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ON $\mathcal{I}^{\mathcal{K}}$-SUPREMUM, $\mathcal{I}^{\mathcal{K}}$-INFIMUM AND RELATED RESULTS

Abstract. In this paper, we introduce the concept of $\mathcal{I}^{\mathcal{K}}$-supremum, $\mathcal{I}^{\mathcal{K}}$-infimum, $\mathcal{I}^{\mathcal{K}}-$limit superior, and $\mathcal{I}^{\mathcal{K}}-$limit inferior and investigate a few implication relationships between them.

Key words: *ideal, filter, $\mathcal{I}^{\mathcal{K}}$-supremum, $\mathcal{I}^{\mathcal{K}}$-infimum, $\mathcal{I}^{\mathcal{K}}$-limit superior, $\mathcal{I}^{\mathcal{K}}$-limit inferior*

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1. Introduction and background. In 1951, Fast [9] and Steinhaus [19] introduced the concept of statistical convergence via the natural density. The natural density of a set $A \subseteq \mathbb{N}$ is defined as

$$d(A) = \lim_{k\to\infty} \frac{card(A \cap \{1, 2, \dots, k\})}{k},$$

provided that the limit exists. A sequence $x = (x_k)$ is called statistically convergent to a real number x_0, if for any $\varepsilon > 0$, the set $\{k \in \mathbb{N}\colon x_k \notin (x_0 - \varepsilon, x_0 + \varepsilon)\}$ has zero natural density. Now, since every finite subsets of $\mathbb{N}$ have zero natural density, statistical convergence has appeared to be one of the generalizations of the usual convergence. Apart from Fast [9] and Steinhaus [19], a lot of investigation and generalizations in this direction has been carried out by Fridy [10], [11], Šalát [17], Tripathy [20] and many others [1], [3], [14].

In an attempt to extend the notion of statistical convergence, $\mathcal{I}$ and $\mathcal{I}^*-$convergence of sequences was introduced in 2001 by Kostyrko et. al. [13] in the metric space setting, where $\mathcal{I}$ represents an ideal in $\mathbb{N}$. A sequence $x = (x_k)$ is called $\mathcal{I}$-convergent to a real number x_0 if for any $\varepsilon > 0$, the set $\{k \in \mathbb{N}\colon x_k \notin (x_0 - \varepsilon, x_0 + \varepsilon)\} \in \mathcal{I}$. Interestingly, $\mathcal{I}$-convergence was appeared not only as a generalization of statistical convergence, but, also, some other important known notions of convergences, such as logarithmic statistical convergence, uniform statistical convergence etc., turned out to be the particular cases of $\mathcal{I}$-convergence. For more details on $\mathcal{I}$-convergence and its several generalizations, [8], [12], [16], [18] can be addressed, where many more references can be found.

On the other hand, the notion of $\mathcal{I}^*$-convergence was further extended in 2011 to $\mathcal{I}^{\mathcal{K}}$-convergence by M. Macaj and M. Sleziak [15]. It should be mentioned that $\mathcal{I}^*$-convergence of a sequence $x = (x_k)$ was defined in terms of usual convergence of the subsequence (x_{m_k}), where $M = \{m_1 < m_2 < \ldots < m_k < \cdots\}$ is an element of the associated filter $\mathcal{F}(\mathcal{I})$. But in the case of $\mathcal{I}^{\mathcal{K}}$-convergence, that usual convergence was replaced by $\mathcal{K}$-convergence, where $\mathcal{K}$ is another ideal. The involvement of two ideals at the same point of time makes this concept more complicated and more interesting. Over the last few years, the study of $\mathcal{I}^{\mathcal{K}}$-convergence of sequences has got much attention from researchers and the research carried out so far shows a strong analogy in the behavior of $\mathcal{I}^{\mathcal{K}}$-convergence of sequences. The relation between $\mathcal{I}$ and $\mathcal{I}^{\mathcal{K}}$-convergence can be found in works by Das et. al. [4] and Macaj and Sleziak [15]. In [5], [6], Das et. al. introduced and investigated $\mathcal{I}^{\mathcal{K}}$-convergence of sequence of function and $\mathcal{I}^{\mathcal{K}}$-Cauchy functions. For more details on $\mathcal{I}^{\mathcal{K}}$-convergence, see [7], where many more references can be found.

When studying some new notion of convergence of sequences, several closely related concepts occur quite naturally, such as cluster points, supremum, infimum, limit superior, limit inferior, etc. In this paper, our aim is to introduce $\mathcal{I}^{\mathcal{K}}$-analogue of the above concepts and investigate some fundamental properties.

2. Definitions and preliminaries.

Definition 1. *[13] Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal in X if*

(i) for every A,$B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$;

(ii) for every $A \in \mathcal{I}$ and $B \subset A$ we have $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subset P(X)$ is called an admissible ideal in X if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$. Some standard examples of ideals are given below:

(i) The set $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathbb{N}$.

(ii) The set $\mathcal{I}_d$ of all subsets of natural numbers having natural density 0 is an admissible ideal in $\mathbb{N}$.

(iii) The set $\mathcal{I}_c = \{A \subseteq \mathbb{N} \colon \sum_{a\in A} a^{-1} < \infty\}$ is an admissible ideal in $\mathbb{N}$.

(iv) Suppose $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ be a decomposition of $\mathbb{N}$ (for $i \neq j$, $D_i \cap D_j = \emptyset$). Then the set $\mathcal{I}$ of all subsets of $\mathbb{N}$, which intersects finitely many D_p's, forms an ideal in $\mathbb{N}$.

More important examples can be found in [12].

Definition 2. *[13] Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter in X if*

(i) for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;

(ii) for each $A \in \mathcal{F}$ and $B \supset A$ we have $B \in \mathcal{F}$.

The filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A \colon A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$.

Remark 1. *If $\mathcal{I}$ and $\mathcal{K}$ are two ideals in $\mathbb{N}$, then the set $\mathcal{I} \vee \mathcal{K} = \{A \cup B \colon A \in \mathcal{I}, B \in \mathcal{K}\}$ forms an ideal in $\mathbb{N}$. Further, if $\mathcal{I} \vee \mathcal{K}$ is non-trivial, then the dual filter of $\mathcal{I} \vee \mathcal{K}$ is denoted and defined by $\mathcal{F}(\mathcal{I} \vee \mathcal{K}) = \{M \cap N \colon M \in \mathcal{F}(\mathcal{I}), N \in \mathcal{F}(\mathcal{K})\}$.*

Definition 3. *[13] Let $\mathcal{I} \subset P(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$. A real-valued sequence $x = (x_k)$ is said to be $\mathcal{I}$-convergent to x_0 if the set $\{k \in \mathbb{N} \colon \mid x_k - x_0 \mid \geqslant \varepsilon\}$ belongs to $\mathcal{I}$ for each $\varepsilon > 0$. In this case, x_0 is called the $\mathcal{I}$-limit of the sequence (x_k) and is written as $\mathcal{I} - \lim x = x_0$.*

Definition 4. *[13] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A real-valued sequence $x = (x_k)$ is said to be $\mathcal{I}^*-$convergent to x_0, if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ in the associated filter $\mathcal{F}(\mathcal{I})$, such that $\lim\limits_{k\in M} x_k = x_0$. In this case, x_0 is called the $\mathcal{I}^*$-limit of the sequence (x_k) and is written as $\mathcal{I}^* - \lim x = x_0$.*

Definition 5. *[15] Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. A real-valued sequence $x = (x_k)$ is said to be $\mathcal{I}^{\mathcal{K}}$-convergent to x_0 if there exists $M \in \mathcal{F}(\mathcal{I})$, such that the sequence $y = (y_k)$ defined by*

$$y_k = \begin{cases} x_k, & k \in M, \\ x_0, & k \notin M \end{cases}$$

is $\mathcal{K}$-convergent to x_0. In this case, x_0 is called the $\mathcal{I}^{\mathcal{K}}$-limit of the sequence (x_k) and is written as $\mathcal{I}^{\mathcal{K}} - \lim x = x_0$.

When $\mathcal{K} = \mathcal{I}_f$, then $\mathcal{I}^{\mathcal{K}}$-convergence concept coincides with $\mathcal{I}^*$-convergence [13].

Definition 6. *[2] A real number l is said to be an $\mathcal{I}$ lower bound of the real-valued sequence $x = (x_k)$, if*

$$\{k \in \mathbb{N} \colon x_k \geqslant l\} \in \mathcal{F}(\mathcal{I}) \quad (\textit{or} \quad \{k \in \mathbb{N} \colon x_k < l\} \in \mathcal{I}).$$

The set of all $\mathcal{I}$ lower bound of the sequence $x = (x_k)$ is denoted by $L_{\mathcal{I}}(x)$.

Theorem 1. *[2] If $l \in \mathbb{R}$ is a usual lower bound of the real-valued sequence $x = (x_k)$, then l is also an $\mathcal{I}$ lower bound of the sequence x.*

Definition 7. *[2] A real number t is said to be the $\mathcal{I}$-infimum of the real-valued sequence $x = (x_k)$ if t is the supremum of $L_{\mathcal{I}}(x)$. In other words,*

$$\mathcal{I} - \inf x = \sup L_{\mathcal{I}}(x).$$

Definition 8. *[2] A real number u is said to be an $\mathcal{I}$ upper bound of the real-valued sequence $x = (x_k)$, if*

$$\{k \in \mathbb{N} \colon x_k \leqslant u\} \in \mathcal{F}(\mathcal{I}) \quad (\textit{or} \quad \{k \in \mathbb{N} \colon x_k > u\} \in \mathcal{I}).$$

The set of all $\mathcal{I}$ upper bounds of the sequence $x = (x_k)$ is denoted by $U_{\mathcal{I}}(x)$.

Definition 9. *[2] A real number s is said to be the $\mathcal{I}$-supremum of the real-valued sequence $x = (x_k)$ if s is the infimum of $U_{\mathcal{I}}(x)$. In other words,*

$$\mathcal{I} - \sup x = \inf U_{\mathcal{I}}(x).$$

Theorem 2. *[2] Let $x = (x_k)$ be any real-valued sequence. Then*

$$\inf x \leqslant \mathcal{I} - \inf x \leqslant \mathcal{I} - \sup x \leqslant \sup x.$$

Definition 10. *[2] (a) Let $x = (x_k)$ be a real-valued sequence. Then the $\mathcal{I}-$limit inferior is denoted and defined by*

$$\mathcal{I} - \liminf x = \mathcal{I} - \sup v,$$

where $v = (v_k)$ is the sequence defined by $v_k = \mathcal{I} - \inf\limits_{n \geqslant k}\{x_n, x_{n+1}, \dots\}$.
(b) Let $x = (x_k)$ be a real-valued sequence. Then the $\mathcal{I}-$limit superior is denoted and defined by

$$\mathcal{I} - \limsup x = \mathcal{I} - \inf w,$$

where $w = (w_k)$ is the sequence defined by $w_k = \mathcal{I} - \sup\limits_{n \geqslant k}\{x_n, x_{n+1}, \dots\}$.

Definition 11. *[13] A real number γ is said to be an $\mathcal{I}$-cluster point of a sequence $x = (x_k)$ if for any $\varepsilon > 0$*

$$\{k \in \mathbb{N} \colon |x_k - \gamma| < \varepsilon\} \notin \mathcal{I}$$

holds.

3. Main results. Throughout the article, $\mathcal{I}$, $\mathcal{K}$, and $\mathcal{I} \vee \mathcal{K}$ denotes the non-trivial admissible ideal in $\mathbb{N}$.

Definition 12. *A real number l is said to be an $\mathcal{I}^*$ lower bound of the real-valued sequence $x=(x_k)$, if there exists $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that l is an usual lower bound of the subsequence (x_{m_k}).*

The set of all $\mathcal{I}^*$ lower bounds of the sequence $x = (x_k)$ is denoted by $L_{\mathcal{I}^*}(x)$.

Definition 13. *A real number t is said to be the $\mathcal{I}^*$-infimum of the real-valued sequence $x = (x_k)$, if t is the supremum of $L_{\mathcal{I}^*}(x)$. In other words,*

$$\mathcal{I}^* - \inf x = \sup L_{\mathcal{I}^*}(x).$$

Definition 14. *A real number u is said to be an $\mathcal{I}^*$ upper bound of the real-valued sequence $x = (x_k)$, if there exists $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$, such that u is an usual upper bound of the subsequence (x_{m_k}).*

The set of all $\mathcal{I}^*$ upper bounds of the sequence $x = (x_k)$ is denoted by $U_{\mathcal{I}^*}(x)$.

Definition 15. *A real number s is said to be the $\mathcal{I}^*$-supremum of the real-valued sequence $x = (x_k)$, if s is the infimum of $U_{\mathcal{I}^*}(x)$. In other words,*

$$\mathcal{I}^* - \sup x = \inf U_{\mathcal{I}^*}(x).$$

Theorem 3. *For any real-valued sequence $x = (x_k)$,*

$$\mathcal{I}^* - \inf x \leqslant \mathcal{I} - \inf x \leqslant \mathcal{I} - \sup x \leqslant \mathcal{I}^* - \sup x.$$

Proof. We first prove that the inclusions $L_{\mathcal{I}^*}(x) \subseteq L_{\mathcal{I}}(x)$ and $U_{\mathcal{I}^*}(x) \subseteq U_{\mathcal{I}}(x)$ hold for the sequence x. Let $l \in L_{\mathcal{I}^*}(x)$. Then, by definition, there exists $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that l is an usual lower bound of (x_{m_k}). In other words, $x_{m_k} \geqslant l$ for any $k \in \mathbb{N}$. But then the inclusion

$$\{k \in \mathbb{N} \colon x_k \geqslant l\} = M \cup \{k \in \mathbb{N} \setminus M : x_k \geqslant l\} \supseteq M$$

holds and, subsequently, $\{k \in \mathbb{N}\colon x_k \geqslant l\} \in \mathcal{F}(\mathcal{I})$. Hence, $l \in L_{\mathcal{I}}(x)$ and the first inclusion is established. Applying a similar technique, the second inclusion can be obtained. Now the inclusions $L_{\mathcal{I}^*}(x) \subseteq L_{\mathcal{I}}(x)$ and $U_{\mathcal{I}^*}(x) \subseteq U_{\mathcal{I}}(x)$ and Theorem 2 altogether give

$$\mathcal{I}^* - \inf x \leqslant \mathcal{I} - \inf x \leqslant \mathcal{I} - \sup x \leqslant \mathcal{I}^* - \sup x.$$

$\square$

Definition 16. *A real number l is said to be an $\mathcal{I}^{\mathcal{K}}$ lower bound of the real-valued sequence $x = (x_k)$, if there exists $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$, such that l is an $\mathcal{K}$ lower bound of the sequence $y = (y_k)$ defined by*

$$y_k = \begin{cases} x_k & k \in M, \\ l & k \notin M. \end{cases}$$

The set of all $\mathcal{I}^{\mathcal{K}}$ lower bounds of the sequence $x = (x_k)$ is denoted by $L_{\mathcal{I}^{\mathcal{K}}}(x)$.

Definition 17. *A real number t is said to be the $\mathcal{I}^{\mathcal{K}}$-infimum of the real-valued sequence $x = (x_k)$, if t is the supremum of $L_{\mathcal{I}^{\mathcal{K}}}(x)$. In other words,*

$$\mathcal{I}^{\mathcal{K}} - \inf x := \sup L_{\mathcal{I}^{\mathcal{K}}}(x).$$

Definition 18. *A real number u is said to be an $\mathcal{I}^{\mathcal{K}}$ upper bound of the real-valued sequence $x = (x_k)$, if there exists $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$, such that u is an $\mathcal{K}$ upper bound of the sequence $y = (y_k)$ defined by*

$$y_k = \begin{cases} x_k & k \in M, \\ u & k \notin M. \end{cases}$$

The set of all $\mathcal{I}^{\mathcal{K}}$ upper bounds of the sequence $x = (x_k)$ is denoted by $U_{\mathcal{I}^{\mathcal{K}}}(x)$.

Definition 19. *A real number s is said to be the $\mathcal{I}^{\mathcal{K}}$-supremum of the real-valued sequence $x = (x_k)$ if s is the infimum of $U_{\mathcal{I}^{\mathcal{K}}}(x)$. In other words,*

$$\mathcal{I}^{\mathcal{K}} - \sup x := \inf U_{\mathcal{I}^{\mathcal{K}}}(x).$$

Theorem 4. *(i) Let $x = (x_k)$ be a real-valued sequence, such that $l \in L_{\mathcal{I}^{\mathcal{K}}}(x)$. If $l' < l$, then $l' \in L_{\mathcal{I}^{\mathcal{K}}}(x)$;*

(ii) Let $x = (x_k)$ be a real-valued sequence, such that $u \in U_{\mathcal{I}^{\mathcal{K}}}(x)$. If $u' > u$, then $u' \in U_{\mathcal{I}^{\mathcal{K}}}(x)$.

Proof. (i) Let $l \in L_{\mathcal{I}^{\mathcal{K}}}(x)$. Then there exists $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that $l \in L_{\mathcal{K}}(y)$, where $y = (y_k)$ is the sequence defined by

$$y_k = \begin{cases} x_k, & k \in M, \\ l, & k \notin M. \end{cases}$$

This implies $\{k \in \mathbb{N} \colon y_k \geqslant l\} \in \mathcal{F}(\mathcal{K})$. Now, since $l' < l$ by the assumption, the inclusion

$$\{k \in \mathbb{N} \colon y_k \geqslant l'\} \supset \{k \in \mathbb{N} \colon y_k \geqslant l\}$$

holds, and, consequently, $\{k \in \mathbb{N} \colon y_k \geqslant l'\} \in \mathcal{F}(\mathcal{K})$, i. e., $l' \in L_{\mathcal{I}^{\mathcal{K}}}(x)$. This completes the proof.
(ii) The proof can be obtained by applying a similar technique. □

Corollary 1. *(i) Let $x = (x_k)$ be a real-valued sequence, such that $l \in L_{\mathcal{I}^*}(x)$. If $l' < l$, then $l' \in L_{\mathcal{I}^*}(x)$;*
(ii) Let $x = (x_k)$ be a real-valued sequence, such that $u \in U_{\mathcal{I}^*}(x)$. If $u' > u$, then $u' \in U_{\mathcal{I}^*}(x)$.

Theorem 5. *For any real-valued sequence $x = (x_k)$,*

$$\mathcal{I}^* - \inf x \leqslant \mathcal{I}^{\mathcal{K}} - \inf x \leqslant \mathcal{I}^{\mathcal{K}} - \sup x \leqslant \mathcal{I}^* - \sup x.$$

Proof. To prove the theorem, we prove the following three inequalities:

$$\mathcal{I}^* - \inf x \leqslant \mathcal{I}^{\mathcal{K}} - \inf x, \tag{1}$$

$$\mathcal{I}^{\mathcal{K}} - \inf x \leqslant \mathcal{I}^{\mathcal{K}} - \sup x, \tag{2}$$

and

$$\mathcal{I}^{\mathcal{K}} - \sup x \leqslant \mathcal{I}^* - \sup x. \tag{3}$$

To prove (1), let $l \in L_{\mathcal{I}^*}(x)$. Then, by definition, there exists $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that l is an usual lower bound of (x_{m_k}). In other words, $l \in L(y)$, where $y = (y_k)$ is defined by

$$y_k = \begin{cases} x_k, & k \in M, \\ l, & k \notin M. \end{cases}$$

Therefore, by Theorem 1, we have $l \in L_{\mathcal{K}}(y)$. This implies $l \in L_{\mathcal{I}^{\mathcal{K}}}(x)$. Hence, we have $L_{\mathcal{I}^*}(x) \subseteq L_{\mathcal{I}^{\mathcal{K}}}(x)$ and, consequently, (1) holds.

To prove (2), assume the contrary. Then there exist some $l' \in L_{\mathcal{I}^{\mathcal{K}}}(x)$ and $u' \in U_{\mathcal{I}^{\mathcal{K}}}(x)$, such that $u' < l'$. But then, by Theorem 4, $l' \in U_{\mathcal{I}^{\mathcal{K}}}(x)$, which is a contradiction.

The proof of (3) is analogous to that of (1), so omitted.

Combining (1), (2), and (3) we obtain the desired inequality. □

Corollary 2. *For any real-valued sequence $x = (x_k)$:*

$$\inf x \leqslant \mathcal{K} - \inf x \leqslant \mathcal{I}^{\mathcal{K}} - \inf x \leqslant \mathcal{I}^{\mathcal{K}} - \sup x \leqslant \mathcal{K} - \sup x \leqslant \sup x.$$

Proof. We omit the proof as it can be easily obtained by combining Theorem 2, Theorem 5, and considering $M = \mathbb{N}$ from $\mathcal{F}(\mathcal{I})$. □

Theorem 6. *Let $\mathcal{I} \vee \mathcal{K}$ be a non-trivial ideal in $\mathbb{N}$ and $x = (x_k)$ be a real-valued sequence. Then:*
(i) if x is a monotonic increasing sequence, then $\mathcal{I}^{\mathcal{K}} - \inf x = \mathcal{I}^* - \sup x$;
(ii) if x is monotonic decreasing, then $\mathcal{I}^{\mathcal{K}} - \sup x = \mathcal{I}^* - \inf x$.

Proof. (i) We divide the entire proof into considering two cases.

Case-I: $\mathcal{I}^* - \sup x < \infty$

Suppose $\mathcal{I}^* - \sup x = s$. Then there exists some $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that $x_{m_k} \leqslant s$ holds for all $k \in \mathbb{N}$. This implies

$$M \subseteq \{k \in \mathbb{N} \colon x_k \leqslant s\}. \tag{4}$$

Also, for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that $x_{m_{k_0}} > s - \varepsilon$. We claim that $s \notin L_{\mathcal{I}^{\mathcal{K}}}(x)$. Otherwise, if $s \in L_{\mathcal{I}^{\mathcal{K}}}(x)$, then there exists $N = \{n_1 < n_2 < \ldots < n_k < \ldots\}$, such that $s \in L_{\mathcal{K}}(y)$, where $y = (y_k)$ is the sequence defined by

$$y_k = \begin{cases} x_k, & k \in N, \\ s, & k \notin N. \end{cases}$$

In other words, $\{k \in N \colon x_k \geqslant s\} \in \mathcal{F}(\mathcal{K})$. Now, as the inclusion

$$\{k \in N \colon x_k \geqslant s\} \subseteq \{k \in \mathbb{N} \colon x_k \geqslant s\}$$

holds, we have $\{k \in \mathbb{N} \colon x_k \geqslant s\} \in \mathcal{F}(\mathcal{K})$. Consequently, from (4), we have $\mathbb{N} \setminus M \in \mathcal{F}(\mathcal{K})$. Now $M \in \mathcal{F}(\mathcal{I})$ and $\mathbb{N} \setminus M \in \mathcal{F}(\mathcal{K})$ together yield

$M \cap (\mathbb{N} \setminus M) \in \mathcal{F}(\mathcal{I} \vee \mathcal{K})$, i. e., $\emptyset \in \mathcal{F}(\mathcal{I} \vee \mathcal{K})$, which is a contradiction. This proves our claim. Now, let $\varepsilon > 0$ be arbitrary. Then we have

$$\{k \in M \colon x_k < s - \varepsilon\} \subseteq \{k \in M \colon x_k < x_{m_{k_0}}\} \subseteq \{1, 2, \ldots, m_{k_0}\}.$$

Since $\mathcal{K}$ is admissible, so $\{1, 2, \ldots, m_{k_0}\} \in \mathcal{K}$ and, as a consequence, $\{k \in M \colon x_k < s - \varepsilon\} \in \mathcal{K}$, i. e., $s - \varepsilon \in L_{\mathcal{I}^{\mathcal{K}}}(x)$. Therefore, by Theorem 4, we obtain $L_{\mathcal{I}^{\mathcal{K}}}(x) = (-\infty, s - \varepsilon]$. Hence, $\mathcal{I}^{\mathcal{K}} - \inf x = \sup L_{\mathcal{I}^{\mathcal{K}}}(x) = s$.

Case-II: $\mathcal{I}^* - \sup x = \infty$

If $\mathcal{I}^* - \sup x = \infty$, then, for any $l \in \mathbb{R}$, there exists $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that $x_{m_{k_0}} \geqslant l$ for some $k_0 \in \mathbb{N}$. Now, since x is monotonic increasing, $x_{m_{k_0}} \leqslant x_k$ for all $k \geqslant m_{k_0}$. Thus, for all $k \geqslant m_{k_0}$, we have $x_k \geqslant l$. Eventually, $\{k \in \mathbb{N} \colon x_k < l\} \subseteq \{1, 2, \ldots, m_{k_0}\} \in \mathcal{K}$, i. e., $l \in L_{\mathcal{K}}(x)$, which further implies that $l \in L_{\mathcal{I}^{\mathcal{K}}}(x)$. Now, since l is arbitrary, $L_{\mathcal{I}^{\mathcal{K}}}(x) = (-\infty, \infty)$. Hence, $\mathcal{I}^{\mathcal{K}} - \inf x = \sup L_{\mathcal{I}^{\mathcal{K}}}(x) = \infty$.
(ii) The proof is analogous to that of (i), so omitted. $\square$

Theorem 7. *Let $x = (x_k)$ be a real-valued sequence and $t \in \mathbb{R}$ be fixed. Then $\mathcal{I}^{\mathcal{K}} - \inf x = t$ if and only if for every $\varepsilon > 0$ there exists $M \in \mathcal{F}(\mathcal{I})$, such that*

$$\{k \in M \colon x_k < t - \varepsilon\} \in \mathcal{K} \text{ and } \{k \in M \colon x_k \geqslant t + \varepsilon\} \notin \mathcal{F}(\mathcal{K}).$$

Proof. Suppose $\mathcal{I}^{\mathcal{K}} - \inf x = t$. Then, for any $l \in L_{\mathcal{I}^{\mathcal{K}}}(x)$, $l \leqslant t$ and for any $\varepsilon > 0$, there exists $l' \in L_{\mathcal{I}^{\mathcal{K}}}(x)$, such that $t - \varepsilon < l'$. So, by Theorem 4(i), $t - \varepsilon \in L_{\mathcal{I}^{\mathcal{K}}}(x)$. This implies that there exists a set $M \in \mathcal{F}(\mathcal{I})$, such that $\{k \in \mathbb{N} \colon y_k < t - \varepsilon\} \in \mathcal{K}$, where $y = (y_k)$ is defined as

$$y_k = \begin{cases} x_k, & k \in M, \\ t - \varepsilon, & k \notin M. \end{cases}$$

Therefore, $\{k \in M : x_k < t - \varepsilon\} \in \mathcal{K}$ holds.

Now, to prove $\{k \in M \colon x_k \geqslant t + \varepsilon\} \notin \mathcal{F}(\mathcal{K})$, we assume the contrary. Then there exists some $\varepsilon_0 > 0$, such that for any $M \in \mathcal{F}(\mathcal{I})$ $\{k \in M \colon x_k \geqslant t + \varepsilon_0\} \in \mathcal{F}(\mathcal{K})$. In particular, if we take $M = \mathbb{N}$, then we have $t + \varepsilon_0 \in L_{\mathcal{K}}(x)$ and, consequently, $t + \varepsilon_0 \in L_{\mathcal{I}^{\mathcal{K}}}(x)$, which is a contradiction to the fact that $t = \sup L_{\mathcal{I}^{\mathcal{K}}}(x)$.

To prove the converse part, assume that $\forall \varepsilon > 0$ there exists $M \in \mathcal{F}(\mathcal{I})$: $\{k \in M \colon x_k < t - \varepsilon\} \in \mathcal{K}$ and $\{k \in M \colon x_k \geqslant t + \varepsilon\} \notin \mathcal{F}(\mathcal{K})$. Then we have $t-\varepsilon \in L_{\mathcal{I}^{\mathcal{K}}}(x)$ and $t+\varepsilon \notin L_{\mathcal{I}^{\mathcal{K}}}(x)$. Therefore, $L_{\mathcal{I}^{\mathcal{K}}}(x) = (-\infty, t-\varepsilon)$. Now, since ε is arbitrary, we have

$$\mathcal{I}^{\mathcal{K}} - \inf x = \sup L_{\mathcal{I}^{\mathcal{K}}}(x) = t.$$

This completes the proof. □

Corollary 3. *Let $x = (x_k)$ be a real-valued sequence and $s \in \mathbb{R}$ be fixed. Then $\mathcal{I}^{\mathcal{K}} - \sup x = s$ if and only if for every $\varepsilon > 0$ there exists $M \in \mathcal{F}(\mathcal{I})$, such that*

$$\{k \in M \colon x_k > s+\varepsilon\} \in \mathcal{K} \text{ and } \{k \in M \colon x_k \leqslant s-\varepsilon\} \notin \mathcal{F}(\mathcal{K}).$$

Proof. The proof is similar to that of Theorem 7, so it is omitted here. □

Theorem 8. *For a real-valued sequence $x = (x_k)$, $\mathcal{I}^{\mathcal{K}} - \lim x = x_0$ if and only if $\mathcal{I}^{\mathcal{K}} - \inf x = x_0 = \mathcal{I}^{\mathcal{K}} - \sup x$.*

Proof. Let $\mathcal{I}^{\mathcal{K}} - \lim x = x_0$. Then there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that for any $\varepsilon > 0$, $\{k \in M \colon |x_k - x_0| \geqslant \varepsilon\} \in \mathcal{K}$. This implies $\{k \in M \colon x_k > x_0+\varepsilon\} \in \mathcal{K}$, i. e., $x_0+\varepsilon \in U_{\mathcal{I}^{\mathcal{K}}}(x)$ and $\{k \in M \colon x_k < x_0-\varepsilon\} \in \mathcal{K}$, i. e., $x_0-\varepsilon \in L_{\mathcal{I}^{\mathcal{K}}}(x)$. By Theorem 4, we have $U_{\mathcal{I}^{\mathcal{K}}}(x) = (x_0, \infty)$ and $L_{\mathcal{I}^{\mathcal{K}}}(x) = (-\infty, x_0)$, which further gives

$$\mathcal{I}^{\mathcal{K}} - \inf x = \sup L_{\mathcal{I}^{\mathcal{K}}}(x) = x_0 = \inf U_{\mathcal{I}^{\mathcal{K}}}(x) = \mathcal{I}^{\mathcal{K}} - \sup x.$$

For the converse part, let $\mathcal{I}^{\mathcal{K}} - \inf x = x_0 = \mathcal{I}^{\mathcal{K}} - \sup x$. Then $\sup L_{\mathcal{I}^{\mathcal{K}}}(x) = x_0 = \inf U_{\mathcal{I}^{\mathcal{K}}}(x)$. Then, by definition of the usual supremum and infimum, for any $\varepsilon > 0$ there exists $l \in L_{\mathcal{I}^{\mathcal{K}}}(x)$ and $u \in U_{\mathcal{I}^{\mathcal{K}}}(x)$, such that $x_0-\varepsilon < l$ and $x_0+\varepsilon > u$. Now, $l \in L_{\mathcal{I}^{\mathcal{K}}}(x)$ and $u \in U_{\mathcal{I}^{\mathcal{K}}}(x)$ imply the existence of two sets $M', M'' \in \mathcal{F}(\mathcal{I})$, such that $\{k \in M' \colon x_k < l\} \in \mathcal{K}$ and $\{k \in M'' \colon x_k > u\} \in \mathcal{K}$. Let M denote the set $M' \cap M''$. Then $M \in \mathcal{F}(\mathcal{I})$ and, since $x_0-\varepsilon < l$ and $x_0+\varepsilon > u$ hold, by the hereditary property of $\mathcal{K}$ we have:

$$\{k \in M \colon x_k \leqslant x_0-\varepsilon\} \subseteq \{k \in M \colon x_k < l\} \subseteq \{k \in M' \colon x_k < l\} \in \mathcal{K},$$

and

$$\{k \in M \colon x_k \geqslant x_0+\varepsilon\} \subseteq \{k \in M \colon x_k > u\} \subseteq \{k \in M'' \colon x_k > u\} \in \mathcal{K}.$$

Consequently,

$$\{k \in M \colon |x_k - x_0| \geqslant \varepsilon\} = \{k \in M \colon x_k \leqslant x_0 - \varepsilon\} \cup \{k \in M \colon x_k \geqslant x_0 + \varepsilon\} \in \mathcal{K}.$$

Hence, $\mathcal{I}^{\mathcal{K}} - \lim x = x_0$. □

Corollary 4. *Let $x = (x_k)$ be a real-valued sequence, such that $\mathcal{K} - \inf x = x_0 = \mathcal{K} - \sup x$. Then, $\mathcal{I}^{\mathcal{K}} - \lim x = x_0$.*

Proof. The proof follows directly from Corollary 2 and Theorem 8, so omitted. □

Theorem 9. *Let $x = (x_k)$ and $y = (y_k)$ be two real-valued sequences, such that there exists a set $M \in \mathcal{F}(\mathcal{I})$ satisfying $\{k \in M \colon x_k \neq y_k\} \in \mathcal{K}$. Then*

$$\mathcal{I}^{\mathcal{K}} - \inf x = \mathcal{I}^{\mathcal{K}} - \inf y \text{ and } \mathcal{I}^{\mathcal{K}} - \sup x = \mathcal{I}^{\mathcal{K}} - \sup y.$$

Proof. We only prove the first part, i. e., $\mathcal{I}^{\mathcal{K}} - \inf x = \mathcal{I}^{\mathcal{K}} - \inf y$. The proof of the second part can be obtained by applying a similar technique.

Let the given conditions hold and suppose $l \in L_{\mathcal{I}^{\mathcal{K}}}(x)$ be arbitrary. Then, by definition, there exists $N \in \mathcal{F}(\mathcal{I})$, such that $\{k \in N \colon x_k < l\} \in \mathcal{K}$. Consequently,

$$\begin{aligned}
&\{k \in M \cap N \colon y_k < l\} = \\
&\quad = \{k \in M \cap N \colon x_k \neq y_k, y_k < l\} \cup \{k \in M \cap N \colon x_k = y_k, y_k < l\} \subseteq \\
&\qquad\qquad \subseteq \{k \in M \colon x_k \neq y_k\} \cup \{k \in N \colon x_k < l\} \in \mathcal{K}.
\end{aligned}$$

From the above inclusion, it is clear that $\{k \in M \cap N \colon y_k < l\} \in \mathcal{K}$. Since $M \cap N \in \mathcal{F}(\mathcal{I})$, we have $l \in L_{\mathcal{I}^{\mathcal{K}}}(y)$. This proves that $L_{\mathcal{I}^{\mathcal{K}}}(x) \subseteq L_{\mathcal{I}^{\mathcal{K}}}(y)$. Similarly, one can establish $L_{\mathcal{I}^{\mathcal{K}}}(y) \subseteq L_{\mathcal{I}^{\mathcal{K}}}(x)$. Hence, $L_{\mathcal{I}^{\mathcal{K}}}(x) = L_{\mathcal{I}^{\mathcal{K}}}(y)$ holds and, finally, $\sup L_{\mathcal{I}^{\mathcal{K}}}(x) = \sup L_{\mathcal{I}^{\mathcal{K}}}(y)$, i. e., $\mathcal{I}^{\mathcal{K}} - \inf x = \mathcal{I}^{\mathcal{K}} - \inf y$. □

Theorem 10. *Let $x = (x_k)$ and $y = (y_k)$ be two real-valued sequences. Then:*
(i) $\mathcal{I}^{\mathcal{K}} - \inf(x + y) = \mathcal{I}^{\mathcal{K}} - \inf x + \mathcal{I}^{\mathcal{K}} - \inf y$;
(ii) $\mathcal{I}^{\mathcal{K}} - \sup(x + y) = \mathcal{I}^{\mathcal{K}} - \sup x + \mathcal{I}^{\mathcal{K}} - \sup y$.

Proof. (i) Let $\mathcal{I}^{\mathcal{K}} - \inf x = t_x$ and $\mathcal{I}^{\mathcal{K}} - \inf y = t_y$. Then, by Theorem 7, for any $\varepsilon > 0$ there exists $M, N \in \mathcal{F}(\mathcal{I})$, such that $\left\{k \in M \colon x_k < t_x - \frac{\varepsilon}{2}\right\} \in \mathcal{K}$ and $\left\{k \in N \colon y_k < t_y - \frac{\varepsilon}{2}\right\} \in \mathcal{K}$. Now, as the inclusion

$$\{k \in M \cap N \colon x_k + y_k < (t_x + t_y) - \varepsilon\} \subseteq \\ \subseteq \left\{k \in M \colon x_k < t_x - \frac{\varepsilon}{2}\right\} \cup \left\{k \in N \colon y_k < t_y - \frac{\varepsilon}{2}\right\}$$

holds and $M \cap N \in \mathcal{F}(\mathcal{I})$, $\{k \in M \cap N \colon x_k + y_k < (t_x + t_y) - \varepsilon\} \in \mathcal{K}$ and, consequently, $\mathcal{I}^{\mathcal{K}} - \inf(x + y) = t_x + t_y = \mathcal{I}^{\mathcal{K}} - \inf x + \mathcal{I}^{\mathcal{K}} - \inf y$. This completes the proof.
(ii) The proof is similar to that of (i), so omitted. □

Definition 20. *A real number γ is said to be the $\mathcal{I}^{\mathcal{K}}$-cluster point of a real-valued sequence $x = (x_k)$ if there exists $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that γ is a $\mathcal{K}$-cluster point of the subsequence (x_{m_k}).*

The set of all $\mathcal{I}^{\mathcal{K}}$-cluster points of a real-valued sequence $x = (x_k)$ is denoted by $\mathcal{I}^{\mathcal{K}} - (\Gamma_x)$.

Theorem 11. *Let $x = (x_k)$ be a real-valued sequence, such that $\mathcal{I}^{\mathcal{K}} - \sup x$ and $\mathcal{I}^{\mathcal{K}} - \inf x$ are finite. Then, $\mathcal{I}^{\mathcal{K}} - \sup x \in \mathcal{I}^{\mathcal{K}} - (\Gamma_x)$ and $\mathcal{I}^{\mathcal{K}} - \inf x \in \mathcal{I}^{\mathcal{K}} - (\Gamma_x)$.*

Proof. Let $\mathcal{I}^{\mathcal{K}} - \sup x = \inf U_{\mathcal{I}^{\mathcal{K}}}(x) = s$. Then, by definition of the usual infimum, for any $\varepsilon > 0$ there exists $t_0 \in U_{\mathcal{I}^{\mathcal{K}}}(x)$, such that $s \leqslant t_0 < s + \varepsilon$. Consequently, there exists $M \in \mathcal{F}(\mathcal{I})$, such that $\{k \in M \colon x_k > t_0\} \in \mathcal{K}$. Now, as the inclusion

$$\{k \in M \colon x_k \geqslant s + \varepsilon\} \subseteq \{k \in M \colon x_k > t_0\}$$

holds,

$$\{k \in M \colon x_k \geqslant s + \varepsilon\} \in \mathcal{K}. \tag{5}$$

Again, $s = \inf U_{\mathcal{I}^{\mathcal{K}}}(x)$ gives $s - \varepsilon \notin U_{\mathcal{I}^{\mathcal{K}}}(x)$, which further implies

$$\{k \in M \colon x_k > s - \varepsilon\} \notin \mathcal{K}. \tag{6}$$

Now, since the following relation

$$\{k \in M \colon x_k > s-\varepsilon\} = \{k \in M \colon s-\varepsilon < x_k < s+\varepsilon\} \cup \{k \in M \colon x_k \geqslant s+\varepsilon\}$$

holds, from (5) and (6) we have $\{k \in M \colon s - \varepsilon < x_k < s + \varepsilon\} \notin \mathcal{K}$, i. e., $s \in \mathcal{I}^{\mathcal{K}} - (\Gamma_x)$. This completes the proof of the first part.
Applying a similar technique, we can show that $\mathcal{I}^{\mathcal{K}} - \inf x \in \mathcal{I}^{\mathcal{K}} - (\Gamma_x)$. □

Definition 21. *(a) Let $x = (x_k)$ be a real-valued sequence. Then $\mathcal{I}^{\mathcal{K}}-$limit inferior is denoted and defined by*

$$\mathcal{I}^{\mathcal{K}} - \liminf x = \mathcal{I}^{\mathcal{K}} - \sup v,$$

where $v = (v_k)$ is the sequence defined by $v_k = \mathcal{I}^{\mathcal{K}} - \inf\limits_{n \geqslant k}\{x_n, x_{n+1}, \ldots\}$.
(b) Let $x = (x_k)$ be a real-valued sequence. Then $\mathcal{I}^{\mathcal{K}}-$limit superior is denoted and defined by

$$\mathcal{I}^{\mathcal{K}} - \limsup x = \mathcal{I}^{\mathcal{K}} - \inf w,$$

where $w = (w_k)$ is the sequence defined by $w_k = \mathcal{I}^{\mathcal{K}} - \sup\limits_{n \geqslant k}\{x_n, x_{n+1}, \ldots\}$.

Corollary 5. *Let $x = (x_k)$ be a real-valued sequence. Then:*
(i) if $v_k = \mathcal{I}^{\mathcal{K}} - \inf\limits_{n \geqslant k}\{x_n, x_{n+1}, \ldots\}$ for all $n \in \mathbb{N}$, then $v = (v_k)$ is a constant sequence and $v_k = \mathcal{I}^{\mathcal{K}} - \inf x$;
(ii) if $w_k = \mathcal{I}^{\mathcal{K}} - \sup\limits_{n \geqslant k}\{x_n, x_{n+1}, \ldots\}$ for all $n \in \mathbb{N}$, then $w = (w_k)$ is a constant sequence and $w_k = \mathcal{I}^{\mathcal{K}} - \sup x$.

Proof. The proof of (i) and (ii) are easy, so omitted. □

Corollary 6. *Let $x = (x_k)$ be a real-valued sequence. Then:*

$$\mathcal{I}^{\mathcal{K}} - \liminf x = \mathcal{I}^{\mathcal{K}} - \inf x \text{ and } \mathcal{I}^{\mathcal{K}} - \limsup x = \mathcal{I}^{\mathcal{K}} - \sup x.$$

Proof. The proof is omitted as it can be easily obtained from the Definition 21 and Corollary 5. □

Corollary 7. *Let $x = (x_k)$ and $y = (y_k)$ be two real-valued sequences. Then:*

(i) $\mathcal{I}^{\mathcal{K}} - \liminf x \leqslant \mathcal{I}^{\mathcal{K}} - \limsup x$;

(ii) $\liminf x \leqslant \mathcal{K} - \liminf x \leqslant \mathcal{I}^{\mathcal{K}} - \liminf x \leqslant$
$$\leqslant \mathcal{I}^{\mathcal{K}} - \limsup x \leqslant \mathcal{K} - \limsup x \leqslant \limsup x;$$

(iii) $\mathcal{I}^{\mathcal{K}} - \liminf(x + y) = \mathcal{I}^{\mathcal{K}} - \liminf x + \mathcal{I}^{\mathcal{K}} - \liminf y$;

(iv) $\mathcal{I}^{\mathcal{K}} - \limsup(x + y) = \mathcal{I}^{\mathcal{K}} - \limsup x + \mathcal{I}^{\mathcal{K}} - \limsup y$.

4. Conclusion.

In this paper, we investigated the notions of $\mathcal{I}^{\mathcal{K}}$-supremum, $\mathcal{I}^{\mathcal{K}}$-infimum, $\mathcal{I}^{\mathcal{K}}$-limit superior, and $\mathcal{I}^{\mathcal{K}}$-limit inferior for a real-valued sequence $x = (x_k)$, and presented some interrelationships between these notions. Theorem 7 and Corollary 3 give the necessary and sufficient conditions for a real number to become $\mathcal{I}^{\mathcal{K}}$-infimum and

$\mathcal{I}^{\mathcal{K}}$-supremum, respectively, of a real-valued sequence. Theorem 8 gives the necessary and sufficient condition regarding the $\mathcal{I}^{\mathcal{K}}$-convergence of a real-valued sequence. Theorem 11 proves the inclusion of the numbers $\mathcal{I}^{\mathcal{K}} - \sup x$ and $\mathcal{I}^{\mathcal{K}} - \inf x$ in the set $\mathcal{I}^{\mathcal{K}} - (\Gamma_x)$. The obtained results may be helpful for future researchers to explore the notion of $\mathcal{I}^{\mathcal{K}}$-convergence in more detail.

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