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ON CAUCHY PROBLEM SOLUTION FOR A HARMONIC FUNCTION IN A SIMPLY CONNECTED DOMAIN

Abstract. Here we present an investigation of the Cauchy problem solvability for the Laplace equation in a simply connected plane domain. The investigation is reduced to solution of two singular integral equations. If the problem is resolvable, its solution can be restored via the integral Cauchy formula. Examples of the solvable and unsolvable problems are presented. The construction involves the auxiliary approximate conformal mapping.

Key words: *Cauchy problem, integral equation, holomorphic function, spline, approximate solution*

2020 Mathematical Subject Classification: *30-08*

1. Introduction. Recall the problem formulation. Let D be a simply connected domain in the XY plane, $\Gamma = \{(x(s), y(s)), 0 \leqslant s \leqslant l\}$ be a one-component smooth arc of the boundary of D with the natural parameter s. Two real-valued functions $\phi(s)$ and $\psi(s)$, $0 \leqslant s \leqslant l$, are given. The problem is to find the harmonic in D and continuous up to the boundary curve Γ function $h(x, y)$ with the properties:

$$h(x(s), y(s)) = \phi(s), \quad \frac{\partial h(x,y)}{\partial n}|_{x=x(s), y=y(s)} = \psi(s), \tag{1}$$

where n is the exterior normal to Γ.

The deep analysis of the problem, its extension, and an observation of the applications are presented in [12], [6]. Reference [6] also indicates the connection of the Cauchy problem with the problem of analytic continuation of functions. It is well-known that this problem is ill-posed. This ill-posed Cauchy problem has wide applications, for example, for interpretation of gravitational and magnetic fields. Usually the problem is solved by the method of regularization [12], [6], [11], [10] or with the help of Carleman's formula [12], [2], [4], [5]. We suggest the method of analysis and approximate solution of the problem applying reduction to two singular integral equations.

2. Analytic continuation. Let $g(x,y)$ be the harmonic in D function, conjugate to $h(x,y)$. Due to (1),

$$g(x(s), y(s)) = \int \psi(s)ds + A \equiv \chi(s) + A, \quad s \in [0, l],$$

with an arbitrary real A. So, the solution of the Cauchy problem is reduced to construction of the holomorphic in D function $f(z) = h(x, y) + \mathrm{i}g(x, y)$, $z = x + \mathrm{i}y$, via its values on the one-component boundary segment. So, the problem can be presented as the problem of analytic continuation of the holomorphic in D function $f(z)$ via the values given at the points of the oriented curve Γ to a domain D on the left of Γ. We call $L = \{(\phi(s), \chi(s)),\ 0 \leqslant s \leqslant l\}$ the curve corresponding to the curve Γ.

Here we assume Γ to be an oriented smooth curve. Let $\phi(s)$, $s \in [0, l]$, be from the Hölder class and $\psi(s)$, $s \in [0, l]$, be continuous. So, the function $f(z(s)) = \phi(s) + \mathrm{i}\chi(s)$, $s \in [0, l]$, is the Hölder class function. Clearly, if the problem of analytic continuation of the holomorphic function $f(z)$ from the boundary segment to the adjacent domain D has a solution, then this solution is unique, due to the well-known theorem of analytic continuation of a function from its zero values on the rectifiable part of the boundary [7].

Consider the domain D that is to the left of Γ. Assume that this domain possesses the smooth boundary $\Gamma \cup \Gamma'$, where $\Gamma' = x(s) + \mathrm{i}y(s)$, $s \in [l, T]$, $x(0) + \mathrm{i}y(0) = x(T) + \mathrm{i}y(T)$. Let $f(z(s))$, $s \in [0, T]$, be the boundary values of $f(z)$, $z \in \Gamma \cup \Gamma'$. Here $f(z(s)) = \phi(s) + \mathrm{i}\chi(s))$, $0 \leqslant s \leqslant l$, is the known function, which defines the corresponding to Γ curve L and $f(z(s)) = \tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s)), l \leqslant s \leqslant T$, is unknown. We denote by L' the curve corresponding to Γ'. If we know $f(z(s))$, $s \in [0, T]$, then the function $f(z)$, $z \in D$, can be restored via the integral Cauchy formula and the solution of the Cauchy problem has the form $h(x, y) = \mathrm{Re}[f(z)]$, $x + \mathrm{i}y = z \in D$.

3. Solvability analysis. The solvability analysis is the main part of our method because if the problem is resolvable then its solution is the Cauchy integral with the known density.

Due to [1], [8], the necessary and sufficient condition for $f(z(s))$, $s \in [0, T]$, to be the boundary values of the holomorphic in D function is the relation

$$f(z(s)) = \frac{1}{\pi \mathrm{i}} \int\limits_0^T \frac{f(z(t))z'(t)}{z(t)-z(s)} dt, \quad s \in [0,T]. \tag{2}$$

We separate the known and unknown functions at the different segments and obtain from equality (2) the following two relations:

$$\phi(s) + \mathrm{i}\chi(s) = \frac{1}{\pi \mathrm{i}} \int\limits_0^l \frac{(\phi(t)+\mathrm{i}\chi(t))z'(t)}{z(t)-z(s)} dt + \\ + \frac{1}{\pi \mathrm{i}} \int\limits_l^T \frac{(\tilde{\phi}(t)+\mathrm{i}\tilde{\chi}(t))z'(t)}{z(t)-z(s)} dt, s \in [0,l], \tag{3}$$

$$\tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s) = \frac{1}{\pi \mathrm{i}} \int\limits_0^l \frac{(\phi(t)+\mathrm{i}\chi(t))z'(t)}{z(t)-z(s)} dt + \\ + \frac{1}{\pi \mathrm{i}} \int\limits_l^T \frac{(\tilde{\phi}(t)+\mathrm{i}\tilde{\chi}(t))z'(t)}{z(t)-z(s)} dt, s \in [l,T]. \tag{4}$$

We can consider relations (3) and (4) as two singular integral equations [8], [1]. The free term $\frac{1}{\pi \mathrm{i}} \int\limits_0^l \frac{(\phi(t)+\mathrm{i}\chi(t))z'(t)}{z(t)-z(s)} dt$ of equation (4) contains the known functions $\phi(s)$ and $\chi(s)$ and is also known. So, we can solve (4) with respect to $\tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s)$, $s \in [l,T]$. After finding $\tilde{\phi}(s) + \mathrm{i}\tilde{\chi}(s)$, we check if the function $\phi(s) + \mathrm{i}\chi(s)$, $s \in [0,l]$, satisfies equation (3). Now, if $\phi(s) + \mathrm{i}\chi(s)$ meets (3), then relation (2) holds, so the function $f(z(t))$ is the boundary value of a holomorphic in D function. Therefore, we can formulate the following

Theorem 1. *Let D be a one-connected domain with the smooth boundary $\Gamma \cup \Gamma'$, where $\Gamma = \{z(s) = x(s) + \mathrm{i}y(s), s \in [0,l]\}$, s being the natural parameter of Γ, $\Gamma' = \{z(s) = x(s) + \mathrm{i}y(s), s \in [l,T]\}$. Let the data defined by formula (1) of the Cauchy problem be given at the points of Γ. Let $\phi(s)$, $s \in [0,l]$, be of the Hölder class and $\psi(s)$, $s \in [0,l]$, be continuous.*

Then this Cauchy problem is solvable if and only if the known function $\phi(s)+\mathrm{i}\int_0^s\psi(t)dt$, $s\in[0,l]$, *satisfies the relation*

$$\phi(s)+\mathrm{i}\int_0^s\psi(t)dt=\frac{1}{\pi\mathrm{i}}\int_0^l\frac{(\phi(t)+\mathrm{i}\int_0^t\psi(\tau)d\tau)z'(t)}{z(t)-z(s)}dt+$$
$$+\frac{1}{\pi\mathrm{i}}\int_l^T\frac{(\tilde\phi(t)+\mathrm{i}\tilde\chi(t))z'(t)}{z(t)-z(s)}dt,\quad s\in[0,l],$$

where $\tilde\phi(s)+\mathrm{i}\tilde\chi(s)$, $s\in[l,T]$, *is the solution of the singular integral equation*

$$\tilde\phi(s)+\mathrm{i}\tilde\chi(s)=\frac{1}{\pi\mathrm{i}}\int_l^T\frac{(\tilde\phi(t)+\mathrm{i}\tilde\chi(t))z'(t)}{z(t)-z(s)}dt+$$
$$+\frac{1}{\pi\mathrm{i}}\int_0^l\frac{(\phi(t)+\mathrm{i}\int_0^t\psi(\tau)d\tau)z'(t)}{z(t)-z(s)}dt,\quad s\in[l,T].$$

The solution of the solvable Cauchy problem has the form

$$h(x,y)=\mathrm{Re}\Big[\frac{1}{2\pi\mathrm{i}}\int_0^l\frac{(\phi(t)+\mathrm{i}\int_0^l\psi(\tau)d\tau)z'(t)}{z(t)-x-\mathrm{i}y}dt+\frac{1}{2\pi\mathrm{i}}\int_l^T\frac{(\tilde\phi(t)+\mathrm{i}\tilde\chi(t))z'(t)}{z(t)-x-\mathrm{i}y}dt\Big].$$

4. Application of the auxiliary conformal mapping. We can overcome the difficulties connected with solution of the singular integral equation by integration over an arbitrary curve and calculation of the Cauchy principal value integral, if we reduce this calculation to integration over circle segments. According to the Riemann mapping theorem, the problem of analytic continuation of the analytic function $f(z)$ to an adjancent domain can be reduced to restoring the holomorphic in the unit disk function from its values on the upper semicircle. Let D be the domain to the left of the curve Γ. Let this domain possess the C^1-smooth boundary $\Gamma\cup\Gamma'$. Construct the analytic function $z=F(\zeta)$ that maps conformally the unit disk $|\zeta| < 1$ onto D, so that the ends of Γ have the preimages 1 and -1. Obtain the function $s = s(\theta)$, $\theta \in (0, \pi)$, then compare the expressions of the curve Γ in terms of parameter s in the form of $x(s) + \mathrm{i}y(s)$, $s \in (0, l)$, with its representation as $F(e^{i\theta})$, $\theta \in (0, \pi)$.

Consider $h(x(s(\theta)), y(s(\theta))) + \mathrm{i}g(x(s(\theta)), y(s(\theta))) \equiv u(\theta) + \mathrm{i}v(\theta)$, $\theta \in [0, \pi]$. Let $\tilde{u}(\theta) + \mathrm{i}\tilde{v}(\theta)$, $\theta \in (\pi, 2\pi)$, be the values of the analytic in the unit disk $|\zeta| < 1$ function $w = f(F(\zeta))$ at the points of the lower semicircle.

We turn to the analogue of formula (2) for the boundary values of the holomorphic function $f(F(\zeta))$, separate the values at the semicircles, and obtain the following relations similar to (4) and (3):

$$\tilde{u}(\theta) + \mathrm{i}\tilde{v}(\theta) = \frac{1}{\pi}\int_0^{\pi} \frac{u(\tau) + \mathrm{i}v(\tau)}{e^{\mathrm{i}\tau} - e^{\mathrm{i}\theta}} e^{\mathrm{i}\tau} d\tau + \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{\tilde{u}(\tau) + \mathrm{i}\tilde{v}(\tau)}{e^{\mathrm{i}\tau} - e^{\mathrm{i}\theta}} e^{\mathrm{i}\tau} d\tau, \theta \in [\pi, 2\pi], \tag{5}$$

and

$$u(\theta) + \mathrm{i}v(\theta) = \frac{1}{\pi}\int_0^{\pi} \frac{u(\tau) + \mathrm{i}v(\tau)}{e^{\mathrm{i}\tau} - e^{\mathrm{i}\theta}} e^{\mathrm{i}\tau} d\tau + \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{\tilde{u}(\tau) + \mathrm{i}\tilde{v}(\tau)}{e^{\mathrm{i}\tau} - e^{\mathrm{i}\theta}} e^{\mathrm{i}\tau} d\tau, \theta \in [0, \pi]. \tag{6}$$

The second integral in the right-hand side of equation (5) and the first integral at the right-hand side of equation (6) are the Cauchy principal value integrals [8], [1] over the unit semicircles. Values of the coefficients in singular integral equations (5) and (6) do not allow us to regularize these equations by reducing them to the Fredholm equations with the help of an auxiliary boundary-value problem solution [8], [1], so we have to find the approximate solutions.

5. Method of analysis and solution applying conformal mapping. The approximate calculation of the singular principal value Cauchy integral over the upper unit semicirle

$$\frac{1}{\pi \mathrm{i}}\int_0^{\pi} \frac{f(e^{\mathrm{i}\theta})\mathrm{i}e^{\mathrm{i}\theta} d\theta}{e^{\mathrm{i}\theta} - e^{\mathrm{i}t_k}}$$

at the intermediate node $t_k = \pi k/N - \pi/(2N)$, $k = 1, \ldots, N$, can be performed by application of the uniform fragmentation of the semicircle via the following approximation formula:

$$\frac{1}{\pi \mathrm{i}}\int_0^{\pi}\frac{f(e^{\mathrm{i}\theta})\mathrm{i}e^{\mathrm{i}\theta}d\theta}{e^{\mathrm{i}\theta}-e^{\mathrm{i}t_k}} \approx \frac{1}{\pi \mathrm{i}}\sum_{j=1,j\neq k}^{N} f\left(\frac{\pi j}{N}-\frac{\pi}{2N}\right)\log\frac{e^{\pi \mathrm{i}(2(j-k)+1)/(2N)}-1}{e^{\pi \mathrm{i}(2(j-k-1)+1)/(2N)}-1}+$$

$$+\frac{1}{\pi \mathrm{i}} f\left(\frac{\pi k}{N}-\frac{\pi}{2N}\right)\log\frac{e^{\pi \mathrm{i}/(2N)}-1}{e^{-\pi \mathrm{i}/(2N)}-1} = \frac{1}{\pi \mathrm{i}}\sum_{j=1}^{k-1} f\left(\frac{\pi j}{N}-\frac{\pi}{2N}\right)\left[\frac{\pi \mathrm{i}}{2N}+\right.$$

$$\left.+\log\frac{\sin(\pi(2(k-j)-1)/(4N))}{\sin(\pi(2(k-j+1)-1)/(4N))}\right] + f\left(\frac{\pi k}{N}-\frac{\pi}{2N}\right)\frac{1}{2N}+$$

$$+\frac{1}{\pi \mathrm{i}}\sum_{j=k+1}^{N} f\left(\frac{\pi j}{N}-\frac{\pi}{2N}\right)\left[\frac{\pi \mathrm{i}}{2N}+\log\frac{\sin(\pi(2(j-k)+1)/(4N))}{\sin(\pi(2(j-k-1)+1)/(4N))}\right].$$

So, we can replace either of singular integral equations (5) or (6) with the linear system over the values of the unknown functions at N intermediate nodes with the corresponding $N \times N$ matrix. Let us investigate the resolvability of this finite system for sufficiently large N. Let N tend to infinity. The finite system tends to the infinite one with the matrix $I-\frac{1}{\pi \mathrm{i}}M$, $M=[a_{kj}]_{k,j}^{\infty}$, where $a_{kj} = -\log(1+2/(2k-2j-1))$ if $j<k$, $a_{kj} = \log(1+2/(2j-2k-1))$ if $j>k$, $a_{kk}=0$, I is the infinite identity matrix. The L_2-norm of this matrix can be compared to that of the analogue of the Hilbert matrix [9].

If M^* is the conjugate matrix of M, then the L_2-norm of the product $M \cdot M^*$ is strictly less than that of the product $H \cdot H^*$, where $H = \left[\frac{1}{j-k-1/2}\right]_{j,k=1}^{\infty}$ due to the inequality $\log(1+x) \leqslant x$, $x>0$, and we can establish the estimates of the elements of the matrix M by those of H. Therefore, due to the well-known results ([9], p.17), since the L_2-norm of the matrix H equals π, the norm of the matrix M is strictly less than π. So, the infinite matrix $I-\dfrac{1}{\pi \mathrm{i}}M$ is invertible, and the corresponding finite system with respect to $f(e^{\mathrm{i}t_k})$, $k=1,\dots,N$, is also resolvable for sufficiently large N. Finally, we conclude that the singular integral equations (5) and (6) can be resolved approximately.

We solve the singular integral equation (5) with respect to $\tilde{u}(\theta)+\mathrm{i}\tilde{v}(\theta)$.

So we reconstruct the values of $\tilde{u}(\theta)+\mathrm{i}\tilde{v}(\theta)$, $\theta \in [\pi, 2\pi]$, and, therefore, we obtain the linear spline curve L' as the curve corresponding to Γ'.

Now we turn to singular integral equation (6) and check if the known function $u(\theta)+\mathrm{i}v(\theta)$, $\theta \in [0,\pi]$, meets this equation, or we solve (6) with respect to $u(\theta)+\mathrm{i}v(\theta)$, $\theta \in [0,\pi]$. If the corresponding curve L'' coincides with the initial curve L, then the function $f(F(\zeta))$ is analytic in the unit disk, so, the function $f(z)$ is analytic in the domain D. If the curve L'' found as the solution of (6) does not coincide with the curve L, then the function $f(z)$ possesses a singular point in the domain D, because the boundary values of the function $f(F(\zeta))$ do not satisfy the necessary and sufficient conditions (5) and (6) for $f(F(\zeta))$ to be analytic in the unit disk. So, the corresponding Cauchy problem is not resolvable in the domain D.

If the curves L and L'' coincide, then the function $f(F(\zeta))$ is analytic in the unit disk and can be restored via the formula

$$f(F(\zeta)) = \frac{1}{2\pi}\int\limits_0^{\pi} \frac{u(\tau)+\mathrm{i}v(\tau)}{e^{\mathrm{i}\tau}-\zeta} e^{\mathrm{i}\tau} d\tau + \frac{1}{2\pi}\int\limits_{\pi}^{2\pi} \frac{\tilde{u}(\tau)+\mathrm{i}\tilde{v}(\tau)}{e^{\mathrm{i}\tau}-\zeta} e^{\mathrm{i}\tau} d\tau, \ \ |\zeta|<1,$$

and the restriction of f to the boundary $f(F(e^{i}\theta))$, $\theta \in [0, 2\pi]$, is a continuous function with almost everywhere finite derivative.

Now the solution of the corresponding Cauchy problem for $(x, y) \in D$ takes the form

$$h(x,y) = \mathrm{Re}\Big(\frac{1}{2\pi}\int\limits_0^{\pi} \frac{u(\tau)+\mathrm{i}v(\tau)}{e^{\mathrm{i}\tau}-F^{-1}(x+\mathrm{i}y)} e^{\mathrm{i}\tau} d\tau + \frac{1}{2\pi}\int\limits_{\pi}^{2\pi} \frac{\tilde{u}(\tau)+\mathrm{i}\tilde{v}(\tau)}{e^{\mathrm{i}\tau}-F^{-1}(x+\mathrm{i}y)} d\tau\Big).$$

Note that in practice the curves L'' and L can differ even when the analytic continuation problem is resolvable due to the approximate property of solution of the singular integral equations. It means that the approximate solution $\bar{u}(\theta) + \mathrm{i}\bar{v}(\theta)$, $\theta \in [0, \pi]$, of the singular equation (6) with the function $\tilde{u}(\theta) + \mathrm{i}\tilde{v}(\theta)$, $\theta \in [\pi, 2\pi]$, which is the approximate solution of the singular equation (5), does not coincide with the given function $u(\theta) + \mathrm{i}v(\theta)$, $\theta \in [0, \pi]$. One can conclude that the problem is resolvable if these functions and the corresponding curves approach each other in respective points, when we correct the approximate solutions by increasing the number of nodes.

6. Examples.

Example 1. Let Γ be the upper half of the unit circle $(\cos(t), \sin(t))$, $t \in [0, \pi]$. Consider the curve $(\cos(t), \sin(t) - \frac{1}{4}\sin(4t))$ for $t \in [0, \pi]$ as L. The initial data for the corresponding Cauchy problem take the form $\phi(t) = \cos(t)$, $\psi(t) = \cos(t) - \cos(4t)$. Let Γ' be the lower half of the unit circle. Here we do not need the additional conform mapping $z = F(\zeta)$ of the unit disk, because D is the unit disk itself and Γ and Γ' are the corresponding semicircles.

Solution of equation (2) allows us to reconstruct the curve L'. Solution of equation (3) allows us to reconstruct the curve L''. Note that the solution L'' of (3) does not coincide with the initial part of the boundary curve L. Fig. 1 shows the domain D with its boundary parts (Γ, Γ') and also L as the image of Γ and the curves L' and L'' as the solutions of equations (2) and (3).

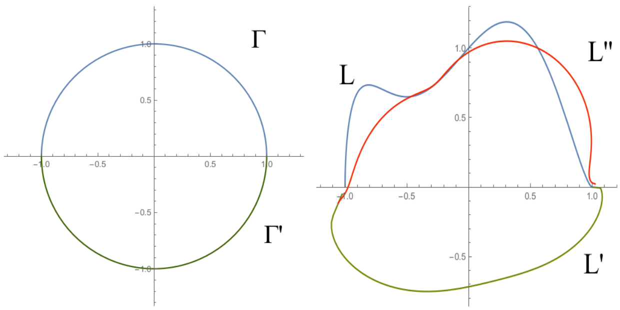

Figure 1: Approximation of the closed contour. Blue lines are Γ and L. Green lines mark the curve Γ' and the corresponding curve L' found from (2). Red line is the curve L'' reconstructed from relation (3).

Note that the solution L'' of (3) does not coincide with the boundary curve L corresponding to Γ.

So the solutions of systems (2) and (3) do not agree.

Example 2. Consider the curve Γ as a part of the unit circle e^{it}, $t \in [0, \pi/4]$, and the curve L given by the relation $(\cos(t), \sin(t)+0.25\sin(4t))$, $t \in [0, \pi/4]$. The curve Γ' is the part of the ellipse

$$x(t) = \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)(\cos(t) - \sin(t)),$$

$$y(t) = \frac{\tan(\pi/8)}{\sqrt{2}} + \frac{\tan(\pi/8)(\sqrt{2}-1)\cos(t)}{2} + \frac{\sin(t)}{2}, \quad t \in [\pi/4, 7\pi/4],$$

that complements Γ to the smooth boundary of the domain D shown on Fig. 2. We apply the method of conformal mapping of the unit disk on the domain D presented in [3]. After we construct the corresponding function $u(\theta) + iv(\theta)$, $\theta \in [0, \pi]$, we solve equation (2) and restore the function $\tilde{u}(\theta) + i\tilde{v}(\theta)$, $\theta \in [\pi, 2\pi]$, so we obtain the curve L'. Now, after we solve (3) with respect to $u(\theta) + iv(\theta)$, we are able to draw the curve L'' close to initial curve L. Fig. 2 also shows the curves L, L' and L''. The curves L and L'' are close to each other, they do not coincide absolutely due to the approximate conformal mapping and approximate solution of equations (2) and (3). We observe behaviour of the curve L'' correcting the auxiliary approximate conformal mapping and solving the singular equations. If the curves L and L'' approach each other, we conclude that $L = L''$. If the corresponding curves do not approach each other, we conclude that $L \neq L''$.

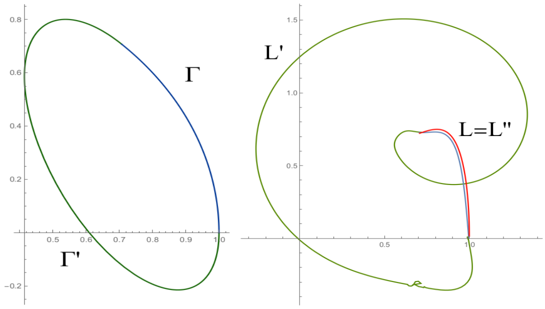

Figure 2: Approximation of the closed contour. Blue lines are Γ and L. Green lines mark the curve Γ' and the corresponding curve L' found from (2). Red line is the curve L'' reconstructed from relation (3).

The main point here is that in the first example we take the domain D that contains the point $z = 0$ and the given boundary values of the analytic function on the curve Γ are the boundary values of the function $z - \frac{1}{8}(z^4 - z^{-4})$, which has the pole at $z = 0$. In the second example, the domain D does not contain $z = 0$ altogether the given boundary data is the boundary values of the analytic in D function $z + \frac{1}{8}(z^4 - z^{-4})$.

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