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VARIABLE LEBESGUE ALGEBRA ON A LOCALLY COMPACT GROUP

Abstract. For a locally compact group H with a left Haar measure, we study the variable Lebesgue algebra $\mathcal{L}^{p(\cdot)}(H)$ with respect to convolution. We show that if $\mathcal{L}^{p(\cdot)}(H)$ has a bounded exponent, then it contains a left approximate identity. We also prove a necessary and sufficient condition for $\mathcal{L}^{p(\cdot)}(H)$ to have an identity. We observe that a closed linear subspace of $\mathcal{L}^{p(\cdot)}(H)$ is a left ideal if and only if it is left translation invariant.

Key words: *variable Lebesgue space, bounded exponent, approximate identity, Haar measure*

2020 Mathematical Subject Classification: *43A10, 43A15, 43A75, 43A77*

1. Introduction. The L^p-space on a locally compact group has a very essential role in Harmonic analysis. It has numerous applications in Mathematics, Physics, Electrical Engineering, and other branches of science. For a locally compact group H and $1 \leqslant p < \infty$, the space $L^p(H)$ is a Banach space and even a Banach algebra with respect to convolution if H is compact [8]. If we replace the constant exponent p with a variable exponent with minimal restrictions, many results on the classical Lebesgue space the hold. The theory of variable-exponent space was initially introduced by Orlicz in the 1930s; these theories were further studied and analyzed by various authors. However, many interesting results were found in variable-exponent space and a spike of interest was found among the authors in recent years.

The variable Lebesgue space is a generalization of the L^p-space for constant exponent. It was first put forward by Russian mathematician Tsennov [14] and further expanded by Sharapudinov [10], [11], [12], [13] and Zhikov [15], [16], [17], [18], [19], [20]. In 1991, Kováčik and J. Rákosník studied some fundamental properties of the variable Lebesgue space in [6], which is considered as one of the foundational paper in this topic. Orlicz space is a particular case of variable Lebesgue space to some extent. In [2] authors established that the variable Lebesgue space is a Banach space. He also proved some elementary results of classical Lebesgue space in variable Lebesgue space. Motivated by the above literature, in this article we show that the variable Lebesgue space forms a Banach algebra with respect to a suitable convolution. We organize this article in the following way:

In Section 2, we furnish some definitions and preliminary results on variable Lebesgue space. In Section 3, we provide the condition under which the variable Lebesgue space on a locally compact group is a Banach algebra with respect to a suitable convolution, which is mentioned as a variable Lebesgue algebra. In Section 4, we show that the variable Lebesgue algebra $\mathcal{L}^{p(\cdot)}(H)$ has a left approximate identity. We also prove a necessary and sufficient condition for the existence of identity in the variable Lebesgue algebra. A familiar result on the weighted Orlicz algebra $L^{\phi}_{w}(H)$ (See [7]) that the closed left ideal of the variable Lebesgue algebra $\mathcal{L}^{p(\cdot)}(H)$ is nothing else the left translation invariant subspace of $\mathcal{L}^{p(\cdot)}(H)$, is also proved.

2. Preliminaries. In this section, we recall some definitions and results on variable Lebesgue spaces. Here we consider a locally compact group H with a Haar measure μ.

Definition 1. *[1] A non-zero regular Borel measure μ on a locally compact group H is called a left Haar measure if it is left invariant under left translation, i.e., $\mu(Sg) = \mu(S)$ for all $g \in H$ and all Borel subset $S \subset H$.*

Similarly, one can define the right Haar measure.

Let $\mathcal{M}$ be the set of all μ-measurable functions from H into $\mathbb{C}$ or into the extended real numbers $[-\infty, \infty]$. An element of $\mathcal{M}$ is called an exponent if it takes the values in $[1, \infty]$. We denote the set of all exponents by $\mathcal{E}$. For a fixed $p \in \mathcal{E}$ and any $f \in \mathcal{L}^{p(\cdot)}(H)$, the functional ρ_p defined by

$$\rho_p(f) := \int_{F_p} |f(x)|^{p(x)} d\mu(x) + \operatorname*{ess\,sup}_{x \in F_p^c} |f(x)|, \tag{1}$$

where $F_p = \{x \in H : p(x) < \infty\}$, is called the p-modular function. There are two other definitions of the modular to define variable Lebesgue space. One of them is defined as

$$\rho_p := \max\Big(\int_{F_p} |f(x)|^{p(x)} d\mu(x), \quad \operatorname*{ess\,sup}_{x \in F_p^c} |f(x)| \Big). \tag{2}$$

It can be easily seen that the modular (2) is equivalent to the modular given by (1) and they admit the same norm. The modular (2) was introduced by Edmunds and Rákosník [4]. Another approach to defining a modular inspired by the theory of Musielak-Orlicz space is given in [3] as

$$\rho_p(f) = \int_H |f(x)|^{p(x)} dx. \tag{3}$$

with the convention that $t^\infty = \infty.\chi_{(1,\infty)}(t)$. This modular is not equivalent to (1), but the resulting norm is equivalent.

For a locally compact group H and $p \in \mathcal{E}$, the variable Lebesgue space $\mathcal{L}^{p(\cdot)}(H)$ is defined as (see [2])

$$\mathcal{L}^{p(\cdot)}(H) = \Big\{ f \in \mathcal{M} : \rho_p\Big(\frac{f}{\lambda}\Big) < \infty, \exists\ \lambda > 0 \Big\}.$$

This space $\mathcal{L}^{p(\cdot)}(H)$ is a Banach space under the norm $\|f\|_{p(\cdot)}$ defined for all $f \in \mathcal{L}^{p(\cdot)}(H)$ by

$$\|f\|_{p(\cdot)} = \inf \Big\{ t > 0 : \rho_p\Big(\frac{f}{t}\Big) \leqslant 1 \Big\}.$$

There is another norm $\|f\|^A_{p(\cdot)}$ (the Amemiya norm) on $\mathcal{L}^{p(\cdot)}(H)$ defined as

$$\|f\|^A_{p(\cdot)} = \inf \Big\{ k > 0 \colon k + k\rho_p\Big(\frac{f}{k}\Big) \Big\}$$

for any $p \in \mathcal{E}$ with $p_+ = \operatorname{ess\,sup}_H p(x) < \infty$. The above two norms are equivalent and $\|f\|_{p(\cdot)} \leqslant \|f\|^A_{p(\cdot)} \leqslant 2\|f\|_{p(\cdot)}$ (for a proof, see [9]).

An exponent $p \in \mathcal{E}$ is called a bounded exponent if $p_+ = \operatorname{ess\,sup}_H p(x) < \infty$. If p is a bounded exponent, then ρ_p-convergence and $\|\cdot\|_{p(\cdot)}$-convergence in $\mathcal{L}^{p(\cdot)}(H)$ are equivalent. Denote by $\mathcal{S}$ the class of simple functions that vanish outside a set of finite measure; then $\mathcal{S}$ is dense in $\mathcal{L}^{p(\cdot)}(H)$, whenever p is a bounded exponent (see [2]). Let $p(\cdot), q(\cdot) \in \mathcal{E}$ be given; if $\mu(H) < \infty$, then $\mathcal{L}^{q(\cdot)}(H) \subset \mathcal{L}^{p(\cdot)}(H)$ when $p(\cdot) \leqslant q(\cdot)$ μ-almost everywhere. Moreover, there exist constants $c_1, c_2 > 0$, such that

$$c_1\|f\|_{p_-} \leqslant \|f\|_{p(\cdot)} \leqslant c_2\|f\|_{p_+}, \tag{4}$$

where $p_- = \operatorname{ess\,inf}_H p(x)$ and $p_+ = \operatorname{ess\,sup}_H p(x)$ (see [2], Theorem 2.26 and Corollary 2.27).

The Hölder inequality holds in variable Lebesgue spaces. If p and q are exponents and r is the function defined by $\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$ is an exponent, then there exists a constant $K \in [1,5]$, such that $\|fg\|_{r(\cdot)} \leqslant K\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}$. The conjugate exponent p' of p is the exponent that satisfies the equation $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for all $x \in X$. The functional $\|\cdot\|'_{p(\cdot)}$ defined on $\mathcal{M}$ given by

$$\|f\|'_{p(\cdot)} := \sup_{\rho_{p'}(g)\leqslant 1} \int\limits_H |fg| d\mu$$

is the conjugate norm to $\|\cdot\|_{p(\cdot)}$. Various other definitions used by different author for $\|\cdot\|'_{p(\cdot)}$ are characterised by the following equalities ([2]):

$$\|f\|'_{p(\cdot)} = \sup_{\|g\|_{p'(\cdot)}\leqslant 1} \int\limits_H |fg| d\mu = \sup_{\rho_{p'}(g)\leqslant 1} \left|\int\limits_H fg d\mu\right| = \sup_{\|g\|_{p'(\cdot)}\leqslant 1} \left|\int\limits_H fg d\mu\right| =$$
$$= \sup_{\rho_{p'}(g)\leqslant 1,\, g\in\mathcal{S}} \int\limits_H |fg| d\mu = \sup_{\|g\|_{p'(\cdot)}\leqslant 1,\, g\in\mathcal{S}} \int_H |fg| d\mu,$$

where $\mathcal{S}$ is the class of simple functions on H. If μ is σ-finite and p is a bounded exponent, then the dual space of $\mathcal{L}^{p(\cdot)}(H)$ is $\mathcal{L}^{p'(\cdot)}(H)$, where p' is the conjugate exponent to p. Moreover, if $p_- = \operatorname{ess\,inf}_H p(x) > 1$, then $\mathcal{L}^{p(\cdot)}(H)$ is reflexive.

Note that a left approximate identity in a Banach algebra $(A, \|\cdot\|)$ is a net $(e_\alpha)_{\alpha\in\Lambda}$ in A if $\lim_\alpha \|e_\alpha x - x\| = 0$ for all $x \in A$; similarly, the right approximate identity is also defined. Now we mention some properties of the norm $\|\cdot\|_{p(\cdot)}$ on $\mathcal{L}^{p(\cdot)}(H)$, which are useful in our studies.

Proposition 1. *[2] Let $p \in \mathcal{E}$ and $f, g \in \mathcal{L}^{(\cdot)}(H)$.*

(i) $\|f\|_{p(\cdot)} \geqslant 0$; *the equality holds iff* $f = 0$, μ-a.e.
(ii) *If* $|f| \leqslant |g|, \mu$ *a.e, then* $\|f\|_{p(\cdot)} \leqslant \|g\|_{p(\cdot)}$.
(iii) $\rho_p\left(f/\|f\|_{p(\cdot)}\right) \leqslant 1$, *provided* $f \neq 0$ μ- *a.e.*
(iv) $\rho_p(f) \leqslant 1$ *iff* $\|f\|_{p(\cdot)} \leqslant 1$.
(v) *If* $\rho_p(f) \leqslant 1$ *or* $\|f\|_{p(\cdot)} \leqslant 1$, *then* $\rho_p(f) \leqslant \|f\|_{p(\cdot)}$.
(vi) $\rho_p(f) \geqslant 1$ *or* $\|f\|_{p(\cdot)} \geqslant 1$, *then* $\|f\|_{p(\cdot)} \leqslant \rho_p(f)$.

3. Variable Lebesgue Algebra. In this section, we show that for a locally compact group H with a Haar measure μ, the variable Lebesgue space $\mathcal{L}^{p(\cdot)}(H)$ forms an algebra with respect to a suitable convolution under the certain condition. Before introducing the variable Lebesgue algebra, we discuss some lemmas.

We define a convolution on $\mathcal{L}^{p(\cdot)}(H)$ as follows. For any $f, g \in \mathcal{L}^{p(\cdot)}(H)$:

$$(f \star g)(x) = \int\limits_H f(y)g(y^{-1}x)d\mu(y). \tag{5}$$

Lemma 1. *$\mathcal{L}^{p(\cdot)}(H) \subseteq L^1(H)$ if and only if there exists a constant $k > 0$ such that $\|f\|_1 \leqslant k\|f\|_{p(\cdot)}$.*

Proof. Assume that $\mathcal{L}^{p(\cdot)}(H) \subseteq L^1(H)$ on $\mathcal{L}^{p(\cdot)}(H)$. It is easy to see that $(\mathcal{L}^{p(\cdot)}(H), \|\cdot\|)$ is a Banach space with $\|f\| = \|f\|_1 + \|f\|_{p(\cdot)}$. Let I be the identity map from $(\mathcal{L}^{p(\cdot)}(H), \|\cdot\|)$ to $(\mathcal{L}^{p(\cdot)}(H), \|\cdot\|_{p(\cdot)})$. Then I is continuous and one to one map, so, by the open mapping theorem, I is an open map; and, so, it is a homeomorphism. Hence, I^{-1} is a bounded map from $(\mathcal{L}^{p(\cdot)}(H), \|\cdot\|_{p(\cdot)})$ to $(\mathcal{L}^{p(\cdot)}(H), \|\cdot\|)$. Thus, $\exists$ a constant $k \geqslant 0$, such that $\|I^{-1}(f)\| \leqslant k\|f\|_{p(\cdot)}$, i.e., $\|f\| \leqslant k\|f\|_{p(\cdot)}$. But $\|f\| = \|f\|_1 + \|f\|_{(\cdot)}$; so, $\|f\|_1 \leqslant \|f\|$ and, hence, $\|f\|_1 \leqslant k\|f\|_{p(\cdot)}$. The converse part follows from the relation $\|f\|_1 \leqslant k\|f\|_{p(\cdot)}$ for $k > 0$. □

Lemma 2. *$\|L_x f\|_{p(\cdot)} = \|f\|_{p(\cdot)}$, for all $f \in \mathcal{L}^{p(\cdot)}(H)$. Here we mean $L_x f(y) = f(x^{-1}y)$ for all $x, y \in H$.*

Theorem 1. *Let H be a locally compact group with a left Haar measure μ. If $\mathcal{L}^{p(\cdot)}(H) \subseteq L^1(H)$ and $|f| \leqslant 1$ for all $f \in \mathcal{L}^{p(\cdot)}(H)$, then the variable Lebesgue space $\mathcal{L}^{p(\cdot)}(H)$ is a Banach algebra with respect to the convolution defined by (5).*

Proof. Let $\mathcal{L}^{p(\cdot)}(H) \subseteq L^1(H)$. By Lemma 1, $\exists\, c > 0$, such that

$$\| f \|_1 \leqslant c \| f \|_{p(\cdot)}, \tag{6}$$

for all $f \in \mathcal{L}^{p(\cdot)}(H)$. Now, for any $f, g \in \mathcal{L}^{p(\cdot)}(H)$ and $t > 0$, we have:

$$\rho_p\Big(\frac{f \star g}{t}\Big) = \rho_p\Big(\frac{1}{t}\int\limits_H f(y)g(y^{-1}x)d\mu(y)\Big) =$$

$$= \int\limits_{F_p}\Big|\frac{1}{t}\int\limits_H f(y)g(y^{-1}x)d\mu(y)\Big|^{p(x)} d\mu(x) + \operatorname*{ess\,sup}_{x\in F_p^c}\Big|\frac{1}{t}\int\limits_H f(y)g(y^{-1}x)d\mu(y)\Big| \leqslant$$

$$\leqslant \int\limits_H |f(y)|d\mu(y)\Big\{\int\limits_{F_p}\Big|\frac{L_y g(x)}{t}\Big|^{p(x)} d\mu(x) + \operatorname*{ess\,sup}_{x\in F_p^c}\Big|\frac{L_y g(x)}{t}\Big|\Big\} =$$

$$= \|f\|_1 \rho_p(L_y(g)/t).$$

The last inequality follows from the Fibuni theorem, and the condition $|f| \leqslant 1$ for all $f \in \mathcal{L}^{p(\cdot)}(H)$. Thus, we have:

$$\rho_p(f \star g/t) \leqslant \|f\|_1 \rho_p(L_y(g)/t). \tag{7}$$

From (6) and (7), it follows that

$$\{t > 0\colon \|f\|_1 \rho_p(L_y(g)/t) \leqslant 1\} \subseteq \{t > 0\colon \rho_p(f \star g/t) \leqslant 1\}.$$

That is, $\|f \star g\|_{p(\cdot)} \leqslant \|f\|_1 \|L_y g\|_{p(\cdot)}$ and, hence, $\|f \star g\|_{p(\cdot)} \leqslant \|f\|_{p(\cdot)} \|g\|_{p(\cdot)}$. □

4. Approximate identity.

Lemma 3. *If p is a bounded exponent and $\mu(H) < \infty$, then the set $C_c(H)$ of continuous functions with compact support is dense in $\mathcal{L}^{p(\cdot)}(H)$.*

Proof. Let $\mathcal{S} = \{s\colon H \to \mathbb{C}|s\}$ be simple, measurable and $\mu\{x|s(x) \neq 0\}$ be finite.

First of all we prove that, $\mathcal{S}$ is dense in $\mathcal{L}^{p(\cdot)}(H)$. Let $B \subset H$ be a measureble set with finite measure. Then $\rho_p(\chi_B) \leqslant \mu(B) < \infty$ which implies that $\chi_B \in L^{p(\cdot)}$. Since every element of $\mathcal{S}$ is a finite linear combination of indicator function on finitely μ-measurable set, so $\mathcal{S} \subset L^{p(\cdot)}(H)$. Now let $f \in L^{p(\cdot)}(H)$ with $f \geqslant 0$, $\exists$ a sequence $\{g_n\}$ of simple function such that $0 \leqslant g_n \leqslant f$ and $g_n \nearrow f$ (see [5]). Take $\lambda > 0$ such that $\rho_p(\frac{f}{\lambda}) < \infty$. Since, $0 \leqslant g_n \leqslant f$, we have

$$\rho_p(g_n) \leqslant \rho_p(f) \leqslant \max\left\{1, \lambda^{p^+}\right\} \rho_p\left(\frac{f}{\lambda}\right) < \infty.$$

It follows that $g_n \in L^{p(\cdot)}(H)$. Since $|f - g_n|^p \to 0$ pointwise and

$$|f - g_n|^p \leqslant (2f)^p \leqslant \max\left\{1, (2\lambda)^{p^+}\right\} \left(\frac{f}{\lambda}\right)^p \in L^1,$$

the dominated convergence theorem implies that $\rho_p(f - g_n) \to 0$ and since ρ_p-convergence and $\|\cdot\|_{p(\cdot)}$ are equivalent, so $\|f - g_n\|_{p(\cdot)} \to 0$, which shows that $\mathcal{S}$ is dense in $L^{p(\cdot)}(H)$. Now, we claim that $C_c(H)$ is dense in $\mathcal{S}$. If $s \in \mathcal{S}$ and $\epsilon > 0$, then by Lusin's theorem $\exists$ $g \in C_c(H)$ such that $\mu\{x \mid g(x) \neq s(x)\} < \epsilon^{p^+}$, where $p^+ = \operatorname*{ess\,sup}_{x \in H} p(x)$ and $|g| \leqslant \|s\|_\infty$. Now since p is bounded exponent, so $p^+ = \operatorname{ess\,sup}_{x \in H} p(x) < \infty$, then by (4), we have

$$\|g - s\|_{p(\cdot)} \leqslant c\|g - s\|_{p^+} = c\Big(\int_G |g - s|^{p^+} d\mu\Big)^{1/p^+} =$$
$$= c\Big(\int_{\{x:\, g(x) \neq s(x)\}} |g - s|^{p^+} d\mu\Big)^{1/p^+} \leqslant 2c\|s\|_\infty\, \varepsilon.$$

This implies that $C_c(H)$ is dense in S and since S is dense in $\mathcal{L}^{p(\cdot)}(H)$, hence $C_c(H)$ is dense in $\mathcal{L}^{p(\cdot)}(H)$. □

Lemma 4. *Let p be a bounded exponent and $\mathcal{L}^{p(\cdot)}(H)$ be a Banach algebra. Then, for any $f \in \mathcal{L}^{p(\cdot)}(H)$ and $\varepsilon > 0$, there exists a neighbourhood U of the identity, such that for all $x \in U$, $\|L_x f - f\|_{p(\cdot)} < \epsilon$.*

Proof. Let $\mathcal{K}$ be a compact neighbourhood of the identity of H. Let $f \in C_c(H)$ with compact support S; then $supp(L_x f) = xS, \ \forall x \in \mathcal{K}$. Let $\mathcal{K}' = \mathcal{K} \cup S \cup \mathcal{K}S$. There exists a constant $c > 0$, such that

$$\|L_x f - f\|_{p(\cdot)} \leqslant c\|L_x f - f\|_{p^+} = c\Big(\int_{\mathcal{K}'} |L_x f - f|^{p^+} d\mu\Big)^{1/p^+} \leqslant$$
$$\leqslant c\|L_x f - f\|_\infty (\mu(\mathcal{K}'))^{1/p^+}.$$

Since $\operatorname{supp}(L_x f - f) \subseteq \mathcal{K}'$, $\|L_x f - f\|_\infty < \dfrac{\varepsilon}{2c(\mu(K))^{1/p+}}$ for any x in a sufficiently small neighbourhood $V \subseteq \mathcal{K}$ of identity. Thus,

$$\|L_x f - f\|_{p(\cdot)} < \frac{\varepsilon}{2} \tag{8}$$

Now, let $f \in \mathcal{L}^{p(\cdot)}(H)$ be arbitrary and U be a compact neighborhood of identity. Since $C_c(H)$ is dense in $\mathcal{L}^{p(\cdot)}(H)$ for a bounded exponent p, for $\varepsilon > 0 \ \exists \ g \in C_c(H)$, such that $\|f - g\|_{p(\cdot)} < \frac{\varepsilon}{4}$. Also, as $g \in C_c(H)$, by (8) we get $\|L_x g - g\|_{p(\cdot)} < \frac{\varepsilon}{2}$ for all $x \in U$. Thus, for any $x \in U$:

$$\|L_x f - f\|_{p(\cdot)} \leqslant \|Lxf - L_x g\|_{p(\cdot)} + \|L_x g - g\|_{p(\cdot)} + \|g - f\|_{p(\cdot)} =$$
$$= 2\|f - g\|_{p(\cdot)} + \|L_x g - g\|_{p(\cdot)} < 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Theorem 2. *Let $\mu(H) < \infty$ and $\mathcal{L}^{p(\cdot)}(H)$ be a variable Lebesgue algebra. Then $\mathcal{L}^{p(\cdot)}(H)$ has a left approximate identity.*

Proof. Let $\mathcal{K}$ be a compact neighbourhood of the identity element $e \in H$ and $\aleph$ be the family of all neighbourhoods $U \subseteq K$ of the identity. Then $\aleph$ is a directed set with the pre-order inclusion. Let $\xi_U = \frac{\chi_U}{\mu(U)}$ for all $U \in \aleph$. Then $\exists\ c > 0$, such that

$$\|\xi_U\|_{p(\cdot)} \leqslant c\|\xi_U\|_{p^+} = c\|\chi_U/\mu(U)\|^{p^+} =$$
$$= c\Big(\int\limits_H |\frac{\chi_U}{\mu(U)}|^{p+}\Big)^{1/p^+} = c\Big(\int\limits_U (1/\mu(U))^{p^+} d\mu\Big)^{1/p^+} = c\Big(\mu(U)\Big)^{\frac{1-p^+}{p^+}}.$$

Therefore, $\|\xi_U\|_{p(\cdot)} < \infty$.

Thus $\rho_p(\xi_U) \leqslant \|\xi_U\|_{p(\cdot)} < \infty$, i.e., $\xi_U \in L^{p(\cdot)}(H)$ and $(\xi_U)_{U\in\aleph}$ is a net in $L^{p(\cdot)}(H)$. Now, given $f \in L^{p(\cdot)}(H)$ and $\epsilon > 0$, by the Lemma 4 we get a neighbourhood $U \in \aleph$, such that $\|L_t f - f\|_{p(\cdot)} < \epsilon$ for all $t \in U$. Then, for all $V \geqslant U$ with $V \in \mathrm{Im}$, we get, by using Fibuni's theorem, that

$$\rho_p\Big(\frac{\xi_V * f - f}{\epsilon}\Big) = \frac{1}{\varepsilon}\Big(\int\limits_{F_p}\Big|\int\limits_V \frac{f(t^{-1}x) - f(x)}{\mu(V)} d\mu(t)\Big|^{p(x)} d\mu(x) +$$
$$+ \operatorname*{ess\,sup}_{x\in F_p^c}\Big|\int\limits_V \frac{f(t^{-1}x) - f(x)}{\mu(V)} d\mu(t)\Big|\Big) \leqslant$$
$$\leqslant \frac{1}{\varepsilon}\frac{1}{\mu(V)}\Big\{\int_{F_p}\int\limits_V \Big|f(t^{-1}x) - f(x)\Big|^{p(x)} d\mu(t)d\mu(x) +$$
$$+ \operatorname*{ess\,sup}_{x\in F_p^c}\Big(\int\limits_V \Big|f(t^{-1}x) - f(x)\Big| d\mu(t)\Big)\Big\} =$$
$$= \frac{1}{\varepsilon}\frac{1}{\mu(V)}\Big\{\int\limits_V\Big(\int\limits_{F_p}\Big|L_t f - f\Big|^{p(x)} d\mu(x) + \operatorname*{ess\,sup}_{x\in F_p^c}|L_t f - f|\Big) d\mu(t)\Big\} =$$
$$= \frac{1}{\varepsilon}\frac{1}{\mu(V)}\int\limits_V \rho_p(L_t f - f)\, d\mu(t) = \frac{1}{\varepsilon}\rho_p(L_t f - f).$$

Since $\rho_p(L_t f - f) \leqslant \|L_t f - f\|_{p(\cdot)} < \varepsilon$, we get $\rho_p\big(\frac{\xi_V * f - f}{\epsilon}\big) \leqslant 1$. Thus, by the definition of the norm $\|\cdot\|_{p(\cdot)}$, we get $\|\xi_V * f - f\|_{p(\cdot)} < \epsilon$. Hence, $\{\xi_V\}_{V\in\aleph}$ is a left approximate identity in $\mathcal{L}^{p(\cdot)}(H)$. $\square$

In the similar approach, we can find the right approximate identity in $\mathcal{L}^{p(\cdot)}(H)$.

Now we shall discuss the condition under which the Banach algebra $\mathcal{L}^{p(\cdot)}(H)$ has an identity.

Theorem 3. *For a bounded exponent p, the variable Lebesgue algebra $\mathcal{L}^{p(\cdot)}(H)$ contains an identity iff H is discrete.*

Proof. Let $\mathcal{L}^{p(\cdot)}(H)$ be a variable Lebesgue algebra that contains an identity h. Then $f * h = h * f = f$ for all $f \in \mathcal{L}^{p(\cdot)}(H)$. Suppose that U is a neighbourhood of the identity $e \in H$ and $\varepsilon > 0$. Since $C_c(H)$ is dense in $\mathcal{L}^{p(\cdot)}(H)$, so $\exists\ f \in C_c(H)$, such that $supp(f) \subset U$ and $\|f * h - h\|_{p(\cdot)} < \varepsilon$. Since $f * h = f$, we have $\|f - h\|_{p(\cdot)} < \varepsilon$. Now,

$$\varepsilon > \|f - h\|_{p(\cdot)} \geqslant \rho_p(f-h) = \\ = \int\limits_U |f-h|^{p(x)}\, d\mu + \int\limits_{H\backslash U} |f-h|^{p(x)} d\mu \geqslant \int\limits_{H\backslash U} |h|^{p(x)} d\mu.$$

This implies that $|h(x)|^{p(x)}$ is zero for all $x \in H\backslash U$. But $1 \leqslant p(x) < \infty$, so $h(x) = 0$ for all $x \in H\backslash U$. Since U is any neighbourhood of the identity in H, so $supp(h) \subset \{e\}$ and $\mu(\{e\}) > 0$, as if $\mu(\{e\}) = 0$; then $h = 0$ a.e. on H, which is a contradiction to fact that h is the identity in $\mathcal{L}^{p(\cdot)}(H)$. Thus H is discrete.

Conversely, let H be a discrete group. In this case, μ is a counting measure on H. The characteristic function χ_e of $\{e\}$ belongs to $\mathcal{L}^{p(\cdot)}(H)$, and we have:

$$(\chi_e * f)(x) = \int\limits_H \chi_e(y) f(y^{-1}x) d\mu(y) = \sum_{y\in H} \chi_e(y) f(y^{-1}x) = f(x)$$

for all $f \in \mathcal{L}^{p(\cdot)}(H)$ and $x \in H$. Thus the function χ_e is an identity of the algebra. $\square$

Let $\mathcal{L}^{p(\cdot)}(H)$ be a variable Lebesgue algebra, where p is a bounded exponent. Then, using the existence of a left approximate identity in $\mathcal{L}^{p(\cdot)}(H)$, we perceive that the closed left ideal of $\mathcal{L}^{p(\cdot)}(H)$ is a left translation invariant.

Theorem 4. *Suppose p is a bounded exponent and μ is σ-finite. Let $\mathcal{L}^{p(\cdot)}(H)$ be a variable Lebesgue algebra and M be a closed linear subspace of $\mathcal{L}^{p(\cdot)}(H)$. Then M is a left ideal in $\mathcal{L}^{p(\cdot)}(H)$ iff $L_x(M) \subseteq M$ for all $x \in H$.*

Proof. Let $M \subseteq \mathcal{L}^{p(\cdot)}(H)$ be a left ideal. For any $f \in M$ and $\epsilon > 0$, using Theorem 2, we get the fact that $(e_V)_{V \in \mathrm{Im}}$ is a left approximate identity in $\mathcal{L}^{p(\cdot)}(H)$, so that $\|e_V * f - f\|_{p(\cdot)} < \epsilon$. Moreover, $(L_x e_V) * f \in M$, since $\mathcal{L}^{p(\cdot)}(H)$ is a left translation invariant and M is a left ideal. Thus, $\|L_x e_V * f - L_x f\|_{p(\cdot)} = \|e_V * f - f\|_{p(\cdot)} < \epsilon$. Therefore, $L_x f \in M$, since M is closed.

Conversely, suppose that M is a left translation invariant subspace of $\mathcal{L}^{p(\cdot)}(H)$, i.e., $L_x(M) \subseteq M$ for all $x \in H$. Let us prove that M is a left ideal. To prove this, we need to show that $j * i \in M$ for all $i \in M$ and $j \in \mathcal{L}^{p(\cdot)}(H)$. Suppose that there exists $i \in M$ and $j \in \mathcal{L}^{p(\cdot)}(H)$, such that $j * i \notin M$. Then, by the consequence of the Hahn-Banach theorem, $\exists$ a bounded linear functional Ψ on $\mathcal{L}^{p(\cdot)}(H))$, such that $\Psi(M) = \{0\}$ and $F(j * i) \neq 0$. Further, since p is a bounded exponent and μ is σ-finite, the dual space of $\mathcal{L}^{p(\cdot)}(H)$ is $\mathcal{L}^{p'(\cdot)}(H)$, where p' is the conjugate exponent of p. So, the bounded linear functional $\Psi \in \left(\mathcal{L}^{p(\cdot)}(H)\right)^*$ can be uniquely determine by $\varphi \in \mathcal{L}^{p'(\cdot)}(H)$, such as

$$\Psi(\xi) = \int_H \xi \varphi d\mu, \ \xi \in \mathcal{L}^{p(\cdot)}(H).$$

Therefore,

$$\begin{aligned}\Psi(j * i) &= \int_H \varphi(x)(j * i)(x) d\mu(x) = \int_H \varphi(x) \int_H \Big(j(y) i(y^{-1}x) d\mu(y)\Big) d\mu(x) = \\ &= \int_H j(y) \Big(\int_H \varphi(x) L_{y^{-1}} i(x) d\mu(x) \Big) d\mu(y) = \int_H j(y) \Psi(L_{y^{-1}} i) d\mu(y) = 0.\end{aligned}$$

So, $L_y f \in M$ and $\Psi(M) = \{0\}$, which contradicts our assumption that $\Psi(j * i) \neq 0$. This completes the proof. □

Corollary. *If M is a subspace of $\mathcal{L}^{p(\cdot)}(H)$ that is closed, then M a right ideal in $\mathcal{L}^{p(\cdot)}(H)$ iff $M \subseteq \mathcal{L}^{p(\cdot)}(H)$ is right translation invariant.*

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