{2s_1}{1+|z_1|} - \dots - \frac{2s_t}{1+|z_t|}\Big\}\max_{|z|=1}|r(z)|.$$

For the proof of these theorems, we need following lemma.

3. Lemma and Proofs of Theorems.

Lemma 1. *Suppose $r \in \mathbb{R}_{m,n}$ has all zeros in $|z| \leqslant 1$; then, for $|z| = 1$, we have:*

$$|r'(z)| \geqslant \frac{1}{2}\Big\{|B'(z)| - (n-m) + \frac{|c_m|-|c_0|}{|c_m|+|c_0|}\Big\}|r(z)|.$$

This lemma was proved by Wali and Shah [15].

Proof of Theorem 3. Since $r(z)$ has a zero of order s at $z = z_0$, therefore, $r(z) = (z - z_0)^s t(z)$, where $t \in \mathbb{R}_{m-s,n}$. This implies

$$r'(z) = (z-z_0)^s t'(z) + s(z-z_0)^{s-1}t(z).$$

Hence,

$$|r'(z)| \geqslant |(z-z_0)^s t'(z)| - s|(z-z_0)^{s-1}t(z)|.$$

This implies

$$\max_{|z|=1}|r'(z)| \geqslant |1-|z_0||^s \max_{|z|=1}|t'(z)| - s|1+|z_0||^{s-1}\max_{|z|=1}|t(z)|. \tag{9}$$

Now, using Lemma 1 for the rational function $t(z)$, we have

$$|t'(z)| \geqslant \frac{1}{2}\Big\{|B'(z)| - (n-m+s) + \frac{|c_{m-s}|-|c_0|}{|c_{m-s}|+|c_0|}\Big\}|t(z)|.$$

Therefore, from (9) we get:

$$\max_{|z|=1}|r'(z)| \geqslant \frac{|1-|z_0||^s}{2}\Big\{|B'(z)| - (n-m+s) + \\ + \frac{|c_{m-s}|-|c_0|}{|c_{m-s}|+|c_0|}\Big\}\max_{|z|=1}|t(z)| - s|1+|z_0||^{s-1}\max_{|z|=1}|t(z)|. \quad (10)$$

Now, for $|z|=1$ we have:

$$|t(z)| = \frac{1}{|z-z_0|^s}|r(z)| \geqslant \frac{1}{|1+|z_0||^s}|r(z)|.$$

This implies

$$\max_{|z|=1}|t(z)| \geqslant \frac{1}{|1+|z_0||^s}\max_{|z|=1}|r(z)|.$$

Therefore, we get, from (10):

$$\max_{|z|=1}|r'(z)| \geqslant \frac{1}{2}\Big\{\Big(\frac{1-|z_0|}{1+|z_0|}\Big)^s\Big(|B'(z)| - (n-m+s) + \frac{|c_{m-s}|-|c_0|}{|c_{m-s}|+|c_0|}\Big) - \\ - \frac{2s}{1+|z_0|}\Big\}\max_{|z|=1}|r(z)|.$$

This completely proves Theorem 3. □

Proof of Theorem 4. Since $r(z)$ has two zeros of order s and d at z_0 and z_1, respectively, with $|z_0|>1, |z_1|>1$, and remaining $m-s-d$ zeros lie in $|z|\leqslant 1$, therefore: $r(z)=(z-z_0)^s(z-z_1)^d t(z)$, where $t \in \mathbb{R}_{m-s-d,n}$ has all zeros in $|z|\leqslant 1$; then

$$r'(z) = (z-z_0)^s(z-z_1)^d t'(z) + s(z-z_0)^{s-1}(z-z_1)^d t(z) + \\ + d(z-z_0)^s(z-z_1)^{d-1}t(z).$$

This implies

$$\max_{|z|=1}|r'(z)| \geqslant \max_{|z|=1}\big|(z-z_0)^s(z-z_1)^d t'(z)\big| - \\ - \max_{|z|=1}\big|(z-z_0)^{s-1}(z-z_1)^{d-1}\big\{s(z-z_1)+d(z-z_0)\big\}t(z)\big|.$$

Also, for $|z|=1$:

$$|1-|z_0|| \leqslant |z-z_0| \leqslant 1+|z_0|$$

and

$$|1-|z_1|| \leqslant |z-z_1| \leqslant 1+|z_1|,$$

we have then:

$$\max_{|z|=1}|r'(z)| \geqslant |1-|z_0||^s|1-|z_1||^d \max_{|z|=1}|t'(z)| - \\ - |1+|z_0||^{s-1}|1+|z_1||^{d-1}\{s|1+|z_1||+d|1+|z_0||\}\max_{|z|=1}|t(z)|. \quad (11)$$

Applying Lemma 1 to the rational function $t(z)$, we have

$$|t'(z)| \geqslant \frac{1}{2}\Big\{|B'(z)|-(n-m+s+d)+\frac{|c_{m-s-d}|-|c_0|}{|c_{m-s-d}|+|c_0|}\Big\}|t(z)|.$$

Therefore, from inequality (11) we get:

$$\max_{|z|=1}|r'(z)| \geqslant \frac{|1-|z_0||^s|1-|z_1||^d}{2}\Big\{|B'(z)|-(n-m+s+d)+ \\ +\frac{|c_{m-s-d}|-|c_0|}{|c_{m-s-d}|+|c_0|}\Big\}\max_{|z|=1}|t(z)| - \\ - |1+|z_0||^{s-1}|1+|z_1||^{d-1}\{s|1+|z_1||+d|1+|z_0||\}\max_{|z|=1}|t(z)|. \quad (12)$$

Also,

$$t(z) = \frac{r(z)}{(z-z_0)^s(z-z_1)^d}.$$

This implies:

$$\max_{|z|=1}|t(z)| = \max_{|z|=1}\left|\frac{r(z)}{(z-z_0)^s(z-z_1)^d}\right| \geqslant \frac{\max_{|z|=1}|r(z)|}{|1+|z_0||^s|1+|z_1||^d}.$$

Therefore, we get from (12):

$$\max_{|z|=1}|r'(z)| \geqslant \frac{1}{2}\Big\{\Big(\frac{1-|z_0|}{1+|z_0|}\Big)^s\Big(\frac{1-|z_1|}{1+|z_1|}\Big)^d\Big(|B'(z)|-(n-m+s+d)+ \\ +\frac{|c_{m-s-d}|-|c_0|}{|c_{m-s-d}|+|c_0|}\Big)-\frac{2s}{1+|z_0|}-\frac{2d}{1+|z_1|}\Big\}\max_{|z|=1}|r(z)|.$$

This completes the proof of Theorem 4. $\square$

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