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A NEW APPROACH TO EGOROV'S THEOREM BY MEANS OF $\alpha\beta$-STATISTICAL IDEAL CONVERGENCE

Abstract. In this work, we introduce the $\alpha\beta$-statistical pointwise ideal convergence, $\alpha\beta$-statistical uniform ideal convergence, and $\alpha\beta$-equi-statistical ideal convergence for sequences of fuzzy-valued functions. With the help of some examples, we present the relationship between these convergence concepts. Moreover, we give the $\alpha\beta$-statistical ideal version of Egorov's theorem for the sequences of fuzzy valued measurable functions.

Key words: *Egorov's theorem, $\alpha\beta$-statistical pointwise ideal convergence, $\alpha\beta$-statistical uniform ideal convergence, $\alpha\beta$-statistical equi-ideal convergence*

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1. Introduction and Preliminaries. The convergence of sequences plays a crucial role in the functional analysis. In the usual convergence, almost all elements of the sequence have to belong to an arbitrarily small neighborhood of the limit, whereas in the statistical one, this condition is relaxed. Basically, statistical convergence demands the validity of the convergence condition only for the majority of elements. The hypothesis of statistical convergence was introduced by Fast [4] and Steinhaus [18], independently. A lot of development has been done in statistical convergence. Generalized statistical convergence for the double sequences was examined by Mursaleen et al. [16]. The ideal convergence is a generalization of the statistical convergence. Balcerzak et al. [3] presented different types of statistical convergence and ideal convergence for sequences of functions. Lacunary statistical convergence in measure for sequences of fuzzy-valued functions was examined by Kişi and Dündar [9]. Recently, Kişi [10] introduced lacunary statistical convergence in measure for double sequences of fuzzy-valued functions.

In summability theory, ideal convergence became a very important concept. The perception of $I-$convergence was studied by Kostyrko et al. [15]. The ideal convergence of continuous functionss was given by Jasinski and Reclaw [8]. The ideal convergence by using fuzzy numbers was investigated by Kumar and Kumar [11]. Later on, lacunary ideal convergence for fuzzy real-valued sequences was studied by Hazarika [6]. These concepts were further generalized by Hazarika [7] by using the ideal convergence. To learn more about ideal convergence,see [12], [14] and [19].
The fuzzy-set theory is a beneficial tool in explaining the situation in which there is a lack of data. Zadeh [20] established the idea of the fuzzy set theory, which was further generalized by Matloka [17]. This study of Zadeh attracted many researchers from different fields of sciences and mathematics. Altin et al. [2] examined pointwise statistical convergence for sequence of fuzzy mappings.

Essentially, motivated by the works mentioned above, we introduced $\alpha\beta$-statistically pointwise ideal convergence, $\alpha\beta$-statistically uniform ideal convergence, and $\alpha\beta$-equi-statistically ideal convergence for sequences of fuzzy-valued functions. With the help of certain examples, we present some relations between these type of convergence. Moreover, from an application point of view, we establish a new form of $\alpha\beta$-statistical ideal Egorov's theorem for sequences of fuzzy-valued measurable functions defined on $(Z, \mathcal{A}, \mu)$.

A fuzzy set is a mapping $\Omega\colon \mathbb{R} \to [0,1]$ that fulfill the following requirements:
(i) Ω is normal,

(ii) Ω is fuzzy convex,

(iii) Ω is upper semi-continuous,

(iv) supp $\Omega = cl\{u \in \mathbb{R}\colon \Omega(u) > 0\}$ is compact.

We represent by $F(\mathbb{R})$ the set of all fuzzy numbers. The set $\mathbb{R}$ is involved in $F(\mathbb{R})$ if we consider $t \in F(\mathbb{R})$ as

$$t(u) = \begin{cases} 1, & \text{if } u = t, \\ \text{supp } \Omega, & \text{if } u \nsucc t. \end{cases}$$

For $\boldsymbol{\alpha} \in (0,1]$, the $\boldsymbol{\alpha}-$cut of Ω is given by $[\Omega]_{\boldsymbol{\alpha}} = \{u \in \mathbb{R}\colon \Omega(u) \geqslant \boldsymbol{\alpha}\}$.

The Hausdorff distance between Ω and q is denoted by $D\colon F(\mathbb{R}) \times F(\mathbb{R}) \to [0,\infty]$ and is given as

$$D(\Omega, q) = \sup_{\alpha\in[0,1]} d([\Omega]_{\boldsymbol{\alpha}}, [q]_{\boldsymbol{\alpha}}) = \sup_{\alpha\in[0,1]} \max\{|\Omega^-_{\boldsymbol{\alpha}} - \Omega^+_{\boldsymbol{\alpha}}|, |q^-_{\boldsymbol{\alpha}} - q^+_{\boldsymbol{\alpha}}|\},$$

where d represents the Hausdorff metric. To learn more about fuzzy sequences, see [5] and [13]. For $T \subset \mathbb{N}$, the natural density of T is given by

$$\delta(T) = \lim_{n\to\infty} \frac{1}{s} |\{k \leqslant s \colon k \in T\}|,$$

if the limit exists; the vertical bars above denote the cardinality of the set. A sequence $z = (z_s)$ is statistically convergent to z if

$$\delta(\{s \in \mathbb{N} \colon |z_s - z| \geqslant \epsilon\}) = 0$$

for every $\epsilon > 0$.

Let Z be a non-empty set. Then a family of sets $I \subseteq 2^Z$ (the power set of Z) is said to be an ideal on Z iff
(i) $\phi \in I$,
(ii) I is additive, that is, $U, V \in I \Rightarrow U \cup V \in I$,
(iii) $U \in I, V \subseteq U \Rightarrow V \in I$.

A non-empty family of sets $F \subseteq 2^Z$ is said to be a filter on Z if and only if $\Phi \notin F$. For $U, V \in F$, we have $U \cap V \in F$, and for each $U \in F$ and $U \subseteq V$ this implies $V \in F$. An ideal $I \subseteq 2^Z$ is called non-trivial if $I \neq 2^Z$. There is a filter $F(I) = \{K \subseteq Z \colon K^c \in I\}$, for each I, where $K^c = Z - K$. A non-trivial $I \subseteq 2^Z$ is called admissible if $\{\{z\} \colon z \in Z\} \subseteq I$. A non-trivial ideal is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. A non-trivial ideal I is called translation invariant ideal if for any $U \in I$ the set $\{k+1 \colon k \in U\} \in I$.

A sequence $z = (z_k)$ is said to $I-$convergent to the number z, if $\forall \epsilon > 0$:

$$\{k \in \mathbb{N} \colon |z_k - z| \geqslant \epsilon\} \in I.$$

A function $\Psi \colon P(\mathbb{N}) \to [0, \infty)$ (where $P(\mathbb{N})$ is the power set of $\mathbb{N}$) is a submeasure on $\mathbb{N}$ if
(i) $\Psi(\phi) = 0$,
(ii) $\Psi(U) \leqslant \Psi(U \cup V) \leqslant \Psi(U + V)\ \forall\, U, V \subset \mathbb{N}$,
(iii) Ψ is lower semicontinuous if $\forall\, U \subset \mathbb{N}$, $\Psi(U) = \lim_{s\to\infty} \Psi(U \cap s)$.

Let $\|\cdot\|_\Psi \colon P(\mathbb{N}) \to [0, \infty)$ be the submeasure defined by

$$\|U\|_\Psi = \lim_{s\to\infty} \sup \Psi(U \backslash s) = \lim_{s\to\infty} \Psi(U \backslash s).$$

Consider $Exh(\Psi) = \{U \subseteq \mathbb{N} \colon \|U\|_\Psi = 0\}$. Therefore, $Exh(\Psi)$ is an ideal for any submeasure Ψ.

Remark. *Let us choose $I = I_\delta = \{T \subseteq \mathbb{N} \colon \delta(T) = 0\}$, where $\delta(T)$ represents the asymptotic density of set T; then I_δ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.*

The concept of $\alpha\beta$-statistical convergence was given by Aktuğlu [1]. Assume that $\alpha(s)$ and $\beta(s)$ are two sequences of positive numbers, which fulfill the following conditions:

(i) α and β are both non-decreasing,

(ii) $\beta(s) \geqslant \alpha(s)$,

(iii) $\beta(s) - \alpha(s) \to \infty$ as $s \to \infty$.

The set of pairs α,β satisfying (i), (ii), and (iii) will be denoted Λ. For each pair $(\alpha, \beta) \in \Lambda$ and $T \subset \mathbb{N}$, we define the density $\delta^{\alpha,\beta}(T)$ as:

$$\delta^{\alpha,\beta}(T) = \lim_{s\to\infty} \frac{T \cap \mathcal{C}_s^{\alpha,\beta}}{\beta(s) - \alpha(s) + 1}, \quad (\mathcal{C}_s^{\alpha,\beta} = [\alpha(s), \beta(s)]).$$

A sequence $z = (z_n)$ is said to be $\alpha\beta$-statistically convergent to z, if for

$$\delta^{\alpha,\beta}(\{k \in \mathcal{C}_s^{\alpha,\beta} \colon |z_s - z| \geqslant \epsilon\}) = \lim_{s\to\infty} \frac{|k \in \mathcal{C}_s^{\alpha,\beta}|}{\beta(s) - \alpha(s) + 1} = 0,$$

for every $\epsilon > 0$; it is denoted by $S_{\alpha\beta}$.

Throughout the text, we denote a sequence of fuzzy-valued functions by $SFVF$ and a fuzzy-valued function by FVF.

Definition 1. *A $SFVF$ (h_ν) is $\alpha\beta$-statistically pointwise ideal convergent to FVF h on $[a, b]$, if for every $z \in [a, b]$, i. e., $\forall\ \epsilon > 0$, $\delta > 0$, $\xi > 0$, $\forall\ z \in [a, b]$*

$$\Big\{r \in \mathbb{N} : \Psi\Big(\Big\{s \in \mathbb{N} \colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta} \colon D(h_\nu(z), h(z)) \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\}\backslash r\Big) \geqslant \xi\Big\}$$

belongs to I. In other words, $\forall z \in [a, b]$, $\forall \epsilon, \delta > 0$, $\forall \xi > 0$ $\exists\ T_z \in I$ s. t.,

$$\Psi\Big(\Big\{s \in \mathbb{N} \colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta} \colon D(h_\nu(z), h(z)) \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\}\backslash r\Big) < \xi,$$

$\forall\ r \in \mathbb{N}\backslash T_z$. *We write* $h_\nu \xrightarrow{S^P_{\alpha\beta}(I)} h$ *on* $[a,b]$ *or* $S_{\alpha\beta}-$*pointwise ideal convergence.*

Definition 2. *A* $SFVF$ (h_ν) *is* $\alpha\beta$*-statistically uniform ideal convergent to* FVF h *on* $[a,b]$*, i.e.,* $\forall\ \epsilon > 0, \delta > 0, \xi > 0$*,* $\forall\ z \in [a,b]$

$$\Big\{r \in \mathbb{N} : \Psi\Big(\Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}^{\alpha,\beta}_s\colon D(h_\nu(z), h(z)) \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\}\backslash r\Big) \geqslant \xi\Big\}$$

belongs to I*. In other words,* $\forall z \in [a,b]$*,* $\forall \epsilon$*,* $\delta > 0$*,* $\forall \xi > 0\ \exists\ T \in I$ *s.t.,*

$$\Psi\Big(\Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}^{\alpha,\beta}_s\colon D(h_\nu(z), h(z)) \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\}\backslash r\Big) < \xi,$$

$\forall\ r \in \mathbb{N}\backslash T$. *We write* $h_\nu \overset{S^U_{\alpha\beta}(I)}{\rightrightarrows} h$ *on* $[a,b]$ *or* $S_{\alpha\beta}$*-uniform ideal convergence.*

Definition 3. *A* $SFVF$ (h_ν) *is* $\alpha\beta$*-equi-statistically ideal convergent to* FVF h *on* $[a,b]$*, iff for all* $\epsilon > 0, \delta > 0, \xi > 0$*, s. t.,*

$$\Big\{r \in \mathbb{N} : \Psi\Big(\Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}^{\alpha,\beta}_s\colon D(h_\nu(z), h(z)) \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\}\backslash r\Big) \geqslant \xi\Big\}$$

w.r.t., $z \in [a,b]$ *is uniformly convergent to zero function. It is denoted by* $h_\nu \twoheadrightarrow S^e_{\alpha\beta}(I)h$ *or* $S_{\alpha\beta}$*-equi ideally convergence.*

2. Relation Between different convergence concepts of sequences of fuzzy-valued functions.

Remark 1. *If* $h_\nu \overset{S^U_{\alpha\beta}(I)}{\rightrightarrows} h$*, then* $h_\nu \xrightarrow{S^P_{\alpha\beta}(I)} h$*. The converse is not necessarily true.*

Let us show this by an example.

Example 1. For any $z \in [0,1]$, let

$$h_\nu(z) = \begin{cases} \Big(\dfrac{\nu z}{1+\nu^2 z^2}\Big), & \text{for } z \in [0,1]; \\ \overline{0}, & \text{otherwise.} \end{cases}$$

Then (h_ν) is $\alpha\beta$-statistically pointwise ideal convergent to h. However, for $\nu \in C_s^{\alpha\beta}$:

$$\sup_{z\in[0,1]} D(h_\nu(z),\overline{0}) = \sup_{z\in[0,1]} \frac{\nu z}{1+\nu^2 z^2} = \sup_{z\in[0,1]} \frac{1}{\frac{1}{\nu z}+\nu z} = \frac{1}{2} \neq 0.$$

Therefore, (h_ν) is not $\alpha\beta$-statistically uniformly convergent nor $\alpha\beta$-statistically uniformly ideal convergent to $\overline{0}$ on $[0,1]$.

Corollary. *Let (h_ν) be a $SFVF$ and h be a FVF on $[a,b]$. Then*

$$h_\nu \overset{S^U_{\alpha\beta}(I)}{\rightrightarrows} h \Rightarrow h_\nu \twoheadrightarrow S^e_{\alpha\beta}(I)h \Rightarrow h_\nu \xrightarrow{S^P_{\alpha\beta}(I)} h,$$

but converse is not true in general.

To show that the converse of $h_\nu \twoheadrightarrow S^e_{\alpha\beta}(I)h \Rightarrow h_\nu \xrightarrow{S^P_{\alpha\beta}(I)} h$ is not true, let us consider an example.

Example 2. Consider the $SFVF$s (h_ν) defined by $h_\nu(z) = e^{-\nu z}$ for $z \in [0,1]$. Then $D(h_\nu(z),\overline{0}) = e^{-\nu z}$ for $z \in [0,1]$ for each $\nu \in \mathbb{N}$. Therefore, (h_ν) is $\alpha\beta$-statistically pointwise ideal convergent to $h = 0$. However, for each $r \in \mathbb{N}$, consider $r \in [\nu, 2\nu-1]$. Therefore, for all $z \in [\nu, 2\nu-1]$, we get

$$D(h_r(z),\overline{0}) = e^{-\nu z} \geqslant e^{-(2\nu-1)z} \geqslant \frac{1}{e} \geqslant \frac{1}{3} \quad \left(\text{as } z \in \left[0, \frac{1}{2\nu-1}\right]\right).$$

Thus, for all $z \in [0, \frac{1}{2\nu-1}]$, we get

$$\frac{1}{\beta(s)-\alpha(s)+1}\left|\left\{r \in [\nu, 2\nu-1] \colon D(h_r(z),\overline{0}) \geqslant \frac{1}{3}\right\}\right| \to 1(\neq 0),$$

so, $h_\nu(z)$ is not $\alpha\beta$-equi-statistically ideal convergent to FVF $\overline{0}$ on $[0,1]$.

To show that the converse of $h_\nu \overset{S^U_{\alpha\beta}(I)}{\rightrightarrows} h \Rightarrow h_\nu \twoheadrightarrow S^e_{\alpha\beta}(I)h$ is not true, let us consider an example.

Example 3. Suppose

$$h_\nu(z) = \begin{cases} \left(\dfrac{1-z^2(\nu+1)^2}{1+z^2}\right), & \text{for } z \in \left[0, \dfrac{1}{\nu+1}\right]; \\ \overline{0}, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1}{\beta(s)-\alpha(s)+1}\Big|\{\nu \in \mathcal{C}_s^{\alpha\beta} \colon D(h_\nu(z),\overline{0}) \geqslant \epsilon\}\Big| \leqslant \frac{1}{\beta(s)-\alpha(s)+1} \to 0$$

as $s \to \infty$. Then (h_ν) is $\alpha\beta$-equi-statistically ideal convergent to $h = \overline{0}$. However, for all $\nu \in \mathcal{C}_s^{\alpha\beta}$:

$$\sup_{z\in[0,1]} D(h_\nu(z),\overline{0}) = \sup_{z\in[0,1]} \frac{1-z^2(s+1)^2}{1+z^2} \neq 0,$$

for all $\nu \in \mathcal{C}_s^{\alpha\beta}$. Hence, (h_ν) is not $\alpha\beta$-statistically uniformly ideally convergence to $\overline{0}$ on $[0,1]$.

Proposition. *$S_{\alpha\beta}$-equi ideally convergence is defined for γ, such that $0 \leqslant \gamma \leqslant 1$.*

Proof. Let us prove this result by contradiction. Consider two lower semicontinuous submeasures Ψ_1 and Ψ_2, such that $Exh(\Psi_1) = Exh(\Psi_2)$ and $\forall\ \epsilon_1 > 0$, $\delta_1 > 0$, $\exists\ r \in \mathbb{N}$, $\forall\ z \in Z$,

$$\Psi_1\Big(\Big\{s \in \mathbb{N} : \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta} : D(h_\nu(z),h(z)) \geqslant \epsilon_1\Big\}\Big| \geqslant \delta_1\Big\}\backslash r\Big) < \epsilon_1,$$

$\forall\ \epsilon_2 > 0, \delta_2 > 0$, $\exists\ r \in \mathbb{N}$, $\forall\ z \in Z$,

$$\Psi_2\Big(\Big\{s \in \mathbb{N} : \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta} : D(h_\nu(z),h(z)) \geqslant \epsilon_2\Big\}\Big| \geqslant \delta_2\Big\}\backslash r\Big) < \epsilon_2.$$

For all $z \in Z$ and for $\gamma \in [0,1]$, we take

$$\rho(z) = \Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta}\colon D(h_\nu(z),h(z)) \geqslant \epsilon_2\Big\}\Big| \geqslant \delta_2\Big\} = \\ = \Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s)-\alpha(s)+1}\Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta}\colon \sup_{\gamma\in[0,1]} \max\Big\{\Big|(h_\nu(z))_\gamma^- - (h(z))_\gamma^-\Big|, \\ \Big|(h_\nu(z))_\gamma^+ - (h(z))_\gamma^+\Big|\Big\} \geqslant \epsilon_2\Big\}\Big| \geqslant \delta_2\Big\}$$

and

$$\rho_1(z) = \Big\{ s \in \mathbb{N}\colon \\ \frac{1}{\beta(s)-\alpha(s)+1} \Big| \Big\{ \nu \in \mathcal{C}_s^{\alpha,\beta}\colon \Big| (h_\nu(z))_\gamma^- - (h(z))_\gamma^- \Big| \geqslant \epsilon_2 \Big\} \Big| \geqslant \delta_2 \Big\},$$

$$\rho_2(z) = \Big\{ s \in \mathbb{N}\colon \\ \frac{1}{\beta(s)-\alpha(s)+1} \Big| \Big\{ \nu \in \mathcal{C}_s^{\alpha,\beta}\colon \Big| (h_\nu(z))_\gamma^+ - (h(z))_\gamma^+ \Big| \geqslant \epsilon_2 \Big\} \Big| \geqslant \delta_2 \Big\}.$$

Now, we take r_1 and z_1 s.t.,

$$\Psi_1(\rho_1(z_1)\backslash r_1) < \frac{1}{2}, \quad \Psi_1(\rho_2(z_1)\backslash r_1) < \frac{1}{2}$$

and

$$\Psi_2(\rho_1(z_1)\backslash r_1) \geqslant \epsilon_2, \quad \Psi_2(\rho_2(z_1)\backslash r_1) \geqslant \epsilon_2.$$

Since submeasure Ψ_2 is lower continuous, we get $r_1^{'}$ with

$$\Psi_2((\rho_1(z_1)\backslash r_1) \cap r_1^{'}) \geqslant \frac{\epsilon_2}{2},$$

$$\Psi_2((\rho_2(z_1)\backslash r_1) \cap r_1^{'}) \geqslant \frac{\epsilon_2}{2}.$$

Suppose that we have found $r_m^{'}$. Choose $r_{m+1} > r_m^{'}$ and z_{m+1}, such that

$$\Psi_1(\rho_1(z_{m+1})\backslash r_{m+1}) \quad < \quad \frac{1}{2^{m+1}}, \, \Psi_1(\rho_2(z_{m+1})\backslash r_{m+1}) < \frac{1}{2^{m+1}}$$

and

$$\Psi_2(\rho_1(z_{m+1})\backslash r_{m+1}) \quad \geqslant \quad \epsilon_2, \, \Psi_2(\rho_2(z_{m+1})\backslash r_{m+1}) \geqslant \epsilon_2.$$

From the lower continuity of Ψ_2, we have $r_{m+1}^{'}$ with

$$\Psi_2((\rho_1(z_{m+1})\backslash r_{m+1}) \cap r_{m+1}^{'}) \geqslant \frac{\epsilon_2}{2},$$

$$\Psi_2((\rho_2(z_{m+1})\backslash r_{m+1}) \cap r_{m+1}^{'}) \geqslant \frac{\epsilon_2}{2}.$$

We put $Q = \cup_{m\in\mathbb{N}}\Big((\rho_1(z_m)\backslash r_m) \cap r'_m)\Big)$, $R = \cup_{m\in\mathbb{N}}\Big((\rho_2(z_m)\backslash r_m) \cap r'_m)\Big)$. Then we get

$$\begin{aligned}\|Q\|_{\Psi_1} &= \lim_{m\to\infty} \Psi_1(Q\backslash r_m) \leqslant \\ &\leqslant \lim_{m\to\infty} \sum_{s=m}^{\infty} \Psi_1(\rho_1(z_m) \cap \{r_m, r'_m\}) \leqslant \\ &\leqslant \lim_{m\to\infty} \sum_{s=m}^{\infty} \frac{1}{2^s} = 0,\end{aligned}$$

and

$$\begin{aligned}\|R\|_{\Psi_1} &= \lim_{m\to\infty} \Psi_1(R\backslash r_m) \leqslant \\ &\leqslant \lim_{m\to\infty} \sum_{s=m}^{\infty} \Psi_1(\rho_2(z_m) \cap \{r_m, r'_m\}) \leqslant \\ &\leqslant \lim_{m\to\infty} \sum_{s=m}^{\infty} \frac{1}{2^s} = 0.\end{aligned}$$

Thus, we have $Q \in Exh(\Psi_1)$ and $R \in Exh(\Psi_1)$. On the other hand, we have

$$\|Q\|_{\Psi_2} = \lim_{m\to\infty} \Psi_2(Q\backslash r_m) \geqslant \frac{\epsilon_2}{2} > 0$$

and

$$\|R\|_{\Psi_2} = \lim_{m\to\infty} \Psi_2(R\backslash r_m) \geqslant \frac{\epsilon_2}{2} > 0,$$

which is a contradiction. □

Theorem 1. *Consider an admissible ideal I. Let (h_ν) be a $SFVF$ and h be a FVF, defined on $[a,b]$. For every $z \in [a,b]$, if $h_\nu(z) \xrightarrow{S^P_{\alpha\beta}(I)} h(z)$, then $[h_\nu(z)]_\gamma \xrightarrow{S^P_{\alpha\beta}(I)} [h(z)]_\gamma$ w.r.t γ.*

Proof. Assume that for every $z \in [a,b]$, $h_\nu(z) \xrightarrow{S^P_{\alpha\beta}(I)} h(z)$. Let $\epsilon > 0$ be given. Then, for each $z \in Z$, there exists an integer $i = i(z,\epsilon) \in \mathbb{N}$ s.t., $\forall\epsilon > 0, \delta > 0$, $\forall\xi > 0$,

$$\Psi\Big(\Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s)-\alpha(s)+1}\Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta}\colon D(h_\nu(z), h(z)) \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\}\backslash r\Big) < \xi \tag{1}$$

for all $r \geqslant i$. Thus, for any $\gamma \in [0,1]$, $\forall \epsilon, \delta, \xi > 0$,

$$\Big\{ s \in \mathbb{N} \colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big| \Big\{ \nu \in \mathcal{C}_s^{\alpha,\beta} \colon \sup_{\gamma \in [0,1]} \max \Big\{ \Big| (h_\nu(z))_\gamma^- - (h(z))_\gamma^- \Big|, \\ \Big| (h_\nu(z))_\gamma^+ - (h(z))_\gamma^+ \Big\} \Big| \geqslant \epsilon \Big\} \Big| \geqslant \delta \Big\} \backslash r \Big) < \xi \quad (2)$$

for all $r \geqslant i$.

For each $\gamma \in [0,1]$, $\forall \epsilon, \delta, \xi > 0$,

$$\Psi\Big(\Big\{ s \in \mathbb{N} \colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big| \Big\{ \nu \in \mathcal{C}_s^{\alpha,\beta} \colon \Big| (h_\nu(z))_\gamma^- - (h(z))_\gamma^- \Big| \geqslant \epsilon \Big\} \Big| \geqslant \delta \Big\} \backslash r \Big) < \xi \quad (3)$$

for all $r \geqslant i$. Also, $\forall \epsilon, \delta, \xi > 0$,

$$\Psi\Big(\Big\{ s \in \mathbb{N} \colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big| \Big\{ \nu \in \mathcal{C}_s^{\alpha,\beta} \colon \Big| (h_\nu(z))_\gamma^+ - (h(z))_\gamma^+ \Big| \geqslant \epsilon \Big\} \Big| \geqslant \delta \Big\} \backslash r \Big) < \xi \quad (4)$$

for all $r \geqslant i$. From (1), we get $\forall \epsilon, \delta, \xi > 0$,

$$\Big\{ r \in \mathbb{N} \colon \Psi\Big(\Big\{ s \in \mathbb{N} \colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big| \Big\{ \nu \in \mathcal{C}_s^{\alpha,\beta} \colon D(h_\nu(z), h(z)) \geqslant \epsilon \Big\} \Big| \geqslant \delta \Big\} \backslash r \Big) \geqslant \xi \Big\} \subset \mathbb{N} \cup \{1, 2, \ldots, i-1\}.$$

From (2), we get, for all $\forall \epsilon, \delta, \xi > 0$:

$$\Big\{ r \in \mathbb{N} \colon \Psi\Big(\Big\{ s \in \mathbb{N} \colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big| \Big\{ \nu \in \mathcal{C}_s^{\alpha,\beta} \sup_{\gamma \in [0,1]} \max \Big\{ \Big| (h_\nu(z))_\gamma^- - (h(z))_\gamma^- \Big|, \\ \Big| (h_\nu(z))_\gamma^+ - (h(z))_\gamma^+ \Big| \Big\} \geqslant \epsilon \Big\} \Big| \geqslant \delta \Big\} \backslash r \Big) \geqslant \xi \Big\} \subset \mathbb{N} \cup \{1, 2, \ldots, i-1\}.$$

for any $\gamma \in [0,1]$. Equation (3) and (4) imply

$$\Big\{r \in \mathbb{N} \colon \Psi\Big(\Big\{s \in \mathbb{N} \colon \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta} \colon \Big|(h_\nu(z))_\gamma^- - (h(z))_\gamma^-\Big| \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\} \backslash r\Big) \geqslant \xi\Big\} \subset \mathbb{N} \cup \{1,2,\ldots,i-1\}$$

and

$$\Big\{r \in \mathbb{N} \colon \Psi\Big(\Big\{s \in \mathbb{N} \colon \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta} \colon \Big|(h_\nu(z))_\gamma^+ - (h(z))_\gamma^+\Big| \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\} \backslash r\Big) \geqslant \xi\Big\} \subset \mathbb{N} \cup \{1,2,\ldots,i-1\}.$$

As I is an admissible ideal, we get $\mathbb{N} \cup \{1,2,\ldots,i-1\} \in I$ and, hence, for each $\epsilon > 0$ and for every $\xi > 0$,

$$\Big\{r \in \mathbb{N} \colon \Psi\Big(\Big\{s \in \mathbb{N} \colon \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta} \colon \Big|(h_\nu(z))_\gamma^- - (h(z))_\gamma^-\Big| \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\} \backslash r\Big) \geqslant \xi\Big\} \in I$$

and

$$\Big\{r \in \mathbb{N} \colon \Psi\Big(\Big\{s \in \mathbb{N} \colon \frac{1}{\beta(s)-\alpha(s)+1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta} \colon \Big|(h_\nu(z))_\gamma^+ - (h(z))_\gamma^+\Big| \geqslant \epsilon\Big\}\Big| \geqslant \delta\Big\} \backslash r\Big) \geqslant \xi\Big\} \in I.$$

Thus, for every $z \in [a,b]$ we get $[h_\nu(z)]_\gamma \xrightarrow{S^P_{\alpha\beta}(I)} [h(z)]_\gamma$ w.r.t. γ. $\square$

Before starting Egorov's theorem, let us choose a finite measurable set Z. Take $E \subset \mathbb{N}$ (finite), consider $[E,s] = \{G \subset \mathbb{N} \colon G \cap s = E\}$. The $[E,s]$ is a base for the Cantor set on $P(\mathbb{N})$.

3. $\alpha\beta$-statistical ideal version of Egorov's theorem for the sequences of fuzzy-valued measurable functions.

Theorem 2. *Let $(Z, \mathcal{A}, \mu)$ be a finite measure space and I be an analytic $P-$ideal. Suppose that FVF h and $SFVF$ (h_ν) are measurable and defined on a.e. on Z. Also, assume that $[h_\nu(z)]_\gamma \xrightarrow{S^P_{\alpha\beta}(I)} [h(z)]_\gamma$ almost everywhere on Z. Then, for each $\epsilon > 0$ $\exists$ $E \subset Z$, such that $\mu(Z\backslash E) < \epsilon$ and $h_{\nu\backslash E} \xrightarrow{S^e_{\alpha\beta}(I)} h_{\backslash E}$ on E.*

Proof. Consider a finite measure space $(Z, \mathcal{A}, \mu)$. Suppose that $\forall\, m \in \mathbb{N}$, h, and (h_ν) are defined everywhere on Z. For any fixed $r, \delta, \xi \in \mathbb{N}$, consider the set

$$F_{r,\delta,\xi} = \Big\{z \in Z\colon \Psi\Big(\Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta}\colon D(h_\nu(z), h(z)) \geqslant \frac{1}{\xi}\Big\}\Big| \geqslant \frac{1}{\delta}\Big\}\backslash r\Big) < \frac{1}{\xi}\Big\}.$$

To show that $F_{r,\delta,\xi}$ is measurable, we need to prove that the complement of every $F_{r,\delta,\xi}$ is measurable.

$$Z\backslash F_{r,\delta,\xi} = \Big\{z \in Z\colon \Big(\Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s) - \alpha(s) + 1} \times \\ \times \Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta}\colon D(h_\nu(z), h(z)) \geqslant \frac{1}{\xi}\Big\}\Big| \geqslant \frac{1}{\delta}\Big\}\backslash r\Big) \in \Psi^{-1}\Big(\Big(\frac{1}{\xi}, \infty\Big)\Big)\Big\}.$$

As Ψ is lower semicontinuous, $\exists$ set $[E_m, s_m]$, such that

$$Z\backslash F_{r,\delta,\xi} = \\ = \Big\{z \in Z\colon \Big(\Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s) - \alpha(s) + 1}\Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta}\colon D(h_\nu(z), h(z)) \geqslant \frac{1}{\xi}\Big\}\Big| \geqslant \\ \geqslant \frac{1}{\delta}\Big\}\backslash r\Big) \in \bigcup_{m\in\mathbb{N}}[E_m, s_m]\Big\} = \\ = \bigcup_{m\in\mathbb{N}} \bigcap_{j=r}^{i_m} \Big\{z \in Z\colon \Big(\Big\{s \in \mathbb{N}\colon \frac{1}{\beta(s) - \alpha(s) + 1}\Big|\Big\{\nu \in \mathcal{C}_s^{\alpha,\beta}\colon D(h_\nu(z), h(z)) \geqslant \\ \geqslant \frac{1}{\xi}\Big\}\Big| \geqslant \frac{1}{\delta}\Big\}\Big)\Big\}^{\chi_{E_m(j)}}.$$

Since (h_ν) and h are measurable, the right-hand side of the equation above is measurable and, so, $Z\backslash F_{r,\delta,\xi}$ is measurable. For every $\xi \in \mathbb{N}$, we have $F_{r,\delta,\xi} \subset F_{r+1,\delta,\xi}$, $Z = \bigcup\limits_{r=1}^{\infty} F_{r,\delta,\xi}$. Thus, $\mu(Z) = \lim_{s\to\infty} \mu(F_{r,\delta,\xi})$. Let $\epsilon > 0$ be given. For every $\xi \in \mathbb{N}$, assume $r(\xi) \in \mathbb{N}$ be s.t., $\mu(Z\backslash F_{r(\xi),\delta,\xi}) < \dfrac{\epsilon}{2^\xi}$. Consider $E_0 = \bigcup\limits_{\xi=1}^{\infty} \big(Z\backslash F_{r(\xi),\delta,\xi}\big)$. So, we have $\mu(E_0) \leqslant \sum\limits_{\xi=1}^{\infty} \mu\big(Z\backslash F_{r(\xi),\delta,\xi}\big) < \epsilon$. Let $E = Z\backslash E_0 = \bigcap\limits_{\xi=1}^{\infty} F_{r(\xi),\delta,\xi}$. Thus, $\mu(Z\backslash E) = \mu(E_0) < \epsilon$. So, we have, for all $\xi > 0$, $\delta > 0$, $\exists\, r(\xi) \in \mathbb{N}$, $\forall z \in E$,

$$\Psi\Big(\big\{s \in \mathbb{N}\colon \frac{1}{\beta(s)-\alpha(s)+1}\big|\big\{\nu \in \mathcal{C}_s^{\alpha,\beta}\colon D(h_\nu(z), h(z)) \geqslant \frac{1}{\xi}\big\}\big| \geqslant \frac{1}{\delta}\big\}\backslash r\Big) < \frac{1}{\xi}.$$

This proves that $h_{\nu\backslash E}$ is $\alpha\beta$-statistically equi-ideal convergent to $h_{\backslash E}$ on E. $\square$

4. Conclusion. Upon the preceding analysis, our interest is to modify the studies of Kişi [10] nd investigate $\alpha\beta$-statistical pointwise ideal convergence, $\alpha\beta$-statistical uniform ideal convergence, $\alpha\beta$-equi-statistically ideal convergence for sequences of FVFs. We demonstrate $\alpha\beta$-statistically ideal version of Egorov's theorem for sequences of fuzzy-valued measurable functions on $(Z, \mathcal{A}, \mu)$. As the future work, we are going to investigate Korovkin-type approximation theorems using $\alpha\beta$-statistically ideal convergence for double sequences.

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