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ON THE STATISTICAL CONVERGENCE OF NESTED SEQUENCES OF SETS

Abstract. In this paper, we show that Wijsman convergence and statistical Wijsman convergence are equivalent to each other if we choose the sequences of sets as monotone. Then, we show that every statistical Wijsman convergent monotone sequence of sets is not only Hausdorff convergent but also statistical Hausdorff convergent to the same set.

Key words: *nested sequences of sets, statistical Wijsman convergence, statistical Hausdorff convergence*

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1. Introduction and Background. There are three basic convergence types for the sequences of sets: Kuratowski convergence, Hausdorff convergence and Wijsman convergence. First, Painleve introduced the concept of convergence of sequences of sets by defining the outer and inner limits of sets. If the inner and outer limits are equal to each other, then this convergence type is known as Kuratowski convergence [7].

Second well-known convergence type for sequences of closed sets was given by Hausdorff as follows: Let (X, ρ_X) be a metric space and $Cl(X)$ denote the nonempty closed subsets of X. The *Hausdorff distance* between two sets A and B of $Cl(X)$ is defined by

$$H\left(A, B\right) = \sup_{x \in X} \left| d(x, A) - d(x, B) \right|,$$

where $d(\cdot, A)\colon X \longrightarrow [0, \infty)$ is the *distance function* defined by $d(x, A) = \inf\{\rho_X(x, y)\colon y \in A\}$ [12].

Equivalently, the Hausdorff distance is given by

$$H\left(A, B\right) = \max\left\{h(A, B), h(B, A)\right\},$$

where $h(A,B) = \sup_{a\in A} d(a,B)$ is the *Hausdorff excess* of the set A with respect to the set B [3].

It is clear that a sequence (A_n) of sets is *Hausdorff convergent* to the set A if

$$\lim_{n\to\infty} H(A_n, A) = 0.$$

In this case, we write $A_n \xrightarrow{H} A$ as $n \to \infty$ [6].

If a sequence (A_n) of sets is Hausdorff convergent to A, then the sequence $\{d(\cdot, A_n)\}_{n\in\mathbb{N}}$ of distance functions is uniform convergent to the distance function $d(\cdot, A)$. Finally, if we replace this type of convergence with the pointwise convergence, we get the Wijsman convergence as follows:

Let $A_n \subset X$ for each $n \in \mathbb{N}$; a sequence (A_n) of sets is said to be *Wijsman convergent* to the set A if

$$\lim_{n\to\infty} d(x, A_n) = d(x, A) \text{ for all } x \in X.$$

In this case, we write $A_n \xrightarrow{W} A$ as $n \to \infty$ ([12], [13]).

A sequence (A_n) of sets is said to be *monotonically increasing* if $A_n \subset A_{n+1}$ for each $n \in \mathbb{N}$ and it is said to be *monotonically decreasing* if $A_{n+1} \subset A_n$ for each $n \in \mathbb{N}$ [7].

The relations among these three types of set convergence have been investigated by several authors. In 1979, Salinetti and Wets [9] showed that every Hausdorff convergent sequence of sets is Kuratowski convergent to the same set. Then, Beer [2] examined that the Wijsman convergence implies the Kuratowski convergence for nonempty closed sets. Also, he gave the conditions for these convergences to be equivalent to each other in [2, Theorem 1]. In 2001, Apreutesei [1] observed that Wijsman convergence and Hausdorff convergence are equivalent to each other if the sequence of compact sets is monotone. In [3], the Hausdorff limit of a sequence of sets was obtained as intersection of closure of the union of terms of this sequence.

In 1951, Fast [4] introduced the concept of statistical convergence. In 2012, the theory of set convergence was generalized to the theory of statistical convergence by Nuray and Rhoades [8]. They investigated the relation between statistical Hausdorff convergence and statistical Wijsman convergence. Furthermore, Talo et al. [10] defined statistical inner and outer limits of sequences of closed sets and they compared the statistical Kuratowski convergence with the statistical Hausdorff convergence.

Recently, Ulusu and Gülle [11] examined statistical Wijsman convergence and statistical Hausdorff convergence of order α for double sequences of sets.

In this paper, we first show that Wijsman convergence and statistical Wijsman convergence are equivalent to each other if we choose the sequences of sets as monotone. Then, we show that every statistical Wijsman convergent monotone sequence of sets is not only Hausdorff convergent, but also statistical Hausdorff convergent to the same set. Finally, we characterize the statistical Hausdorff limit by using the concept of ideal. Note that we used the similar proof tecniques of [1].

2. Main Results. Before giving our results, we recall some definitions about the statistical convergence.

Let K be a subset of the set of positive integers $\mathbb{N}$ and $|\{k \leqslant n : k \in K\}|$ denote the number of elements of $K \cap [1, n]$. The *natural density* of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n \colon k \in K\}|$$

if this limit exists. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(K^c) = 1 - \delta(K)$, where $K^c := \mathbb{N} \setminus K$ [5].

Let $A, A_n \in Cl(X)$. The sequence (A_n) is said to be *statistical Hausdorff convergent* to the set A if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n \colon \max\{h(A_k, A), h(A, A_k)\} \geqslant \varepsilon\}| = 0.$$

In this case, we write $A_n \xrightarrow{st-H} A$ [8].

Let $A_n \subset X$ for each $n \in \mathbb{N}$. Then the sequence (A_n) of sets is said to be *statistical Wijsman convergent* to a set A if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : |d(x, A_k) - d(x, A)| \geqslant \varepsilon\}| = 0$$

for each $x \in X$. In this case, we write $A_n \xrightarrow{st-W} A$ [8].

Theorem 1. *Let $A, A_n \in Cl(X)$ $(n \in \mathbb{N})$.*
(i) *Let $A_1 \subset A_2 \subset \ldots \subset A_n \ldots$. Then the sequence (A_n) is Wijsman convergent to the set A if and only if it is statistical Wijsman convergent to the set A.*
(ii) *Let $A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$. Then the sequence (A_n) is Wijsman convergent to the set A if and only if it is statistical Wijsman convergent to the set A.*

Proof. The necessity parts of (i) and (ii) are provided for all sequences of sets (see [8]).

i) Assume that $A_n \xrightarrow{st-W} A$. Firstly, we show that $A_n \subset A$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and let $u \in A_n$. From $A_n \xrightarrow{st-W} A$, we have $\delta\left(K(u,\varepsilon)\right) = 1$ for each $\varepsilon > 0$, where

$$K(u,\varepsilon) := \{m \in \mathbb{N} : |d(u, A_m) - d(u, A)| < \varepsilon\}.$$

For each $\varepsilon > 0$, there exists an $m_\varepsilon \in \mathbb{N}$, which is $m_\varepsilon \in K(u,\varepsilon)$ and $m_\varepsilon \geqslant n$. Since (A_n) is monotonically increasing, we have $A_n \subset A_{m_\varepsilon}$ and $u \in A_{m_\varepsilon}$. Therefore, we get $d(u, A_{m_\varepsilon}) = 0$ and so,

$$d(u, A) = |d(u, A_{m_\varepsilon}) - d(u, A)| < \varepsilon$$

for each $\varepsilon > 0$. We get $u \in A$, since the set A is closed.

Now, let us take any $x \in X$. Hence, we can write $d(x, A) \leqslant d(x, A_n)$ and so:

$$d(x, A_n) - d(x, A) \geqslant 0 \tag{1}$$

for each $x \in X$ and each $n \in \mathbb{N}$.

We show that for each $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$, such that $d(x, A_n) - d(x, A) < \varepsilon$ for every $n \geqslant n_0$. Let $\varepsilon > 0$. Since $A_n \xrightarrow{st-W} A$, we have $\delta\left(L(x,\varepsilon)\right) = 1$, where

$$L(x,\varepsilon) := \{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| < \varepsilon\}. \tag{2}$$

Let $n_0(x,\varepsilon) := \min L(x,\varepsilon)$. By the monotonicity of the sequence (A_n), we have

$$d(x, A_n) \leqslant d(x, A_{n_0}) \tag{3}$$

for every $n \geqslant n_0$. If we combine the expressions (2) and (3), we get

$$d(x, A_n) - d(x, A) \leqslant d(x, A_{n_0}) - d(x, A) < \varepsilon \tag{4}$$

for every $n \geqslant n_0$.

By the inequalities (1) and (4), we get

$$|d(x, A_n) - d(x, A)| < \varepsilon$$

for every $n \geqslant n_0$.

Since x is arbitrary, we obtain

$$\lim_{n\to\infty} d(x, A_n) = d(x, A)$$

for every $x \in X$ and $A_n \xrightarrow{W} A$.

ii) Let us assume that $A_n \xrightarrow{st-W} A$. Fix $n \in \mathbb{N}$ and take $u \in A$. Since $A_n \xrightarrow{st-W} A$, we have $\delta\left(K(u,\varepsilon)\right) = 1$ for each $\varepsilon > 0$, where

$$K(u,\varepsilon) := \left\{m \in \mathbb{N} : |d(x, A_m) - d(x, A)| < \varepsilon\right\}.$$

Then there exists an $m_\varepsilon \in \mathbb{N}$, which is $m_\varepsilon \in K(u,\varepsilon)$ and $m_\varepsilon \geqslant n$ for each $\varepsilon > 0$. Since (A_n) is monotonically decreasing, we have $A_{m_\varepsilon} \subset A_n$ and $d\left(u, A_n\right) \leqslant d\left(u, A_{m_\varepsilon}\right)$. Moreover, since $d(u,A) = 0$, we have

$$d(u, A_{m_\varepsilon}) = |d(u, A_{m_\varepsilon}) - d(u, A)| < \varepsilon$$

for each $\varepsilon > 0$. Hence, we get $d\left(u, A_n\right) < \varepsilon$ for every $\varepsilon > 0$ and, so, $u \in A_n$ since the sets A_n are closed.

Hence, we can write $d(x, A_n) \leqslant d(x, A)$ and, so,

$$d(x, A_n) - d(x, A) \leqslant 0 \tag{5}$$

for each $x \in X$ and each $n \in \mathbb{N}$.

Let $x \in X$ and $\varepsilon > 0$. Since $A_n \xrightarrow{st-W} A$, we have $\delta\left(L(x,\varepsilon)\right) = 1$, where

$$L(x,\varepsilon) := \left\{n \in \mathbb{N}\colon\ |d(x, A_n) - d(x, A)| < \varepsilon\right\}. \tag{6}$$

Define $n_0(x,\varepsilon) := \min L(x,\varepsilon)$. Using the monotonicity of the sequence (A_n), we have

$$d(x, A_{n_0}) \leqslant d(x, A_n) \tag{7}$$

for every $n \geqslant n_0$. By the expressions (6) and (7), we get

$$-\varepsilon < d(x, A_{n_0}) - d(x, A) \leqslant d(x, A_n) - d(x, A) \tag{8}$$

for every $n \geqslant n_0$.

Using the inequalities (5) and (8), we get

$$|d(x, A_n) - d(x, A)| < \varepsilon$$

for every $n \geqslant n_0$.

Since x is arbitrary, we obtain

$$\lim_{n\to\infty} d(x, A_n) = d(x, A).$$

Consequently, we get $A_n \xrightarrow{W} A$. □

Corollary.
(i) *Let $A_1 \subset A_2 \subset \ldots \subset A_n \ldots$ $(n \in \mathbb{N})$. If there exists a compact set A, such that $A_n \xrightarrow{st-W} A$, then*

$$A_n \subset A \text{ for every } n \in \mathbb{N} \text{ and } A_n \xrightarrow{st-H} A.$$

(ii) *Let $A_1 \supset A_2 \supset \ldots \supset A_n \ldots$ $(n \in \mathbb{N})$. If there exists a closed set A, such that $A_n \xrightarrow{st-W} A$, then*

$$A \subset A_n \text{ for every } n \in \mathbb{N} \text{ and } A_n \xrightarrow{st-H} A.$$

Proof. i) Assume that $A_n \xrightarrow{st-W} A$. By Theorem 1(i), we write $A_n \xrightarrow{W} A$. From [1, Theorem 3.1.(i)], we obtain $A_n \subset A$ for every $n \in \mathbb{N}$ and $A_n \xrightarrow{H} A$. Since $A_n \xrightarrow{H} A$ implies $A_n \xrightarrow{st-H} A$, the proof is completed.

ii) By Theorem 1(ii) and [1, Theorem 3.1.(ii)], the proof is obvious. □

Remark. *Assume that the hypotheses of Corollary (i) (or (ii)) are valid. If $A_n \xrightarrow{st-W} A$, then we have $A_n \xrightarrow{H} A$.*

As can be seen from the following theorem, the hypothesis of Corollary can be weakened using the concept of natural density.

Theorem 2. *Let $A, A_n \in Cl(X)$, $n \in \mathbb{N}$ and $K = \{n_1 < n_2 < \ldots < n_k < \ldots\}$ be a subset of $\mathbb{N}$, such that $\delta(K) = 1$.*

i) *Let the subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ be monotonically increasing according to the inclusion relation, i. e., $A_{n_k} \subset A_{n_{k+1}}$ for each $k \in \mathbb{N}$. If $A_n \xrightarrow{st-W} A$ and A is compact, then*

$$A_{n_k} \subset A \text{ for every } k \in \mathbb{N} \text{ and } A_n \xrightarrow{st-H} A.$$

(ii) *Let the subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ be monotonically decreasing according to the inclusion relation. If A_{n_k}'s are compact and $A_n \xrightarrow{st-W} A$, then*

$$A \subset A_{n_k} \text{ for every } k \in \mathbb{N} \text{ and } A_n \xrightarrow{st-H} A.$$

Proof. i) Fix $n_k^* \in K$ and $u \in A_{n_k^*}$. We have $u \in A_{n_k}$ for every $n_k \in K$ with $n_k \geqslant n_k^*$; hence, we have $d(u, A_{n_k}) = 0$. Since $A_n \xrightarrow{st-W} A$, we have $\delta(L(u, \varepsilon)) = 1$ for every $\varepsilon > 0$, where

$$L(u, \varepsilon) := \{n \in \mathbb{N} : |d(u, A_n) - d(u, A)| < \varepsilon\}.$$

Then, for each $\varepsilon > 0$ there exists an $n_k \in K \cap L(u, \varepsilon)$, with $n_k \geqslant n_k^*$ such that

$$d(u, A) = |d(u, A_{n_k}) - d(u, A)| < \varepsilon.$$

Hence, we get $d(u, A) < \varepsilon$ for every $\varepsilon > 0$. So, $u \in A$ since the set A is closed. Therefore, we obtain $A_{n_k} \subset A$ for every $n_k \in K$. Then we get

$$h(A_{n_k}, A) = \sup \{d(x, A) : x \in A_{n_k}\} = 0 \tag{9}$$

for every $n_k \in K$.

We have $d(x, A_{n_{k_2}}) \leqslant d(x, A_{n_{k_1}})$ for every $k_1, k_2 \in \mathbb{N}$ with $k_2 \geqslant k_1$ and for every $x \in X$. Therefore, we have

$$d(a, A_{n_{k_2}}) \leqslant d(a, A_{n_{k_1}}) \text{ for every } a \in A$$

and for every $k_1, k_2 \in \mathbb{N}$ with $k_2 \geqslant k_1$ and, so, we get

$$h(A, A_{n_{k_2}}) \leqslant h(A, A_{n_{k_1}}).$$

Define $\alpha_k = h(A, A_{n_k})$ for each $k \in \mathbb{N}$. Since $(\alpha_k)_{k \in \mathbb{N}}$ is a decreasing sequence of positive real numbers, it is convergent. Let us call this "the limit as $\alpha \geqslant 0$". Since the function $d(., A_{n_k})$ is continuous and A is compact, for every $k \in \mathbb{N}$ there exists an $a_k \in A$, such that

$$\alpha_k = \sup \{d(a, A_{n_k}) \colon a \in A\} = d(a_k, A_{n_k}).$$

By the compactness of A, the sequence $(a_k)_{k \in \mathbb{N}}$ has a subsequence $\left(a_{k_j}\right)_{j \in \mathbb{N}}$, converging to a point $a_0 \in A$. By the triangle inequality, we get

$$0 \leqslant d(a_{k_j}, A_{n_{k_j}}) \leqslant \rho(a_{k_j}, a_0) + d(a_0, A_{n_{k_j}}).$$

Since $A_n \stackrel{st-W}{\longrightarrow} A$, we have

$$|d(a_0, A_n) - d(a_0, A)| < \frac{\varepsilon}{2}$$

for every $n \in K \cap L(a_0, \varepsilon)$. Since $d(a_0, A) = 0$, we get $|d(a_0, A_{n_{k_j}})| < \varepsilon/2$. By $\lim\limits_{j \to \infty} a_{k_j} = a_0$, there exists a $j_0 \in \mathbb{N}$, such that

$$\rho(a_{k_j}, a_0) < \frac{\varepsilon}{2} \text{ for every } j \geqslant j_0.$$

Then we get

$$0 \leqslant d(a_{k_j}, A_{n_{k_j}}) < \varepsilon$$

for every $n_{k_j} \in K \cap L(a_0, \varepsilon)$ with $j \geqslant j_0$. Hence, the subsequence $\left(\alpha_{k_j}\right)$ of (α_k) is convergent to 0, therefore, we obtain $\alpha = 0$. Then we have $\lim\limits_{k\to\infty} h(A, A_{n_k}) = 0$. Consequently, we get $A_n \overset{st-H}{\longrightarrow} A$.

ii) Let $u \in A$. Fix $n_k \in K$. Since $A_n \overset{st-W}{\longrightarrow} A$, we have $\delta(L(u, \varepsilon)) = 1$ for every $\varepsilon > 0$ where

$$L(u, \varepsilon) := \{n \in \mathbb{N}\colon \ |d(u, A_n) - d(u, A)| < \varepsilon\}.$$

Then, for each $\varepsilon > 0$, there exists an $n_k^\varepsilon \in K \cap L(u, \varepsilon)$ with $n_k^\varepsilon \geqslant n_k$, such that

$$d(u, A_{n_k^\varepsilon}) = \left|d(u, A_{n_k^\varepsilon}) - d(u, A)\right| < \varepsilon.$$

Also, we have $d(u, A_{n_k}) \leqslant d(u, A_{n_k^\varepsilon})$, since $n_k^\varepsilon \geqslant n_k$. Hence, we get $d(u, A_{n_k}) < \varepsilon$ for every $\varepsilon > 0$ and, so, $u \in A_{n_k}$ from the closeness of A_{n_k}. Therefore, we obtain $A \subset A_{n_k}$ for every $n_k \in K$.

Hence, we get

$$h(A, A_{n_k}) = \sup\{d(x, A_{n_k})\colon x \in A\} = 0 \tag{10}$$

for every $n_k \in K$.

Take $\beta_k = h(A_{n_k}, A)$ for each $k \in \mathbb{N}$. Since $h(A_{n_{k_2}}, A) \leqslant h(A_{n_{k_1}}, A)$ for every $k_1, k_2 \in \mathbb{N}$ with $k_2 \geqslant k_1$, the sequence $(\beta_k)_{k\in\mathbb{N}}$ is a decreasing sequence of positive real numbers and therefore it is convergent. Say its limit $\beta \geqslant 0$. Since the function $d(., A)$ is continuous and A_{n_k} are compact, for every $k \in \mathbb{N}$ there exists an $a_k \in A_{n_k}$, such that

$$\beta_k = \sup\{d(a, A)\colon a \in A_{n_k}\} = d(a_k, A).$$

Hence, we have $a_k \in A_{n_1}$ for every $k \in \mathbb{N}$ due to monotone decrease of (A_{n_k}). From the compactness of A_{n_1}, the sequence $(a_k)_{k\in\mathbb{N}}$ has a subsequence $(a_{k_j})_{j\in\mathbb{N}}$ converging to the point $a_0 \in A_{n_1}$. Since the subsequence $(A_{n_{k_j}})$ is decreasing and $A_{n_{k_j}}$ are closed, we get

$$\lim_{j\to\infty} a_{k_j} = a_0 \in A_{n_{k_j}} \text{ for every } j \in \mathbb{N}.$$

Using the triangle inequality, we have

$$0 \leqslant d(a_{k_j}, A) \leqslant \rho(a_{k_j}, a_0) + d(a_0, A)$$

for each $j \in \mathbb{N}$.

Take $\varepsilon > 0$. Since $A_n \overset{st-W}{\longrightarrow} A$, we have

$$|d(a_0, A_n) - d(a_0, A)| < \frac{\varepsilon}{2}$$

for every $n \in K \cap L\,(a_0, \varepsilon)$. Since $d(a_0, A_{n_{k_j}}) = 0$ for every $j \in \mathbb{N}$ we get $|d(a_0, A)| < \varepsilon/2$. Also, from $\lim\limits_{j\to\infty} a_{k_j} = a_0$, there is a $j_0 \in \mathbb{N}$, such that

$$\rho(a_{k_j}, a_0) < \frac{\varepsilon}{2} \text{ for every } j \geqslant j_0.$$

Then we get

$$0 \leqslant d(a_{k_j}, A) < \varepsilon$$

for every $n_{k_j} \in K \cap L\,(a_0, \varepsilon)$ with $j \geqslant j_0$. Hence, the subsequence (β_{k_j}) of (β_k) is convergent to 0, and, therefore, we obtain $\beta = 0$. Then we have $\lim\limits_{k\to\infty} h\,(A_{n_k}, A) = 0$. Consequently, we get $A_n \overset{st-H}{\longrightarrow} A$. $\Box$

Theorem 3. *Let $\mathcal{I}_\delta = \{I \subset \mathbb{N} : \delta(I) = 0\}$ be an ideal connected with statistical convergence. If $A_n \overset{st-H}{\longrightarrow} A$, then we have*
(i) $A = \bigcap\limits_{I\in\mathcal{I}_\delta} \overline{\bigcup\limits_{m\in\mathbb{N}\setminus I} A_m}$ *and*
(ii) $A = \bigcap\limits_{\varepsilon>0} \bigcup\limits_{I\in\mathcal{I}_\delta} \bigcap\limits_{m\in\mathbb{N}\setminus I} B(A_m, \varepsilon)$, *where* $B(A_m, \varepsilon) = \{x \in X : d(x, A_m) \leqslant \varepsilon\}$.

Proof. i) Define $B := \bigcap\limits_{I\in\mathcal{I}_\delta} \overline{\bigcup\limits_{m\in\mathbb{N}\setminus I} A_m}$. Firstly, we show that $A \subseteq B$. Let $x \in A$ and $I \in \mathcal{I}_\delta$. Since $A_n \overset{st-H}{\longrightarrow} A$, we have

$$I_\varepsilon := \{n \in \mathbb{N} \colon h\,(A, A_n) \geqslant \varepsilon \text{ or } h\,(A_n, A) \geqslant \varepsilon\} \in \mathcal{I}_\delta$$

for every $\varepsilon > 0$. Moreover, since $I \cup I_\varepsilon \in \mathcal{I}_\delta$, the set $\mathbb{N}\setminus(I \cup I_\varepsilon) = (\mathbb{N}\setminus I) \cap (\mathbb{N}\setminus I_\varepsilon)$ is non-empty. Hence, there exists an $m \in (\mathbb{N} \setminus I) \cap (\mathbb{N} \setminus I_\varepsilon)$ for each $\varepsilon > 0$, and we get

$$h\,(A, A_m) < \varepsilon$$

and, so,

$$d\,(x, A_m) < \varepsilon.$$

Since A_m is a closed set, there exists an $x_m \in A_m$, such that

$$\rho\,(x, x_m) = d\,(x, A_m) < \varepsilon.$$

Also, since $x_m \in \bigcup_{m\in\mathbb{N}\setminus I} A_m$, we get $x \in \overline{\bigcup_{m\in\mathbb{N}\setminus I} A_m}$. Since I is arbitrary, we get $x \in B$.

Now let us show that $B \subseteq A$. Take $x \in B = \bigcap_{I\in\mathcal{I}_\delta} \overline{\bigcup_{m\in\mathbb{N}\setminus I} A_m}$. Then we have $x \in \overline{\bigcup_{m\in\mathbb{N}\setminus I} A_m}$ for each $I \in \mathcal{I}_\delta$. Since $A_n \xrightarrow{st-H} A$, we have

$$I_\varepsilon := \{n \in \mathbb{N}\colon h(A, A_n) \geqslant \varepsilon \text{ or } h(A_n, A) \geqslant \varepsilon\} \in \mathcal{I}_\delta$$

for every $\varepsilon > 0$. Take $\varepsilon > 0$. Then we have $x \in \overline{\bigcup_{m\in\mathbb{N}\setminus I_\varepsilon} A_m}$. Hence, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \bigcup_{m\in\mathbb{N}\setminus I_\varepsilon} A_m$, such that $x_n \longrightarrow x$. In this case, there exists an $n_0(\varepsilon) \in \mathbb{N}$, such that $\rho(x_n, x) < \varepsilon$ for every $n \geqslant n_0(\varepsilon)$. We can choose an $n_1 \geqslant n_0$ and an $m_1 \in \mathbb{N} \setminus I_\varepsilon$, such that $x_{n_1} \in A_{m_1}$. Then we get

$$\begin{aligned} d(x, A) &\leqslant \rho(x, x_{n_1}) + d(x_{n_1}, A) \leqslant \\ &\leqslant \rho(x, x_{n_1}) + d(x_{n_1}, A_{m_1}) + h(A_{m_1}, A) < \\ &< \varepsilon + 0 + \varepsilon = 2\varepsilon. \end{aligned}$$

Since ε is arbitrary and A is closed, we get $x \in A$.

ii) Define $B := \bigcap_{\varepsilon>0} \bigcup_{I\in\mathcal{I}_\delta} \bigcap_{m\in\mathbb{N}\setminus I} \mathrm{B}(A_m, \varepsilon)$. Let $x \in A$. Choose $\varepsilon > 0$ arbitrarily. Since $A_n \xrightarrow{st-H} A$, we have

$$I_\varepsilon := \{n \in \mathbb{N} : h(A, A_n) \geqslant \varepsilon \text{ or } h(A_n, A) \geqslant \varepsilon\} \in \mathcal{I}_\delta.$$

We can write

$$h(A, A_m) < \varepsilon \Longrightarrow d(x, A_m) < \varepsilon \Longrightarrow x \in \mathrm{B}(A_m, \varepsilon),$$

for every $m \in \mathbb{N} \setminus I_\varepsilon$. Consequently, we obtain $x \in B$ and, therefore, $A \subseteq B$.

Now, we show that $B \subseteq A$. Let $x \in B$. Hence, for every $\varepsilon > 0$ there exists an $I_\varepsilon \in \mathcal{I}_\delta$, such that $x \in B(A_m, \varepsilon)$ for every $m \in \mathbb{N} \setminus I_\varepsilon$. Also, from $A_n \xrightarrow{st-H} A$ we have

$$J_\varepsilon := \{n \in \mathbb{N}\colon h(A, A_n) \geqslant \varepsilon \text{ or } h(A_n, A) \geqslant \varepsilon\} \in \mathcal{I}_\delta.$$

Then $I_\varepsilon \cup J_\varepsilon \in \mathcal{I}_\delta$ and there exists an $m_0 = m_0(\varepsilon) \in \mathbb{N} \setminus (I_\varepsilon \cup J_\varepsilon)$. By $x \in \mathrm{B}(A_{m_0}, \varepsilon)$, we have

$$d(x, A_{m_0}) < \varepsilon.$$

Since A_{m_0} is a closed set, there is a $y \in A_{m_0}$, such that

$$\rho(x, y) < \varepsilon. \tag{11}$$

By $h(A_n, A) \geqslant \varepsilon$, we have

$$d(y, A) < \varepsilon.$$

Since A is a closed set, there is a $z \in A$, such that

$$\rho(y, z) < \varepsilon. \tag{12}$$

Using the inequalities (11) and (12), we obtain

$$\rho(x, z) \leqslant \rho(x, y) + \rho(y, z) < 2\varepsilon.$$

Since ε is arbitrary and A is closed, we get $x \in A$. □

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