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ON A SUM INVOLVING CERTAIN ARITHMETIC FUNCTIONS ON PIATETSKI–SHAPIRO AND BEATTY SEQUENCES

Abstract. Let c, α, $\beta \in \mathbb{R}$ be such that $1 < c < 2$, $\alpha > 1$ is irrational and with bounded partial quotients, $\beta \in [0, \alpha)$. In this paper, we study asymptotic behaviour of the summations of the form $\sum\limits_{n \leqslant N} \frac{f(\lfloor n^c \rfloor)}{\lfloor n^c \rfloor}$ and $\sum\limits_{n \leqslant N} \frac{f(\lfloor \alpha n + \beta \rfloor)}{\lfloor \alpha n + \beta \rfloor}$, where f is the Euler totient function ϕ, Dedekind function Ψ, sum-of-divisors function σ, or the alternating sum-of-divisors function σ_{alt}.

Key words: *arithmetic function, Beatty sequence, Piatetski–Shapiro sequence*

2020 Mathematical Subject Classification: *11N37, 11N69*

1. Introduction and results. Let f be any arithmetic function and $\lfloor t \rfloor$ denote the integral part of any $t \in \mathbb{R}$. There is an interesting problem on the sum involving the integer part of the form

$$\sum_{n \leqslant N} \frac{f(\lfloor \frac{x}{n} \rfloor) \lfloor \frac{x}{n} \rfloor}{;}$$

see, for example, [2], [7], [11], [12]. A popular example appeared in [2, Corollary 2.4]: Bordellés-Dai-Heyman-Pan-Shparlinski proved that

$$\sum_{n \leqslant x} \frac{\phi(\lfloor x/n \rfloor)}{\lfloor x/n \rfloor} = x \sum_{n=1}^{\infty} \frac{\phi(n)}{n^2(n+1)} + O(x^{1/2}). \tag{1}$$

Later, it has been improved by many authors, see [7], [11], [12]. For fixed real numbers α and β, the associated non-homogeneous Beatty sequence is the sequence of integers, defined by

$$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}.$$

For $1 < c \notin \mathbb{N}$, the Piatetski–Shapiro sequences are sequences of the form

$$\mathcal{N}^{(c)} := (\lfloor n^c \rfloor)_{n=1}^{\infty}.$$

The Beatty sequence and the Piatetski–Shapiro sequence are special sequences involving the integer part, which is very important in analytic number theory. Thus, it would be interesting to study the sums of

$$\sum_{n\leqslant N} \frac{\phi(\lfloor n^c \rfloor)}{\lfloor n^c \rfloor} \quad \text{and} \quad \sum_{n\leqslant N} \frac{\phi(\lfloor \alpha n + \beta \rfloor)}{\lfloor \alpha n + \beta \rfloor}. \tag{2}$$

Surprisingly, these summations (2) appeared in [6]. Deshouillers, Hassani, and Nasiri–Zare extended previous results on the Luca–Schinzel question in [5] to Piatetski–Shapiro sequences, and proved that for any real $c > 1$ the sequence $\left(\sum_{n\leqslant m} \frac{\phi(\lfloor n^c \rfloor)}{\lfloor n^c \rfloor}\right)_{m\geqslant 1}$ is dense modulo 1. Moreover, they also gave a challenging question about finding the values of c for which the sequence $\left(\sum_{n\leqslant m} \frac{\phi(\lfloor n^c \rfloor)}{\lfloor n^c \rfloor}\right)_{m\geqslant 1}$ is uniformly distributed modulo 1. Thus, the study of these summations (2) could give more informations to answer the question in [5]. Moreover, Ma and Sun [7] classified the function $\phi(n)/n$ as belonging to the same class of other functions, such as the Dedekind function $\Psi(n)/n$, sum-of-divisors function $\sigma(n)/n$, and the alternating sum-of-divisors function $\sigma_{alt}(n)/n$. Thus, in this paper, we will study the summations

$$\sum_{n\leqslant N} \frac{f(\lfloor n^c \rfloor)}{\lfloor n^c \rfloor},$$

and

$$\sum_{n\leqslant N} \frac{f(\lfloor \alpha n + \beta \rfloor)}{\lfloor \alpha n + \beta \rfloor},$$

where f is the Euler totient function ϕ, Dedekind function Ψ, sum-of-divisors function σ, and the alternating sum-of-divisors function σ_{alt}. Denote by $\mu(n)$ the Möbius function and by $\lambda(n)$ the Liouville function, respectively. From the following relations:

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d}, \quad \frac{\sigma_{alt}(n)}{n} = \sum_{d|n} \frac{\lambda(d)}{d}, \quad \frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}, \quad \frac{\Psi(n)}{n} = \sum_{d|n} \frac{\mu^2(d)}{d},$$

we define

$$g(n)=\begin{cases}1, & \text{for } f=\sigma;\\ \mu(n), & \text{for } f=\phi;\\ \lambda(n), & \text{for } f=\sigma_{alt};\\ \mu^2(n), & \text{for } f=\Psi.\end{cases}$$

In our results, we prove the following theorem:

Theorem 1. *If $f=id*g$, then we have*

$$\sum_{n\leqslant N}\frac{f(\lfloor n^c\rfloor)}{\lfloor n^c\rfloor}=N\sum_{n=1}^{\infty}\frac{g(n)}{n^2}+\begin{cases}O\big(N^{(c+4)/7}\big), & \text{for } 5/4\leqslant c<2;\\ O\big(N^{3/4}\big), & \text{for } 1<c\leqslant 5/4.\end{cases} \tag{3}$$

Moreover, by using the exponent pair method, we obtain the following theorem:

Theorem 2. *Let (κ,λ) be any exponent pair and $1<c<\frac{1}{\lambda}$. If $f=id*g$, then we have*

$$\sum_{n\leqslant N}\frac{f(\lfloor n^c\rfloor)}{\lfloor n^c\rfloor}=N\sum_{n=1}^{\infty}\frac{g(n)}{n^2}+O\big(N^{\frac{c\lambda+\kappa}{1+\kappa}}\log N\big). \tag{4}$$

By choosing the exponent pair

$$(\kappa,\lambda)=BA^4\Big(\frac{13}{84}+\epsilon,\frac{55}{84}+\epsilon\Big)=\Big(\frac{131}{289}+\epsilon,\frac{440}{867}+\epsilon\Big)$$

in Theorem 2 and combining with Theorem 1, we obtain the following Corollary.

Corollary. *If $f=id*g$, then we have*

$$\sum_{n\leqslant N}\frac{f(\lfloor n^c\rfloor)}{\lfloor n^c\rfloor}=N\sum_{n=1}^{\infty}\frac{g(n)}{n^2}+\begin{cases}O\big(N^{(c+4)/7}\big), & 327/260\leqslant c<2;\\ O\big(N^{(440c+393)/1260}\log N\big), & 1<c\leqslant 327/260.\end{cases}$$

Theorem 3. *Let $\alpha>1$ be irrational and with bounded partial quotients, $\beta\in[0,\alpha)$. If $f=id*g$, then we have*

$$\sum_{n\leqslant N}\frac{f(\lfloor \alpha n+\beta\rfloor)}{\lfloor \alpha n+\beta\rfloor}=N\sum_{n=1}^{\infty}\frac{g(n)}{n^2}+O\big(\sqrt{\alpha N+\beta}\log^3 N\big).$$

2. Lemmas and notation. Throughout this paper, implied constants in symbols O and $\ll$ may depend on the parameters, α, β, c, ϵ, but are absolute otherwise. For given functions F and G, the notations $F \ll G$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leqslant C|G|$ holds with some constant $C > 0$.

The proof of Theorem 1 makes use of the following estimate, due originally to Rieger [8]. It is for the number of integers n up to x, such that $\lfloor n^c \rfloor$ belongs to an arithmetic progression. This lemma is proved in [4].

Lemma 1. [4, Theorem 1] *For $1 < c < 2$, let x be a positive real number, and let q and a be two integers, such that $0 \leqslant a < q \leqslant x^c$. Then*

$$\sum_{\substack{n \leqslant x \\ \lfloor n^c \rfloor \equiv a \ (\mathrm{mod}\ q)}} 1 = \frac{x}{q} + \begin{cases} O\left(\frac{x^{(c+4)/7}}{q^{1/7}}\right), & \text{for } q < x^{c-5/4}; \\ O\left(\frac{x^{(c+1)/3}}{q^{1/3}}\right), & \text{for } x^{c-5/4} \leqslant q < x^{c-1/2}; \\ O\left(\frac{x^c}{q}\right), & \text{for } x^{c-1/2} \leqslant q < x^c. \end{cases}$$

On the other hand, using the method of Cao and Zhai in [3] can improve Rieger's result in [8]. It is used to study square-full and (k, r)-integers in Piatetski-Shapiro sequence in [9], [10], respectively. Thus, it would be interesting to apply the method of Cao and Zhai in the present paper.

Lemma 2. [3, Lemma 3] *Let $y > 0$, $X > 1$, $0 \leqslant \sigma < 1$, $g(n) = (n + \sigma)^\gamma$. Then, for any exponent pair (κ, λ),*

$$\sum_{n \sim X} \psi(yg(n)) \ll y^{\frac{\kappa}{1+\kappa}} X^{\frac{\lambda+\gamma\kappa}{1+\kappa}} + y^{-1} X^{1-\gamma}.$$

The following lemma is the result of A. V. Begunts and D. V. Goryashin [1]; this is the main ingredient of the proof of Theorem 3.

Lemma 3. *For an irrational $\alpha > 1$ with bounded partial quotients, $\beta \in [0; \alpha)$, and a positive integer $d \geqslant 2, 0 \leqslant a < d$, we have:*

$$\sum_{\substack{n \leqslant x \\ \lfloor \alpha n + \beta \rfloor \equiv a (\bmod d)}} 1 = \frac{x}{d} + O(d \log^3 x) \quad \text{as } x \to \infty.$$

For a growing difference d, the result is non-trivial, provided that $d \ll \sqrt{x} \log^{-3/2-\varepsilon} x$, for $\varepsilon > 0$.

3. Proof of Theorems.

Proof of Theorem 1. For $1 < c < 2$, we have:

$$\sum_{n\leqslant N}\frac{f(\lfloor n^c\rfloor)}{\lfloor n^c\rfloor}=\sum_{n\leqslant N}\sum_{d\mid\lfloor n^c\rfloor}\frac{g(d)}{d}=\sum_{d\leqslant N^c}\frac{g(d)}{d}\sum_{\substack{n\leqslant N\\ \lfloor n^c\rfloor\equiv 0\ (\bmod d)}}1.$$

Using Lemma 1, we get, for all $d\geqslant 1$, $|g(d)|\leqslant 1$,

$$\begin{aligned}\sum_{n\leqslant N}\frac{f(\lfloor n^c\rfloor)}{\lfloor n^c\rfloor}&=N\sum_{d\leqslant N^c}\frac{g(d)}{d^2}+O\Big(N^{(c+4)/7}\Big|\sum_{d\leqslant N^{c-5/4}}\frac{1}{d^{8/7}}\Big|\Big)+\\&+O\Big(N^{(c+1)/3}\Big|\sum_{N^{c-5/4}<d\leqslant N^{c-1/2}}\frac{1}{d^{4/3}}\Big|\Big)+O\Big(N^c\Big|\sum_{N^{c-1/2}<d\leqslant N^c}\frac{1}{d^2}\Big|\Big)=\\&=N\sum_{d\leqslant N^c}\frac{g(d)}{d^2}+O\Big(N^{(c+4)/7}\Big)+O\Big(N^{3/4}\Big).\end{aligned}\tag{5}$$

Since $\sum\limits_{d=1}^{\infty}\frac{g(d)}{d^2}$ converges, we have

$$N\sum_{d\leqslant N^c}\frac{g(d)}{d^2}=N\sum_{d=1}^{\infty}\frac{g(d)}{d^2}+O\Big(N^{1-c}\Big).\tag{6}$$

Inserting (6) in (5), Theorem 1 follows. □

Proof of Theorem 2. Note that $\lfloor n^c\rfloor\in\mathbb{Z}$, if and only if there is an integer m, such that $m^\gamma\leqslant n<(m+1)^\gamma$, where $\gamma=\frac{1}{c}$. Therefore,

$$\begin{aligned}\sum_{n\leqslant N}\frac{f(\lfloor n^c\rfloor)}{\lfloor n^c\rfloor}&=\sum_{m\leqslant N^c}\frac{f(m)}{m}\Big(\lfloor -m^\gamma\rfloor-\lfloor -(m+1)^\gamma\rfloor\Big)+O(1)=\\&=\sum_{m\leqslant N^c}\frac{f(m)}{m}\Big((m+1)^\gamma-m^\gamma\Big)+E_c(N),\end{aligned}$$

where

$$E_c(N)=\sum_{m\leqslant N^c}\frac{f(m)}{m}\Big(\psi(-(m+1)^\gamma)-\psi(-m^\gamma)\Big)+O(1),\tag{7}$$

and $\psi(z) = z - \lfloor z \rfloor - \frac{1}{2}$. From $(m+1)^\gamma - m^\gamma = \gamma m^{\gamma-1} + O(m^{\gamma-2})$, we have:

$$\sum_{m\leqslant N^c} \frac{f(m)}{m}\Big((m+1)^\gamma - m^\gamma\Big) = \gamma \sum_{m\leqslant N^c} \frac{f(m)}{m^{2-\gamma}} + O\Big(\Big|\sum_{m\leqslant N^c} \frac{f(m)}{m^{3-\gamma}}\Big|\Big).$$

Noting that

$$\sum_{n\leqslant x} \frac{f(n)}{n} = \sum_{n\leqslant x}\sum_{d|n} \frac{g(d)}{d} = \sum_{d\leqslant x} \frac{g(d)}{d} \sum_{\substack{n\leqslant x \\ n\equiv 0(\bmod d)}} 1 =$$
$$= \sum_{d\leqslant x} \frac{g(d)}{d}(\frac{x}{d} + O(1)) =$$
$$= \sum_{d=1}^{\infty} \frac{g(d)}{d^2}x + O(\log x),$$

and using the partial summation, we have

$$\sum_{m\leqslant N^c} \frac{f(m)}{m}\Big((m+1)^\gamma - m^\gamma\Big) = N\sum_{d=1}^{\infty} \frac{g(d)}{d^2} + O(N^{1-c}\log N).$$

Now it remains to bound the sum in (7). We write:

$$\sum_{m\leqslant N^c} \frac{f(m)}{m}\Big(\psi(-(m+1)^\gamma) - \psi(-m^\gamma)\Big) =$$
$$= \sum_{m\leqslant N^c}\sum_{d|m} \frac{g(d)}{d}\Big(\psi(-(m+1)^\gamma) - \psi(-m^\gamma)\Big) =$$
$$= \sum_{de\leqslant N^c} \frac{g(d)}{d}\Big(\psi(-(de+1)^\gamma) - \psi(-d^\gamma e^\gamma)\Big) =$$
$$= \sum_{d\leqslant N^c} \frac{g(d)}{d} \sum_{e\leqslant N^c d^{-1}} \Big(\psi(-(de+1)^\gamma) - \psi(-d^\gamma e^\gamma)\Big).$$

In view of Lemma 2, we have

$$\sum_{d\leqslant N^c} \frac{g(d)}{d} \sum_{e\leqslant N^c d^{-1}} \Big(\psi(-(de+1)^\gamma) - \psi(-d^\gamma e^\gamma)\Big) \ll$$
$$\ll \log N \sum_{d\leqslant N^c} \frac{1}{d}\Big((d^\gamma)^{\frac{\kappa}{1+\kappa}}\Big(\frac{N^c}{d}\Big)^{\frac{\lambda+\gamma\kappa}{1+\kappa}} + (d^\gamma)^{-1}\Big(\frac{N^c}{d}\Big)^{1-\gamma}\Big) \ll$$

$$\ll N^{\frac{c\lambda+\kappa}{1+\kappa}} \log N \sum_{d \leqslant N^c} d^{-1-\frac{\lambda}{1+\kappa}} + N^{c-1} \log N \sum_{d \leqslant N^c} d^{-2} \ll$$
$$\ll N^{\frac{c\lambda+\kappa}{1+\kappa}} \log N + N^{c-1} \log N.$$

Thus, we have

$$\sum_{n \leqslant N} \frac{f(\lfloor n^c \rfloor)}{\lfloor n^c \rfloor} = N \sum_{d=1}^{\infty} \frac{g(d)}{d^2} + O\Big(\max\{N^{\frac{c\lambda+\kappa}{1+\kappa}} \log N, N^{c-1} \log N\}\Big).$$

The complete proof follows from $\frac{c\lambda+\kappa}{1+\kappa} > c-1$ when $1 < c < 2$. □

Proof of Theorem 3. The proof follows similarly to that of Theorem 1. For irrational $\alpha > 1$ with bounded partial quotients, $\beta \in [0, \alpha)$, we have:

$$\sum_{n \leqslant N} \frac{f(\lfloor \alpha n + \beta \rfloor)}{\lfloor \alpha n + \beta \rfloor} = \sum_{n \leqslant N} \sum_{d | \lfloor \alpha n + \beta \rfloor} \frac{g(d)}{d} =$$
$$= \sum_{d \leqslant \alpha N + \beta} \frac{g(d)}{d} \sum_{\substack{n \leqslant N \\ \alpha n + \beta \equiv 0 (\bmod d)}} 1 =$$
$$= \Big(\sum_{d \leqslant \sqrt{\alpha N + \beta}} + \sum_{\sqrt{\alpha N + \beta} < d \leqslant \alpha N + \beta} \Big) \frac{g(d)}{d} \sum_{\substack{n \leqslant N \\ \alpha n + \beta \equiv 0 (\bmod d)}} 1.$$

In view of Lemma 3, we have

$$\sum_{d \leqslant \sqrt{\alpha N + \beta}} \frac{g(d)}{d} \sum_{\substack{n \leqslant N \\ \alpha n + \beta \equiv 0 (\bmod d)}} 1 = \sum_{d \leqslant \sqrt{\alpha N + \beta}} \frac{g(d)}{d} \Big(\frac{N}{d} + O(d \log^3 N) \Big) =$$
$$= N \sum_{d \leqslant \sqrt{\alpha N + \beta}} \frac{g(d)}{d^2} + O\Big(\sqrt{\alpha N + \beta} \log^3 N \Big).$$

Note that

$$\sum_{\sqrt{\alpha N + \beta} < d \leqslant \alpha N + \beta} \frac{g(d)}{d} \sum_{\substack{n \leqslant N \\ \alpha n + \beta \equiv 0 (\bmod d)}} 1 \ll N \sum_{\sqrt{\alpha N + \beta} < d \leqslant \alpha N + \beta} \frac{1}{d^2} \ll N^{1/2},$$

and $\sum\limits_{d=1}^{\infty} \frac{g(d)}{d^2}$ converges. This proves Theorem 3. □

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