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TIME-HYBRID HEAT AND WAVE EQUATIONS ON SCATTERED N-DIMENSIONAL COUPLED-JUMPING TIME SCALES

Abstract. In this paper, the exponential, hyperbolic, and trigonometric functions on n-dimensional coupled-jumping time scales (CJTS for short) are introduced. Based on this, we introduce the Laplace transform on n-dimensional CJTS and establish their related properties. Moreover, the homogeneous time-hybrid heat and wave equations are solved on scattered n-demensional CJTS using this Laplace transform.

Key words: *Coupled-jumping time scales; Multivariable calculus; Partial dynamic equation; Laplace transform.*

2020 Mathematical Subject Classification: *26E70; 35A08; 44A10*

1. Introduction and preliminaries. The notion of the time scale was introduced by Stefan Hilger in 1988. It unifies the continuous and discrete analysis on various hybrid domains (see [6], [7]). There have been many results related to different topics on time scales, such as the general theory of dynamic equations (see [3]), Sturm-Liouville eigenvalue problems (see [2]), the theory of translation closedness of time scales, and the related functions (see [10], [11]). Moreover, some new studies of various applications and theories on time scales were conducted (see [13] – [19]), such as quaternion dynamic equations (see [9]), the Lebesgue measure integral (see [4]), and partial dynamic equations and applications (see [1], [5], [8]).

In [12], the authors initiated the notion of coupled-jumping timescale space (CJTS for short) and introduced the theory of calculus and fundamental functions. Based on this theory, the initial-value problem of time-hybrid dynamic equations whose initial value is given in the time scale $\mathbb{T}_2$ and the unique solution is located in the time scale $\mathbb{T}_1$ was introduced and discussed including some important integral transforms, such as the convolution and the Laplace transform. The coupled-jumping timescale theory largely deepens and includes the Hilger theory and brings a completely new significance of dynamic equations on time scales (see [12]).

In this paper, based on the basic concepts and properties from [12], we introduce the n-dimensional coupled jump operators and basic functions on CJTS. For more details of coupled-jumping time scale theory, consult [12].

Definition 1. *[3]*

(i) A time scale $\mathbb{T}$ *is an arbitrary nonempty closed subset of the real line. Let* $\sigma\colon \mathbb{T} \to \mathbb{T}$ *be the forward jump operator with* $\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}$ *and* $\rho\colon \mathbb{T} \to \mathbb{T}$ *the backward jump operator with* $\rho(t) = \sup\{s \in \mathbb{T}\colon s < t\}$.

(ii) Let $f\colon \mathbb{T} \to \mathbb{R}$ *on* $t \in \mathbb{T}^{\kappa}$, *where* $\mathbb{T}^{\kappa} = \mathbb{T}\backslash[\rho(\sup\mathbb{T}), \sup\mathbb{T}]$ *for* $\sup\mathbb{T} < \infty$ *and* $\mathbb{T}^{\kappa} = \mathbb{T}$, *otherwise. Then, we define* $f^{\triangle}(t)$ *to be a real number (provided it exists) with the property that for every* $\epsilon > 0$, *there exists a neighborhood* U *of* t *(i. e.,* $U = (t-\delta, t+\delta) \cap \mathbb{T}$*) for some* $\delta > 0$*), such that* $\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| \leqslant \epsilon|\sigma(t) - s|$ *for all* $s \in U$. *We call* $f^{\Delta}(t)$ *the delta (or Hilger) derivative of* f *at* t.

Definition 2. *[12] Let* $\mathbb{T}_{m_1}$ *and* $\mathbb{T}_{m_2}$ *be a pair of time scales. For* $t \in \mathbb{T}_{m_1} \cup \mathbb{T}_{m_2}$, *we define the coupled-forward jump operator between* $\mathbb{T}_{m_1}$ *and* $\mathbb{T}_{m_2}$ *by* $\sigma_{\mathbb{T}_l}(t) = \inf\{s \in \mathbb{T}_l\colon \geqslant t\}$ *and define the coupled-backward jump operator between* $\mathbb{T}_{m_1}$ *and* $\mathbb{T}_{m_2}$ *by* $\rho_{\mathbb{T}_l}(t) = \sup\{s \in \mathbb{T}_l\colon \leqslant t\}$, $l \in \{m_1, m_2\}$, $m \in \{1, 2, \ldots, n\}$.

Definition 3. *[12] Let* $\mathbb{T}_{m_1}$ *and* $\mathbb{T}_{m_2}$ *be a pair of time scales. Define* $\mathbb{T}_k^{\acute{\kappa}}$, $\mathbb{T}_k^{\grave{\kappa}}$ *and* $\mathbb{T}_k^{\overline{\kappa}}$ *as follows:*

$$\mathbb{T}_k^{\acute{\kappa}} = \begin{cases} \mathbb{T}_k\backslash(\sup\mathbb{T}_j, +\infty), & \sup\mathbb{T}_j < \infty, \\ \mathbb{T}_k, & \textit{otherwise}, \end{cases}$$

$$\mathbb{T}_k^{\grave{\kappa}} = \begin{cases} \mathbb{T}_k\backslash(-\infty, \inf\mathbb{T}_j), & \inf\mathbb{T}_j < \infty, \\ \mathbb{T}_k, & \textit{otherwise}, \end{cases}$$

$\mathbb{T}_k^{\overline{\kappa}} = \mathbb{T}_k^{\acute{\kappa}} \cap \mathbb{T}_k^{\grave{\kappa}}$, *where* $k,j \in \{m_1, m_2\}$ *and* $k \neq j$, $m \in \{1, 2, \ldots, n\}$.

Definition 4. *Let $f\colon \{\mathbb{T}_{1_1}\cup\mathbb{T}_{1_2}\}\times\{\mathbb{T}_{2_1}\cup\mathbb{T}_{2_2}\}\times\ldots\times\{\mathbb{T}_{n_1}\cup\mathbb{T}_{n_2}\}\to\mathbb{R}$. Define a hybrid-composition integral of $f(\mathbf{t})$ with respect to t_m as follows:*

$$\int_a^b f(\mathbf{t}_m)\Delta_{hm}\tau=\begin{cases}\alpha_m\int_{[\sigma_{\mathbb{T}_{m_1}}(a),\,\rho_{\mathbb{T}_{m_1}}(b)]_{\mathbb{T}_{m_1}}} f(\mathbf{t}_m)\Delta_{m_1}\tau+\\ +(1-\alpha_m)\int_{[\sigma_{\mathbb{T}_{m_2}}(a),\,\rho_{\mathbb{T}_{m_2}}(b)]_{\mathbb{T}_{m_2}}} f(\mathbf{t}_m)\Delta_{m_2}\tau, & a<b,\\ -\alpha_m\int_{[\sigma_{\mathbb{T}_{m_1}}(b),\,\rho_{\mathbb{T}_{m_1}}(a)]_{\mathbb{T}_{m_1}}} f(\mathbf{t}_m)\Delta_{m_1}\tau-\\ -(1-\alpha_m)\int_{[\sigma_{\mathbb{T}_{m_2}}(b),\,\rho_{\mathbb{T}_{m_2}}(a)]_{\mathbb{T}_{m_2}}} f(\mathbf{t}_m)\Delta_{m_2}\tau, & a>b,\end{cases}$$

where $\mathbf{t}=(t_1,\ldots,t_m,\ldots,t_n)$, $a,b\in\mathbb{T}_{m_1}\cup\mathbb{T}_{m_2}$, $0\leqslant\alpha_m\leqslant1$, $\mathbf{t}_m=(t_1,\ldots,t_{m-1},\tau,t_{m+1},\ldots,t_n)$, $t_m\in\mathbb{T}_{m_1}\cup\mathbb{T}_{m_2}$, $1\leqslant m\leqslant n$.

Definition 5. *[12] Let $t,s\in\mathbb{T}_1\cup\mathbb{T}_2$, $f\colon\mathbb{T}_1\cup\mathbb{T}_2\to\mathbb{C}$, $0\leqslant\alpha\leqslant1$. Define the hybrid-composition exponential function by*

$$\bar{e}_f(t,s)=\begin{cases}\exp\Big(\alpha\int_{[\sigma_{\mathbb{T}_1}(s),\,\rho_{\mathbb{T}_1}(t)]_{\mathbb{T}_1}}\frac{\mathrm{Log}(1+\mu_1(\tau)f(\tau))}{\mu_1(\tau)}\Delta_1\tau+\\ +(1-\alpha)\int_{[\sigma_{\mathbb{T}_2}(s),\,\rho_{\mathbb{T}_2}(t)]_{\mathbb{T}_2}}\frac{\mathrm{Log}(1+\mu_2(\tau)f(\tau))}{\mu_2(\tau)}\Delta_2\tau\Big), & s<t,\\ \exp\Big(-\alpha\int_{[\sigma_{\mathbb{T}_1}(t),\,\rho_{\mathbb{T}_1}(s)]_{\mathbb{T}_1}}\frac{\mathrm{Log}(1+\mu_1(\tau)f(\tau))}{\mu_1(\tau)}\Delta_1\tau-\\ -(1-\alpha)\int_{[\sigma_{\mathbb{T}_2}(t),\,\rho_{\mathbb{T}_2}(s)]_{\mathbb{T}_2}}\frac{\mathrm{Log}(1+\mu_2(\tau)f(\tau))}{\mu_2(\tau)}\Delta_2\tau\Big), & s>t.\end{cases}$$

If $1\pm\mu_l(t)f(t)\neq0$ and $1\pm i\mu_l(t)f(t)\neq0$ for any $t\in\mathbb{T}_1^{\kappa}\cup\mathbb{T}_2^{\kappa}$, $l\in\{1,2\}$, then we define the hybrid-composition hyperbolic functions $\overline{\sinh}_f(t,s)$ and $\overline{\cosh}_f(t,s)$ by

$$\overline{\sinh}_f(t,s)=\frac{\bar{e}_f(t,s)-\bar{e}_{-f}(t,s)}{2}$$

and

$$\overline{\cosh}_f(t,s)=\frac{\bar{e}_f(t,s)+\bar{e}_{-f}(t,s)}{2};$$

we define the hybrid-composition trigonometric functions $\overline{\sin}_f(t,s)$ and $\overline{\cos}_f(t,s)$ by

$$\overline{\sin}_f(t,s)=\frac{\bar{e}_{if}(t,s)-\bar{e}_{-if}(t,s)}{2i}$$

and

$$\overline{\cos}_f(t,s)=\frac{\bar{e}_{if}(t,s)+\bar{e}_{-if}(t,s)}{2},$$

where i is the unit imaginary number.

Definition 6. $\mathbb{T}_{m_1}, \mathbb{T}_{m_2}$ *are called coupled-jumping equivalent time scales on* $\mathbb{T}^{\overline{\kappa}}_{m_1}$ *if* $\rho_{\mathbb{T}_{m_2}}(\sigma_{m_1}(t)) = \sigma_{m_2}(\rho_{\mathbb{T}_{m_2}}(t))$ *and* $\mu_{m_1}(t) = \mu_{m_2}(\rho_{\mathbb{T}_{m_2}}(t)) = \mu_{m_1}(\rho_{\mathbb{T}_{m_1}}(\rho_{\mathbb{T}_{m_2}}(t)))$ *for any* $t \in \mathbb{T}^{\overline{\kappa}}_{m_1}$. *For convenience, the coupled-jumping equivalent time scales* $\mathbb{T}_{m_1}$ *and* $\mathbb{T}_{m_2}$ *are denoted by* $\mathbb{T}_{m_1} \sim \mathbb{T}_{m_2}$.

2. Hybrid-composition Laplace transform on n-dimensional CJTS. In this section, we introduce the hybrid-composition Laplace transform and establish some properties of the hybrid-composition Laplace transform on n-dimensional CJTS. We always assume that $\sup \mathbb{T}_{m_1} = \infty$ and $\sup \mathbb{T}_{m_2} = \infty$ for all $m \in \{1, 2, \ldots, n\}$.

Definition 7. *[12] Let* $m \in \{1, 2, \ldots, n\}$ *be fixed,* $f\colon \{\mathbb{T}_{1_1} \cup \mathbb{T}_{1_2}\} \times \{\mathbb{T}_{2_1} \cup \mathbb{T}_{2_2}\} \times \ldots \times \{\mathbb{T}_{n_1} \cup \mathbb{T}_{n_2}\} \to \mathbb{R}$ *is regulated with respect to* t_m, $t_{m_0} \in \mathbb{T}^{\overline{\kappa}}_{m_1} \cup \mathbb{T}^{\overline{\kappa}}_{m_2}$. *We define the hybrid-composition Laplace transform of* f *with respect to* t_m *by*

$$\overline{\mathscr{L}}(f)(\mathbf{t}^{(m)}) = \alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_1}(\tau), t_{m_0}\big) f(\mathbf{t}_m) \Delta_{m_1}\tau + \\ + (1-\alpha_m) \int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_2}(\tau), t_{m_0}\big) f(\mathbf{t}_m) \Delta_{m_2}\tau, \quad (1)$$

where $t_m \in \mathbb{T}_{m_1} \cup \mathbb{T}_{m_2}$, $0 \leqslant \alpha_m \leqslant 1$, $z_m \in \mathbb{C}$, $1 + \mu_{m_k}(t_{m_k}) \ominus z_m \neq 0$, $\mathbf{t}^{(m)} = (t_1, \ldots, t_{m-1}, z_m, t_{m+1}, \ldots, t_n)$, $\mathbf{t}_m = (t_1, \ldots, t_{m-1}, \tau, t_{m+1}, \ldots, t_n)$ *and* $\ominus z_m = \frac{-z_m}{1+\mu_{m_k}(t_{m_k}) z_m}$ *for all* $t_{m_k} \in \mathbb{T}^{\acute{\kappa}}_{m_k}$, $k \in \{1, 2\}$ *and the improper integral* (1) *exists.*

In what follows, we establish some properties of the hybrid-composition Laplace transform on scattered n-dimensional CJTS.

Theorem 1. *Let* $f\colon \{\mathbb{T}_{1_1} \cup \mathbb{T}_{1_2}\} \times \{\mathbb{T}_{2_1} \cup \mathbb{T}_{2_2}\} \times \ldots \times \{\mathbb{T}_{n_1} \cup \mathbb{T}_{n_2}\} \to \mathbb{R}$, $m, j \in \{1, \ldots, n\}$, $m \neq j$, $t_{m_0} \in \mathbb{T}^{\overline{\kappa}}_{m_1} \cup \mathbb{T}^{\overline{\kappa}}_{m_2}$, *then*

$$\big[\overline{\mathscr{L}}(f)(\mathbf{t}^{(m)})\big]^{\Delta_{j_k}} = \alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \overline{e}^{\sigma_{m_1}}_{\ominus z_m}(\tau, t_{m_0}) f^{\Delta_{j_k}}(\mathbf{t}_m) \Delta_{m_1}\tau +$$

$$+ (1-\alpha_m) \int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \overline{e}^{\sigma_{m_2}}_{\ominus z_m}(\tau, t_{m_0}) f^{\Delta_{j_k}}(\mathbf{t}_m)\Delta_{m_2}\tau,$$

where $k = 1, 2$.

Proof. Let $t_{j_k} \in \mathbb{T}^{\overline{\kappa}}_{j_k}$. We have

$$\left[\overline{\mathscr{L}}(f)(\mathbf{t}^{(m)})\right]^{\Delta_{j_k}} = \frac{1}{\mu_{j_k}(t_{j_k})}\left[\alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}(\sigma_{m_1}(\tau), t_{m_0}) f^{\sigma_{j_k}}(\mathbf{t}_m)\Delta_{m_1}\tau - \right.$$

$$\left. - \alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_1}(\tau), t_{m_0}\big) f(\mathbf{t}_m)\Delta_{m_1}\tau\right] +$$

$$\frac{1-\alpha_m}{\mu_{j_k}(t_{j_k})} \times \left[\int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_2}(\tau), t_{m_0}\big) f^{\sigma_{j_k}}(\mathbf{t}_m)\Delta_{m_2}\tau - \right.$$

$$\left. - \int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_2}(\tau), t_{m_0}\big) f(\mathbf{t}_m)\Delta_{m_2}\tau\right] =$$

$$= \alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_1}(\tau), t_{m_0}\big) \frac{f^{\sigma_{j_k}}(\mathbf{t}_m) - f(\mathbf{t}_m)}{\mu_{j_k}(t_{j_k})}\Delta_{m_1}\tau +$$

$$+ (1-\alpha_m) \int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_2}(\tau), t_{m_0}\big) \frac{f^{\sigma_{j_k}}(\mathbf{t}_m) - f(\mathbf{t}_m)}{\mu_{j_k}(t_{j_k})}\Delta_{m_2}\tau =$$

$$= \alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_1}(\tau), t_{m_0}\big) f^{\Delta_{j_k}}(\mathbf{t}_m)\Delta_{m_1}\tau +$$

$$+ (1-\alpha_m) \int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_2}(\tau), t_{m_0}\big) f^{\Delta_{j_k}}(\mathbf{t}_m)\Delta_{m_2}\tau.$$

The proof is completed. □

Lemma 1. *[12] Let* $t, t_{m_0} \in \mathbb{T}^{\overline{\kappa}}_{m_1} \cup \mathbb{T}^{\overline{\kappa}}_{m_2}$, $t \geqslant t_{m_0}$. *If* $\mathbb{T}_{m_1} \sim \mathbb{T}_{m_2}$. *Then* $\overline{e}^{\Delta_t}_{\ominus z_m}(t, t_{m_0}) = \ominus z_m \overline{e}^{\Delta_t}_{\ominus z_m}(t, t_{m_0})$.

Remark 1. *Let* $c \in \mathbb{C}$, $t, t_{m_0} \in \mathbb{T}_{m_1}^{\overline{\kappa}} \cup \mathbb{T}_{m_2}^{\overline{\kappa}}$, $t > t_{m_0}$, $\mathbb{T}_{m_1} \sim \mathbb{T}_{m_2}$. *Then*

$$\overline{e}_c^{\Delta_t}(t, t_{m_0}) = c\,\overline{e}_c(t, t_{m_0}), \quad \overline{\cosh}_c^{\Delta_t}(t, t_{m_0}) = c\,\overline{\sinh}_c(t, t_{m_0}),$$

$$\overline{\sinh}_c^{\Delta_t}(t, t_{m_0}) = c\,\overline{\cosh}_c(t, t_{m_0}), \quad \overline{\cos}_c^{\Delta_t}(t, t_{m_0}) = -c\,\overline{\sin}_c(t, t_{m_0}),$$

$$\overline{\sin}_c^{\Delta_t}(t, t_{m_0}) = c\,\overline{\cos}_c(t, t_{m_0}).$$

Moreover,

$$\overline{e}_c^{\Delta_t^2}(t, t_{m_0}) = c^2\,\overline{e}_c(t, t_{m_0}), \quad \overline{\cosh}_c^{\Delta_t^2}(t, t_{m_0}) = c^2\,\overline{\cosh}_c(t, t_{m_0}),$$

$$\overline{\sinh}_c^{\Delta_t^2}(t, t_{m_0}) = c^2\,\overline{\sinh}_c(t, t_{m_0}), \quad \overline{\cos}_c^{\Delta_t^2}(t, t_{m_0}) = c^2\,\overline{\cos}_c(t, t_{m_0}),$$

$$\overline{\sin}_c^{\Delta_t^2}(t, t_{m_0}) = c^2\,\overline{\sin}_c(t, t_{m_0}).$$

Lemma 2. *[12] Let* $t_m, t_{m_0} \in \mathbb{T}_{m_1}^{\overline{\kappa}} \cup \mathbb{T}_{m_2}^{\overline{\kappa}}$, $t_m > t_{m_0}$, $f\colon \{\mathbb{T}_{1_1} \cup \mathbb{T}_{1_2}\}\times \times\{\mathbb{T}_{2_1}\cup\mathbb{T}_{2_2}\}\times\ldots\times\{\mathbb{T}_{n_1}\cup\mathbb{T}_{n_2}\}\to\mathbb{R}$, $\mathbb{T}_{m_1}\sim\mathbb{T}_{m_2}$ *and* $F_m(\mathbf{t})=\int\limits_{t_{m_0}}^{t_m} f(\mathbf{t}_m)\Delta_{hm}\tau$. *Then*

$$F_m^{\Delta_{t_m}}(\mathbf{t})=\alpha_m f(t_1,\ldots,\rho_{\mathbb{T}_{m_1}}(t_m),\ldots,t_n)+(1-\alpha_m)f(t_1,\ldots,\rho_{\mathbb{T}_{m_2}}(t_m),\ldots,t_n).$$

Lemma 3. *[12] Let* $f\colon \{\mathbb{T}_{1_1} \cup \mathbb{T}_{1_2}\}\times\{\mathbb{T}_{2_1}\cup\mathbb{T}_{2_2}\}\times\ldots\times\{\mathbb{T}_{n_1}\cup\mathbb{T}_{n_2}\}\to\mathbb{R}$, $1 \leqslant m \leqslant n$, $t_{m_0}, t_m \in \mathbb{T}_{m_1}^{\overline{\kappa}} \cup \mathbb{T}_{m_2}^{\overline{\kappa}}$, $F_m(\mathbf{t}) = \int\limits_{t_{m_0}}^{t_m} f(\mathbf{t}_m)\Delta_{hm}\tau$. *If* $\lim\limits_{t_m\to\infty} \overline{e}_{\ominus z_m}(t_m, t_{m_0})F_m(\mathbf{t}) = 0$, $\mathbb{T}_{m_1} \sim \mathbb{T}_{m_2}$. *Then*

$$\overline{\mathscr{L}}(F_m)(\mathbf{t}^{(m)}) = \frac{1}{z_m}\overline{\mathscr{L}}(F_m^{\Delta_{m_k}})(\mathbf{t}^{(m)})$$

for $k = 1, 2$.

Using Lemmas 1, 2, and 3, we can prove the following result:

Theorem 2. *Let* $f\colon \{\mathbb{T}_{1_1} \cup \mathbb{T}_{1_2}\}\times\{\mathbb{T}_{2_1}\cup\mathbb{T}_{2_2}\}\times\ldots\times\{\mathbb{T}_{n_1}\cup\mathbb{T}_{n_2}\}\to\mathbb{R}$, $1 \leqslant m \leqslant n$, $t_{m_0}, t_m \in \mathbb{T}_{m_1}^{\overline{\kappa}} \cup \mathbb{T}_{m_2}^{\overline{\kappa}}$, $F_m(\mathbf{t}) = \int\limits_{t_{m_0}}^{t_m} f(\mathbf{t}_m)\Delta_{hm}\tau$. *If* $\lim\limits_{t_m\to\infty} \overline{e}_{\ominus z_m}(t_m, t_{m_0})f(\mathbf{t}) = 0$ *and* $\lim\limits_{t_m\to\infty} \overline{e}_{\ominus z_m}(t_m, t_{m_0})F_m(\mathbf{t}) = 0$, $\mathbb{T}_{m_1} \sim \mathbb{T}_{m_2}$. *Then*

$$\overline{\mathscr{L}}\Big(F_m^{\Delta^2_{m_k}}\Big)(\mathbf{t}^{(m)}) = z_m^2\overline{\mathscr{L}}\big(F_m\big)(\mathbf{t}^{(m)}) - \alpha_m f\big(t_1,\dots,\sigma_{\mathbb{T}_{m_1}}(t_{m_0}),\dots,t_n\big)- \\ -(1-\alpha_m)f\big(t_1,\dots,\sigma_{\mathbb{T}_{m_2}}(t_{m_0}),\dots,t_n\big),$$

where $k = 1, 2$.

Proof. Since $\mathbb{T}_{m_1} \sim \mathbb{T}_{m_2}$, we have

$$\begin{aligned}
\overline{\mathscr{L}}\Big(F_m^{\Delta^2_{m_k}}\Big)(\mathbf{t}^{(m)}) &= \alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_1}(s),t_{m_0}\big)F_m^{\Delta^2_{m_1}}(\mathbf{t}_m)\Delta_{m_1}s+ \\
&+(1-\alpha_m)\int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_2}(s),t_{m_0}\big)F_m^{\Delta^2_{m_2}}(\mathbf{t}_m)\Delta_{m_2}s= \\
&= \alpha_m \lim_{t_m\to\infty}\overline{e}_{\ominus z_m}(t_m,t_{m_0})F_m^{\Delta_{m_1}}\big(t_1,\dots,t_m,\dots,t_n\big)- \\
&- \alpha_m\overline{e}_{\ominus z_m}\big(\sigma_{\mathbb{T}_{m_1}}(t_{m_0}),t_{m_0}\big)F_m^{\Delta_{m_1}}\big(t_1,\dots,\sigma_{\mathbb{T}_{m_1}}(t_{m_0}),\dots,t_n\big)+ \\
&+(1-\alpha_m)\lim_{t_m\to\infty}\overline{e}_{\ominus z_m}(t_m,t_{m_0})F_m^{\Delta_{m_2}}\big(t_1,\dots,t_m,\dots,t_n\big)- \\
&-(1-\alpha_m)\overline{e}_{\ominus z_m}(\sigma_{\mathbb{T}_{m_2}}(t_{m_0}),t_{m_0})F_m^{\Delta_{m_2}}(t_1,\dots,\sigma_{\mathbb{T}_{m_2}}(t_{m_0}),\dots,t_n)- \\
&- \alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \ominus z_m\overline{e}_{\ominus z_m}(s,t_{m_0})F_m^{\Delta_{m_1}}(\mathbf{t}_m)\Delta_{m_1}s- \\
&-(1-\alpha_m)\int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \ominus z_m\overline{e}_{\ominus z_m}(s,t_{m_0})F_m^{\Delta_{m_2}}(\mathbf{t}_m)\Delta_{m_2}s= \\
&= -\alpha_m F_m^{\Delta_{m_1}}\big(t_1,\dots,\sigma_{\mathbb{T}_{m_1}}(t_{m_0}),\dots,t_n\big)- \\
&-(1-\alpha_m)F_m^{\Delta_{m_2}}(t_1,\dots,\sigma_{\mathbb{T}_{m_2}}(t_{m_0}),\dots,t_n)+ \\
&+ z_m\alpha_m \int\limits_{\sigma_{\mathbb{T}_{m_1}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_1}(s),t_{m_0}\big)F_m^{\Delta_{m_1}}(\mathbf{t}_m)\Delta_{m_1}s+ \\
&+ z_m(1-\alpha_m)\int\limits_{\sigma_{\mathbb{T}_{m_2}}(t_{m_0})}^{\infty} \overline{e}_{\ominus z_m}\big(\sigma_{m_2}(s),t_{m_0}\big)F_m^{\Delta_{m_2}}(\mathbf{t}_m)\Delta_{m_2}s= \\
&= -\alpha_m F_m^{\Delta_{m_1}}\big(t_1,\dots,\sigma_{\mathbb{T}_{m_1}}(t_{m_0}),\dots,t_n\big)-
\end{aligned}$$

$$-(1-\alpha_m)F_m^{\Delta_{m_2}}(t_1,\dots,\sigma_{\mathbb{T}_{m_2}}(t_{m_0}),\dots,t_n)+z_m\overline{\mathscr{L}}\big(F_m^{\Delta_{m_k}}\big)(\mathbf{t}^{(m)})=$$
$$=-\alpha_m\big[\alpha_m f\big(t_1,\dots,\rho_{\mathbb{T}_{m_1}}(\sigma_{\mathbb{T}_{m_1}}(t_{m_0})),\dots,t_n\big)+$$
$$+(1-\alpha_m)f\big(t_1,\dots,\rho_{\mathbb{T}_{m_2}}(\sigma_{\mathbb{T}_{m_1}}(t_{m_0})),\dots,t_n\big)\big]-$$
$$-(1-\alpha_m)\big[\alpha_m f\big(t_1,\dots,\rho_{\mathbb{T}_{m_1}}(\sigma_{\mathbb{T}_{m_2}}(t_{m_0})),\dots,t_n\big)+$$
$$+(1-\alpha_m)f\big(t_1,\dots,\rho_{\mathbb{T}_{m_2}}(\sigma_{\mathbb{T}_{m_2}}(t_{m_0})),\dots,t_n\big)\big]+z_m^2\overline{\mathscr{L}}\big(F_m\big)(\mathbf{t}^{(m)})=$$
$$=-\alpha_m f\big(t_1,\dots,\sigma_{\mathbb{T}_{m_1}}(t_{m_0}),\dots,t_n\big)-(1-\alpha_m)f\big(t_1,\dots,\sigma_{\mathbb{T}_{m_2}}(t_{m_0}),\dots,t_n\big)+$$
$$+z_m^2\overline{\mathscr{L}}\big(F_m\big)(\mathbf{t}^{(m)}).$$

The proof is completed. □

3. Time-hybrid homogeneous heat and wave hybrid equations on scattered CJTS. In this section, we will solve the time-hybrid homogeneous heat and wave hybrid equations on scattered CJTS using the hybrid-composition Laplace transform.

Consider the time-hybrid homogeneous heat and wave equation on CJTS as follows:

$$F^{\Delta_{1_1}}(t_1,t_2)=c^2F^{\Delta_{2_1}^2}(t_1,t_2) \tag{2}$$

with the initial boundary-value conditions

$$\begin{cases} F^{\Delta_{1_1}}(t_1,t_2)=c^2F^{\Delta_{2_1}^2}(t_1,t_2),\\ F(t_{1_0},t_2)=0,\ \alpha F(t_1,a)+\beta F^{\Delta_{2_1}}(t_1,a)=g(t_1),\\ \gamma F\big(t_1,\sigma_{2_1}^2(b)\big)+\delta F^{\Delta_{2_1}}\big(t_1,\sigma_{2_1}(b)\big)=h(t_1),\end{cases}$$

where $c,\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $t_1\in\mathbb{T}_{1_1}^{\overline{\kappa}}$, $t_2,b\in\mathbb{T}_{2_1}^{\overline{\kappa}}$, $t_{1_0}\in\mathbb{T}_{1_1}^{\overline{\kappa}}\cup\mathbb{T}_{1_2}^{\overline{\kappa}}$, $a\in\mathbb{T}_{2_1}^{\overline{\kappa}}\cup\mathbb{T}_{2_2}^{\overline{\kappa}}$, $f\colon(\mathbb{T}_{1_1}\cup\mathbb{T}_{1_2})\times(\mathbb{T}_{2_1}\cup\mathbb{T}_{2_2})\to\mathbb{R}$, $F(t_1,t_2)=\int\limits_{t_{1_0}}^{t_1}f(\tau,t_2)\Delta_{h1}\tau$, $a<b$. Using the hybrid-composition Laplace transform, the initial boundary-value problem (2) can be rewritten as

$$\begin{cases} X^{\Delta_{2_1}^2}(z_1,t_2)=\dfrac{z_1}{c^2}X(z_1,t_2),\\ \alpha X(z_1,a)+\beta X^{\Delta_{2_1}}(z_1,a)=G(z_1),\\ \gamma X\big(z_1,\sigma_{2_1}^2(b)\big)+\delta X^{\Delta_{2_1}}\big(z_1,\sigma_{2_1}(b)\big)=H(z_1),\end{cases} \tag{3}$$

where $X(z_1,t_2)=\overline{\mathscr{L}}\big(F\big)(z_1,t_2)$, $G(z_1)=\overline{\mathscr{L}}(g)(z_1)$, $H(z_1)=\overline{\mathscr{L}}(h)(z_1)$, $z_1\in\mathbb{C}$.

Theorem 3. *If $\mathbb{T}_{m_1} \sim \mathbb{T}_{m_2}$, $m = 1, 2$. Then the solution of (3) can be given as:* $X(z_1, t_2) = c_1 \overline{e}_{\frac{\sqrt{z_1}}{c}}(t_2, a) + c_2 \overline{e}_{-\frac{\sqrt{z_1}}{c}}(t_2, a)$, *where*

$$[c_1, c_2]^T = \\ = \begin{bmatrix} \alpha + \beta\frac{\sqrt{z_1}}{c} & \alpha - \frac{\sqrt{z_1}}{c}\beta \\ \gamma\overline{e}_{\frac{\sqrt{z_1}}{c}}(\sigma_{2_1}^2(b), a) + \delta\frac{\sqrt{z_1}}{c}\overline{e}_{\frac{\sqrt{z_1}}{c}}(\sigma_{2_1}(b), a) & \gamma\overline{e}_{-\frac{\sqrt{z_1}}{c}}(\sigma_{2_1}^2(b), a) - \delta\frac{\sqrt{z_1}}{c}\overline{e}_{-\frac{\sqrt{z_1}}{c}}(\sigma_{2_1}(b), a) \end{bmatrix}^{-1} \times \\ \times \begin{bmatrix} G(z_1) \\ H(z_1) \end{bmatrix}.$$

Proof. By Remark 1, we have

$$X^{\Delta_{2_1}}(z_1, t_2) = c_1\frac{\sqrt{z_1}}{c}\overline{e}_{\frac{\sqrt{z_1}}{c}}(t_2, a) - \frac{\sqrt{z_1}}{c}c_2\overline{e}_{-\frac{\sqrt{z_1}}{c}}(t_2, a),$$

$$X^{\Delta_{2_1}^2}(z_1, t_2) = c_1\frac{z_1}{c^2}\overline{e}_{\frac{\sqrt{z_1}}{c}}(t_2, a) + \frac{z_1}{c^2}c_2\overline{e}_{-\frac{\sqrt{z_1}}{c}}(t_2, a) = \frac{z_1}{c^2}X(z_1, t_2).$$

Hence, $X(z_1, t_2) = c_1\overline{e}_{\frac{\sqrt{z_1}}{c}}(t_2, a) + c_2\overline{e}_{-\frac{\sqrt{z_1}}{c}}(t_2, a)$ is a solution of (3). Moreover, through using the initial boundary-value conditions of (3), we have

$$\alpha\Big[c_1\overline{e}_{\frac{\sqrt{z_1}}{c}}(a, a) + c_2\overline{e}_{-\frac{\sqrt{z_1}}{c}}(a, a)\Big] + \beta\Big[c_1\frac{\sqrt{z_1}}{c}\overline{e}_{\frac{\sqrt{z_1}}{c}}(a, a) - \frac{\sqrt{z_1}}{c}c_2\overline{e}_{-\frac{\sqrt{z_1}}{c}}(a, a)\Big] = G(z_1),$$

i. e.,

$$\alpha(c_1 + c_2) + \beta\Big(c_1\frac{\sqrt{z_1}}{c} - \frac{\sqrt{z_1}}{c}c_2\Big) = G(z_1);$$

$$\gamma\Big[c_1\overline{e}_{\frac{\sqrt{z_1}}{c}}(\sigma_{2_1}^2(b), a) + c_2\overline{e}_{-\frac{\sqrt{z_1}}{c}}(\sigma_{2_1}^2(b), a)\Big] + \\ + \delta\Big[c_1\frac{\sqrt{z_1}}{c}\overline{e}_{\frac{\sqrt{z_1}}{c}}(\sigma_{2_1}(b), a) - \frac{\sqrt{z_1}}{c}c_2\overline{e}_{-\frac{\sqrt{z_1}}{c}}(\sigma_{2_1}(b), a)\Big] = H(z_1).$$

The proof is completed. □

Example. Let $\mathbb{T}_{1_1} = \mathbb{Z}$, $\mathbb{T}_{1_2} = \{n + \frac{1}{2}\colon \in \mathbb{Z}\}$ and $\mathbb{T}_{2_1} = \{2n\colon \in \mathbb{N}\}$, $t_1 > t_{1_0} = 0$. Then the solution of (2) satisfies the equation

$$\alpha_1 f(t_1, t_2) + (1 - \alpha_1)f\big(t_1 - \frac{1}{2}, t_2\big) =$$

$$= \frac{\alpha_1}{4}\sum_{k=0}^{t_1-1}\big[f(k,t_2+4)-2f(k,t_2+2)+f(k,t_2)\big]+$$

$$+\frac{1-\alpha_1}{4}\sum_{k=1}^{t_1-1}\Big[f\Big(k-\frac{1}{2},t_2+4\Big)-2f\Big(k-\frac{1}{2},t_2+2\Big)+f\Big(k-\frac{1}{2},t_2\Big)\Big],$$

where $\alpha_1 \in [0,1]$. In fact,

$$F^{\Delta_{1_1}}(t_1,t_2)=\frac{\alpha_1\int\limits_{\sigma_{\mathbb{T}_{1_1}}(t_{1_0})}^{\sigma_{1_1}(t_1)} f(\tau,t_2)\Delta_{1_1}\tau-\alpha_1\int\limits_{\sigma_{\mathbb{T}_{1_1}}(t_{1_0})}^{t_1} f(\tau,t_2)\Delta_{1_1}\tau}{\mu_{1_1}(t_1)}+$$

$$+\frac{(1-\alpha_1)\int\limits_{\sigma_{\mathbb{T}_{1_2}}(t_{1_0})}^{\rho_{\mathbb{T}_{1_2}}(\sigma_{1_1}(t_1))} f(\tau,t_2)\Delta_{1_2}\tau-(1-\alpha_1)\int\limits_{\sigma_{\mathbb{T}_{1_2}}(t_{1_0})}^{\rho_{\mathbb{T}_{1_2}}(t_{1_0})} f(\tau,t_2)\Delta_{1_2}\tau}{\mu_{1_1}(t_1)}=$$

$$=\alpha_1\int\limits_{t_1}^{t_1+1} f(\tau,t_2)\Delta_{1_1}\tau+(1-\alpha_1)\int\limits_{t_1-\frac{1}{2}}^{t_1+\frac{1}{2}} f(\tau,t_2)\Delta_{1_2}\tau=$$

$$=\alpha_1 f(t_1,t_2)+(1-\alpha_1)f\Big(t_1-\frac{1}{2},t_2\Big),$$

$$F^{\Delta^2_{2_1}}(t_1,t_2)=\frac{\frac{F(t_1,\sigma^2_{2_1}(t_2))-F(t_1,\sigma_{2_1}(t_2))}{\mu_{2_1}(t_2)}-\frac{F(t_1,\sigma_{2_1}(t_2))-F(t_1,t_2)}{\mu_{2_1}(t_2)}}{\mu_{2_1}(t_2)}=$$

$$=\frac{F(t_1,t_2+4)-2F(t_1,t_2+2)+F(t_1,t_2)}{4}=$$

$$=\frac{1}{4}\Bigg(\alpha_1\int\limits_0^{t_1} f(\tau,t_2+4)\Delta_{1_1}\tau+(1-\alpha_1)\int\limits_{\frac{1}{2}}^{t_1-\frac{1}{2}} f(\tau,t_2+4)\Delta_{1_2}\tau-$$

$$-2\alpha_1\int\limits_0^{t_1} f(\tau,t_2+2)\Delta_{1_1}\tau-2(1-\alpha_1)\int\limits_{\frac{1}{2}}^{t_1-\frac{1}{2}} f(\tau,t_2+2)\Delta_{1_2}\tau+$$

$$+\alpha_1\int_0^{t_1} f(\tau,t_2)\Delta_{1_1}\tau+(1-\alpha_1)\int\limits_{\frac{1}{2}}^{t_1-\frac{1}{2}} f(\tau,t_2)\Delta_{1_2}\tau\Bigg)=$$

$$= \frac{\alpha_1}{4}\sum_{k=0}^{t_1-1}\big[f(k,t_2+4)-2f(k,t_2+2)+f(k,t_2)\big]+$$
$$+\frac{1-\alpha_1}{4}\sum_{k=1}^{t_1-1}\Big[f\big(k-\frac{1}{2},t_2+4\big)-2f\big(k-\frac{1}{2},t_2+2\big)+f\big(k-\frac{1}{2},t_2\big)\Big].$$

Now consider the time-hybrid homogeneous heat and wave equation on CJTS as follows:

$$F^{\Delta^2_{1_1}}(t_1,t_2)=c^2F^{\Delta^2_2}(t_1,t_2), \tag{4}$$

with the initial boundary-value conditions

$$\begin{cases} F^{\Delta^2_{1_1}}(t_1,t_2)=c^2F^{\Delta^2_2}(t_1,t_2),\\ F(t_{1_0},t_2)=0,\\ F^{\Delta_{1_1}}(t_{1_0},t_2)=\alpha_1 f\big(\sigma_{\mathbb{T}_{1_1}}(t_{1_0}),t_2\big)+(1-\alpha_1)f\big(\sigma_{\mathbb{T}_{1_2}}(t_{1_0}),t_2\big)=r(t_2),\\ \alpha F(t_1,a)+\beta F^{\Delta_2}(t_1,a)=g(t_1),\\ \gamma F\big(t_1,\sigma_2^2(b)\big)+\delta F^{\Delta_2}\big(t_1,\sigma_2(b)\big)=h(t_1), \end{cases}$$

where $c,\alpha,\beta,\gamma,\delta\in\mathbb{C}$, $\alpha_1\in[0,1]$, $t_1\in\mathbb{T}^{\kappa}_{1_1}$, $t_2,a,b\in\mathbb{T}^{\kappa}_2$, $t_{1_0}\in\mathbb{T}^{\kappa}_{1_1}\cup\mathbb{T}^{\kappa}_{1_2}$, $r\colon\mathbb{T}_2\to\mathbb{R}$, $f\colon(\mathbb{T}_{1_1}\cup\mathbb{T}_{1_2})\times\mathbb{T}_2\to\mathbb{R}$, $F(t_1,t_2)=\int_{t_{1_0}}^{t_1}f(\tau,t_2)\Delta_{h1}\tau$, $a<b$. Through using the hybrid-composition Laplace transform, the initial boundary-value problem (4) can be rewritten as

$$\begin{cases} X^{\Delta^2_2}(z_1,t_2)=\frac{z_1^2}{c^2}X(z_1,t_2)-\frac{1}{c^2}r(t_2),\\ \alpha X(z_1,a)+\beta X^{\Delta_2}(z_1,a)=G(z_1),\\ \gamma X\big(z_1,\sigma_2^2(b)\big)+\delta X^{\Delta_2}\big(z_1,\sigma_2(b)\big)=H(z_1), \end{cases} \tag{5}$$

where $X(z_1,t_2)=\overline{\mathscr{L}}(F)(z_1,t_2)$, $G(z_1)=\overline{\mathscr{L}}(g)(z_1)$, $H(z_1)=\overline{\mathscr{L}}(h)(z_1)$, $z_1\in\mathbb{C}$.

Lemma 4. *Let*

$$R(z_1,t_2,a):=$$
$$:=\frac{1}{2cz_1}\int_a^{t_2}\frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(t_2,a)-\overline{e}_{\frac{z_1}{c}}(t_2,a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}r(\tau)\Delta_2\tau,$$

where $\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a) = e^{\int\limits_a^{\sigma_2(\tau)} \frac{\mathrm{Log}(1+\mu_2(t)\frac{z_1}{c})}{\mu_2(t)}\Delta_2 t}$. Then

$$R^{\Delta_2^2}(z_1,t_2,a) = \frac{z_1^2}{c^2}R(z_1,t_2,a) - \frac{1}{c^2}r(t_2).$$

Proof. Let $R(z_1,t_2,a) := H(t_2)$. We have

$$H^{\Delta_2}(t_2) = \frac{1}{\mu_2(t_2)}\times$$

$$\times\Bigg[\frac{1}{2cz_1}\int\limits_a^{\sigma_2(t_2)} \frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(t_2),a) - \overline{e}_{\frac{z_1}{c}}(\sigma_2(t_2),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau -$$

$$-\frac{1}{2cz_1}\int\limits_a^{t_2} \frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(t_2,a) - \overline{e}_{\frac{z_1}{c}}(t_2,a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau\Bigg] =$$

$$= \frac{1}{\mu_2(t_2)}\times$$

$$\times\Bigg[\frac{1}{2cz_1}\int\limits_{t_2}^{\sigma_2(t_2)} \frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(t_2),a) - \overline{e}_{\frac{z_1}{c}}(\sigma_2(t_2),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau +$$

$$+\frac{1}{2cz_1}\int\limits_a^{t_2} \frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(t_2),a) - \overline{e}_{\frac{z_1}{c}}(\sigma_2(t_2),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau -$$

$$-\frac{1}{2cz_1}\int\limits_a^{t_2} \frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(t_2,a) - \overline{e}_{\frac{z_1}{c}}(t_2,a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau\Bigg] =$$

$$= \frac{1}{2cz_1}\frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(t_2),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(t_2),a) - \overline{e}_{\frac{z_1}{c}}(\sigma_2(t_2),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(t_2),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(t_2),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(t_2),a)} r(t_2) +$$

$$+\frac{1}{2cz_1}\int\limits_a^{t_2} \frac{-\frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(t_2,a) - \frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(t_2,a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau =$$

$$= \frac{1}{2cz_1}\int\limits_a^{t_2} \frac{-\frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(t_2,a) - \frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(t_2,a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau.$$

Similarly, $X^{\Delta_2^2}(t_2) = \dfrac{1}{2cz_1}\dfrac{-2z_1}{c}r(t_2) + \dfrac{z_1^2}{c^2}X(t_2) = \dfrac{z_1^2}{c^2}X(t_2) - \dfrac{1}{c^2}r(t_2)$.

The proof is completed. □

Theorem 4. *If $\mathbb{T}_{1_1} \sim \mathbb{T}_{1_2}$, then the solution of (5) can be given as:*

$$X(z_1,t_2) = c_1\overline{e}_{\frac{z_1}{c}}(t_2,a) + c_2\overline{e}_{-\frac{z_1}{c}}(t_2,a) + $$
$$+\frac{1}{2cz_1}\int_a^{t_2} \frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(t_2,a) - \overline{e}_{\frac{z_1}{c}}(t_2,a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau. \quad (6)$$

where

$$A(z_1) = H(z_1) - \gamma R(z_1,\sigma_2^2(b),a) - \delta\frac{1}{2cz_1}\times$$
$$\times\int_a^{\sigma_2(b)}\left[\frac{-\frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(b),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} - \frac{\frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(\sigma_2(b),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}\right] r(\tau)\Delta_2\tau,$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} =$$
$$=\begin{bmatrix} \alpha+\beta\frac{z_1}{c} & \alpha-\beta\frac{z_1}{c} \\ \gamma\overline{e}_{\frac{z_1}{c}}(\sigma_2^2(b),a)+\delta\frac{z_1}{c}c_1\overline{e}_{\frac{z_1}{c}}(\sigma_2(b),a) & \gamma\overline{e}_{-\frac{z_1}{c}}(\sigma_2^2(b),a)-\frac{z_1}{c}\delta\overline{e}_{-\frac{z_1}{c}}(\sigma_2(b),a) \end{bmatrix}^{-1}\times$$
$$\times\begin{bmatrix} G(z_1) \\ A(z_1) \end{bmatrix}.$$

Proof. From Lemma 4, it follows that (6) is a solution of (5). Since

$$X^{\Delta_2}(z_1,t_2) = \frac{z_1}{c}c_1\overline{e}_{\frac{z_1}{c}}(t_2,a) - \frac{z_1}{c}c_2\overline{e}_{-\frac{z_1}{c}}(t_2,a) + $$
$$+\frac{1}{2cz_1}\int_a^{t_2} \frac{-\frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(t_2,a) - \frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(t_2,a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)} r(\tau)\Delta_2\tau,$$

for the initial boundary-conditions of (5), we have

$$X(z_1,a) = c_1 + c_2,\, X^{\Delta_2}(z_1,a) = \frac{z_1}{c}c_1 - \frac{z_1}{c}c_2,$$

$$X\big(z_1,\sigma_2^2(b)\big) = c_1\overline{e}_{\frac{z_1}{c}}\big(\sigma_2^2(b),a\big) + c_2\overline{e}_{-\frac{z_1}{c}}\big(\sigma_2^2(b),a\big) +$$

$$+\frac{1}{2cz_1}\int\limits_a^{\sigma_2^2(b)}\frac{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2^2(b),a)-\overline{e}_{\frac{z_1}{c}}(\sigma_2^2(b),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}r(\tau)\Delta_2\tau,$$

$$X^{\Delta_2}\big(z_1,\sigma_2(b)\big)=\frac{z_1}{c}c_1\times\overline{e}_{\frac{z_1}{c}}\big(\sigma_2(b),a\big)-\frac{z_1}{c}c_2\overline{e}_{-\frac{z_1}{c}}\big(\sigma_2(b),a\big)+$$

$$+\frac{1}{2cz_1}\int\limits_a^{\sigma_2(b)}\frac{-\frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(b),a)-\frac{z_1}{c}\overline{e}_{\frac{z_1}{c}}(\sigma_2(b),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}{\overline{e}_{\frac{z_1}{c}}(\sigma_2(\tau),a)\overline{e}_{-\frac{z_1}{c}}(\sigma_2(\tau),a)}r(\tau)\Delta_2\tau.$$

Hence,

$$\alpha(c_1+c_2)+\beta\Big(\frac{z_1}{c}c_1-\frac{z_1}{c}c_2\Big)=G(z_1),$$

$$\gamma\big[c_1\overline{e}_{\frac{z_1}{c}}\big(\sigma_2^2(b),a\big)+c_2\overline{e}_{-\frac{z_1}{c}}\big(\sigma_2^2(b),a\big)\big]+ \\ +\delta\Big[\frac{z_1}{c}c_1\overline{e}_{\frac{z_1}{c}}\big(\sigma_2(b),a\big)-\frac{z_1}{c}c_2\overline{e}_{-\frac{z_1}{c}}\big(\sigma_2(b),a\big)\Big]=A(z_1),$$

i. e.,

$$\begin{bmatrix}\alpha+\beta\frac{z_1}{c} & \alpha-\beta\frac{z_1}{c}\\ \gamma\overline{e}_{\frac{z_1}{c}}(\sigma_2^2(b),a)+\delta\frac{z_1}{c}c_1\overline{e}_{\frac{z_1}{c}}(\sigma_2(b),a) & \gamma\overline{e}_{-\frac{z_1}{c}}(\sigma_2^2(b),a)-\frac{z_1}{c}\delta\overline{e}_{-\frac{z_1}{c}}(\sigma_2(b),a)\end{bmatrix}\times \\ \times\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}G(z_1)\\ A(z_1)\end{bmatrix}.$$

The proof is completed. □

Remark. *Let* $\lambda\in\mathbb{C}$, $\mathbb{T}_{m_1}\sim\mathbb{T}_{m_2}$, $t_m\in\mathbb{T}_{m_1}^{\kappa}$, $t_{m_0}\in\mathbb{T}_{m_1}^{\kappa}\cup\mathbb{T}_{m_2}^{\kappa}$, $t_m>t_{m_0}$ *where* $m\in\{1,2\}$,

$$s_\lambda(\cdot)\in\{\overline{e}_{c\lambda}(t_1,t_{1_0}),\overline{\sinh}_{c\lambda}(t_1,t_{1_0}),\overline{\cosh}_{c\lambda}(t_1,t_{1_0}),\overline{\sin}_{c\lambda}(t_1,t_{1_0}),\overline{\cos}_{c\lambda}(t_1,t_{1_0})\},$$

$$l_\lambda(\cdot)\in\{\overline{e}_{\lambda}(t_2,t_{2_0}),\overline{\sinh}_{\lambda}(t_2,t_{2_0}), \\ \overline{\cosh}_{\lambda}(t_2,t_{2_0}),\overline{\sin}_{\lambda}(t_2,t_{2_0}),\overline{\cos}_{\lambda}(t_2,t_{2_0})\}.$$

Then

$$F(t_1,t_2)=s_\lambda(t_1)l_\lambda(t_2)$$

are the solutions of (4).

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