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EXPONENTIAL APPROXIMATION OF FUNCTIONS IN LEBESGUE SPACES WITH MUCKENHOUPT WEIGHT

Abstract. Using a transference result, several inequalities of approximation by entire functions of exponential type in $\mathcal{C}(\mathbf{R})$, the class of bounded uniformly continuous functions defined on $\mathbf{R} \coloneqq (-\infty, +\infty)$, are extended to the Lebesgue spaces $L^p(\varrho dx)$ $1 \leqslant p < \infty$ with Muckenhoupt weight ϱ. This gives us a different proof of Jackson type direct theorems and Bernstein-Timan type inverse estimates in $L^p(\varrho dx)$. Results also cover the case $p = 1$.

Key words: *Lebesgue spaces, Muckenhoupt weight, entire functions of exponential type, one-sided Steklov operator, best approximation, direct theorem, inverse theorem, modulus of smoothness, Marchaud-type inequality, K-functional.*

2020 Mathematical Subject Classification: *41A10, 41A25, 41A27, 41A65, 41A81*

1. Introduction. An entire function $f(z)$ is called of exponential type $\sigma \in [0, \infty)$ (briefly e.f.e.t$\leqslant \sigma$) if

$$\limsup_{|z|=r\to\infty}(r^{-1}\ln(\left(\max_{|z|=r}|f(z)|\right)) \leqslant \sigma.$$

Sometimes e.f.e.t$\leqslant \sigma$ are also recalled as band-limited functions. See, e.g., paper [4].

Studies on e.f.e.t$\leqslant \sigma$ have intensified with problems related to approximation of non-periodic continuous functions (see Bernstein's paper [8] of the year 1912) defined on the real axis $\mathbf{R} := (-\infty, +\infty)$.

It is well known that trigonometric polynomials are not suitable enough for approximation of non-periodic functions defined on $\mathbf{R}$, while the class of e.f.e.t$\leqslant \sigma$ serves as a correct class for non-periodic functions on $\mathbf{R}$.

Consider the class $A_{\mathbf{p}}$ of Muckenhoupt's weights [18]. The main aim of this paper is to obtain central inequalities of approximation by e.f.e.t$\leqslant \sigma$

for functions from the Lebesgue spaces $L^p(\varrho dx)$ on $\mathbf{R}$ with Muckenhoupt's weights $\varrho \in A_{\mathbf{p}}$, $1 \leqslant p < \infty$.

Before presenting the main results, we give some historical remarks and achievements in the particular case of non-weighted classical Lebesgue spaces $L^p(\mathbf{R}) := L^p(\varrho dx)$ with $\varrho \equiv 1$ and $1 \leqslant p \leqslant \infty$.

After the results of S. N. Bernstein in [8], some systematic studies on e.f.e.t$\leqslant \sigma$ continued, chronologically, by N. I. Ackhieser [2], S. M. Nikolski [20], I. I. Ibragimov [15]. All these reference books contain several inequalities of e.f.e.t$\leqslant \sigma$ in spaces $L^p(\mathbf{R})$ with $1 \leqslant p \leqslant \infty$.

On the other hand, many other works also include results of approximation by e.f.e.t$\leqslant \sigma$. See, for example, papers by F. G. Nasibov [19]; S. Artamonov, K. Runovski, H. J. Schmeisser [4]; D. P. Dryanov, M. A. Qazi, and Q. I. Rahman [12]; Z. Ditzian, K. G. Ivanov [11].

Recently, in the paper [13], D. V. Gorbachev, V. I. Ivanov, and S. Yu. Tikhonov have studied approximation by spherical e.f.e.t$\leqslant \sigma$ for functions given in $L^p(\varrho dx)$, $1 \leqslant p < \infty$, with Dunkl weights ϱ on $\mathbf{R}$.

After these historical remarks, we can return to the case of $L^p(\varrho dx)$, $1 \leqslant p < \infty$ with the Muckenhoupt weight ϱ on $\mathbf{R}$.

Some results on trigonometric approximation are known for periodic $\varrho \in A_{\mathbf{p}}$, $1 < p < \infty$ and periodic $f \in L^p(\varrho dx)$. See, e.g., papers by S. Z. Jafarov [16], [17]; A. Guven and V. Kokilashvili [14]; Y. E. Yildirir and D. M. Israfilov [23]; F. Abdullaev, A. Shidlich and S. Chaichenko [1], and A. H. Avşar and H. Koç [5].

Recently, the author has proved in [3] a transference result to obtain norm inequalities for functions in variable exponent Lebesgue spaces on the real axis $L^{p(x)}$.

In the present work, we deal with non-periodic weighted case in $L^p(\varrho dx)$, $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$.

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For $j \in \mathbb{N}$, all constants $\mathbb{C}_j := \mathbb{C}_j(a, b, \ldots)$ are positive numbers that depend on the parameters $a, b, \ldots$ and change only when parameters $a, b, \ldots$ change. Absolute constants are denoted by $c_i > 0$ $(i \in \mathbb{N})$ and do not change in each occurrences.

2. Preliminary notations and the transference result. A function $\varrho \colon \mathbf{R} \to [0, \infty]$ is called weight if ϱ is measurable and positive a.e. on $\mathbf{R}$. Define $\langle \varrho \rangle_{\mathbf{A}} := \int_{\mathbf{A}} \varrho(t) dt$ for $\mathbf{A} \subset \mathbf{R}$. A weight ϱ belongs to the

Muckenhoupt class $A_{\mathbf{p}}$, $1 \leqslant p < \infty$, if

$$|J|^{-1}\langle\varrho\rangle_J \leqslant [\varrho]_1 \left(essinf_{x\in J}\,\varrho(x)\right), \text{ a.e. on } \mathbf{R}, \quad (p=1), \tag{1}$$

$$[\varrho]_p := \sup_{J\subset\mathbf{R}} |J|^{-p}\langle\varrho\rangle_J \left\langle\varrho^{1/(1-p)}\right\rangle_J^{p-1} < \infty, \quad (1<p<\infty) \tag{2}$$

with some finite constants independent of J.

For a weight ϱ on $\mathbf{R}$, denote by $L^p(\varrho dx)$, $1 \leqslant p \leqslant \infty$ the class of real-valued measurable functions, defined on $\mathbf{R}$, such that

$$\|f\|_{p,\varrho} := \left(\int_{\mathbf{R}} |f(x)|^p \varrho(x)\,dx\right)^{1/p} < \infty, \quad (1 \leqslant p < \infty)$$

and

$$\|f\|_{\infty,\varrho} := ess.\sup_{x\in\mathbf{R}} |f(x)|, \quad (p=\infty).$$

Let $C(\mathbf{R})$ (respectively, $\mathcal{C}(\mathbf{R})$) be the class of continuous (bounded uniformly continuous) functions defined on $\mathbf{R}$. Denote by C_c (respectively, S_c) the collection of real-valued continuous (respectively, simple) functions f on $\mathbf{R}$, such that support supp(f) of f is a compact set in $\mathbf{R}$.

For $1<p<\infty$, set $(1/p)+(1/p')=1$.

Lemma 1. *([7, (2.7) p. 933 and (2.10) p. 934]) If $p \in [1,\infty)$, $\varrho \in A_{\mathbf{p}}$, and $f\chi_A \in L^p(\varrho dx)$, then*

$$\|f\chi_A\|_1 \leqslant [\varrho]_p^{1/p}\langle\varrho\rangle_A^{-1/p}\|f\chi_A\|_{p,\varrho}$$

holds for any compact subset A of $\mathbf{R}$.

Lemma 2. *If $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $f \in L^p(\varrho dx)$ and $g \in L^{p'}(\varrho dx)$, then, Hölder's inequality*

$$\int_{\mathbf{R}} |f(x)g(x)|\,\varrho(x)\,dx \leqslant \|f\|_{p,\varrho}\|g\|_{p',\varrho} \tag{3}$$

holds.

Proof. In Theorems 16.14 and 16.40 of [22], we replace $d\mu$ by $\varrho(x)\,dx$ and get Hölder's inequality (3). □

Definition 1. *Suppose that $0<\lambda<\infty$ and $\tau \in \mathbf{R}$. Define the family of the translated Steklov operators $\{\mathbf{S}_{\lambda,\tau}f\}$, by*

$$\mathbf{S}_{\lambda,\tau}f(x) := \lambda \int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(t)\,dt, \quad x \in \mathbf{R} \tag{4}$$

for a locally integrable function f defined on $\mathbf{R}$.

Definition 2. *Let $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $f \in L^p(\varrho dx)$. For $u \in \mathbf{R}$, we define*

$$F_f(u) := \int_{\mathbf{R}} \mathbf{S}_{1,u} f(x) \left|G(x)\right| \varrho(x)\, dx \quad (1 \leqslant p < \infty) \tag{5}$$

with $G \in L^{p'}(\varrho dx)$ satisfying $\|G\|_{p',\varrho} \leqslant 1$.

Definition 3. *(a) A family Q of measurable sets $E \subset \mathbb{R}$ is called locally N-finite ($N \in \mathbb{N}$) if*

$$\sum_{E \in Q} \chi_E(x) \leqslant N$$

almost everywhere in $\mathbb{R}$, where χ_U is the characteristic function of the set U.

(b) A family Q of open bounded sets $U \subset \mathbb{R}$ is locally 1-finite if and only if the sets $U \in Q$ are pairwise disjoint.

Theorem 1. *Suppose that $1 \leqslant p < \infty$ and $\varrho \in A_p$. Then the family of Steklov Mean Operators $\{\mathbf{S}_{1,\tau}\}_{\tau \in \mathbf{R}}$ is uniformly bounded (in τ) in $L^p(\varrho dx)$, namely,*

$$\|\mathbf{S}_{1,\tau} f\|_{p,\varrho} \leqslant 3 \cdot 3^{\frac{2}{p}} [\varrho]_p^{1/p} \|f\|_{p,\varrho} \quad \text{for } \tau \in \mathbf{R}.$$

Proof. Let Q be 1-finite family of open bounded subsets P_i of $\mathbb{R}$, having Lebesgue measure 1, such that $(\cup_i P_i) \cup A = \mathbf{R}$ for some null-set A. Since $\tau \in \mathbf{R}$, there exists $m \in \mathbb{Z}$, such that $m \leqslant \tau < (m+2)$. Let $P + m$ be the translation of the set P by m. Set $(P_i + m)^{\pm} := (P_{i-1} \cup P_i \cup P_{i+1}) + m$. Then

$$\|\mathbf{S}_{1,\tau} f\|_{p,\varrho}^p = \sum_{P_i \in Q} \int_{P_i} \left|\lambda \int_{x+\tau-1/2}^{x+\tau+1/2} f(t)dt\right|^p \varrho(x)dx \leqslant$$

$$\leqslant \sum_{P_i \in Q} \int_{P_i} \Big[\int_{x+\tau-1/2}^{x+\tau+1/2} \varrho^{\frac{1}{p}}(t) \left|f(t)\right| \frac{1}{\varrho^{\frac{1}{p}}(t)} dt \Big]^p \varrho(x)dx \leqslant$$

$$\leqslant \sum_{P_i \in Q} \int_{P_i} \Big(\Big(\int_{x+\tau-1/2}^{x+\tau+1/2} \varrho(t) \left|f(t)\right|^p dt \Big)^{\frac{1}{p}} \Big(\int_{x+\tau-1/2}^{x+\tau+1/2} \varrho^{\frac{-p'}{p}}(t) dt \Big)^{\frac{1}{p'}} \Big)^p \varrho(x)dx =$$

$$= \sum_{P_i \in Q} \int_{P_i} \int_{x+\tau-1/2}^{x+\tau+1/2} \varrho(t) \left| f(t) \right|^p dt \Big(\int_{x+\tau-1/2}^{x+\tau+1/2} \varrho^{\frac{-p'}{p}}(t) dt \Big)^{\frac{p}{p'}} \varrho(x) dx =$$

$$= \sum_{P_i \in Q} \int_{P_i} \int_{x+\tau-1/2}^{x+\tau+1/2} \varrho(t) \left| f(t) \right|^p dt \Big(\int_{x+\tau-1/2}^{x+\tau+1/2} \varrho^{-\frac{1}{p-1}}(t) dt \Big)^{p-1} \varrho(x) dx =$$

$$= \sum_{P_i \in Q} \int_{P_i} \Big(\int_{x+\tau-1/2}^{x+\tau+1/2} \varrho^{-\frac{1}{p-1}}(t) dt \Big)^{p-1} \Big(\int_{x+\tau-1/2}^{x+\tau+1/2} \varrho(t) \left| f(t) \right|^p dt \Big) \varrho(x) dx \leqslant$$

$$\leqslant 3^p \sum_{P_i \in Q} \Big(\frac{1}{\left| P_i^{\pm} \right|} \int_{P_i^{\pm}} \varrho(x) dx \Big) \Big(\frac{1}{\left| (P_i + m)^{\pm} \right|} \int_{(P_i+m)^{\pm}} \varrho^{-\frac{1}{p-1}}(t) dt \Big)^{p-1} \times$$

$$\times \int_{(P_i+m)^{\pm}} \varrho(t) \left| f(t) \right|^p dt \leqslant$$

$$\leqslant 3^p \left[\varrho \right]_p \sum_{P_i \in Q} \Big\{ \int_{P_{i-1}+m} + \int_{P_i+m} + \int_{P_{i+1}+m} \Big\} \varrho(t) \left| f(t) \right|^p dt \leqslant$$

$$\leqslant 3^{p+1} \left[\varrho \right]_p \int_{\mathbf{R}} \varrho(t) \left| f(t) \right|^p \times$$

$$\times \Big\{ \sum_{P_i \in Q} \chi_{P_{i-1}+m}(t) + \sum_{P_i \in Q} \chi_{P_i+m}(t) + \sum_{P_i \in Q} \chi_{P_{i+1}+m}(t) \Big\} dt =$$

$$= 3^{p+2} \left[\varrho \right]_p \int_{\mathbf{R}} \varrho(t) \left| f(t) \right|^p dt = 3^{p+2} \left[\varrho \right]_p \left\| f \right\|_{p,\varrho}^p \text{ and}$$

$$\left\| \mathbf{S}_{1,\tau} f \right\|_{p,\varrho} \leqslant 3 \cdot 3^{\frac{2}{p}} \left[\varrho \right]_p^{1/p} \left\| f \right\|_{p,\varrho}.$$

For $p = 1$, we find:

$$\left\| \mathbf{S}_{1,\tau} f \right\|_{1,\varrho} = \sum_{P_i \in Q} \int_{P_i} \Big| \int_{x+\tau-1/2}^{x+\tau+1/2} f(t) dt \Big| \varrho(x) dx \leqslant$$

$$\leqslant \sum_{P_i \in Q} \int_{P_i} \int_{x+\tau-1/2}^{x+\tau+1/2} \varrho(t) \left| f(t) \right| \frac{1}{\varrho(t)} dt \varrho(x) dx \leqslant$$

$$\leqslant 3\sum_{P_i\in Q}\frac{1}{\left|(P_i+m)^{\pm}\right|}\int\limits_{(P_i+m)^{\pm}}\varrho(x)dx\Big(\underset{t\in(P_i+m)^{\pm}}{esssup}\frac{1}{\varrho(t)}\Big)\int\limits_{(P_i+m)^{\pm}}\varrho(t)\left|f(t)\right|dt\leqslant$$

$$\leqslant 3\left[\gamma\right]_1\sum_{P_i\in Q}\Big\{\int\limits_{P_{i-1}+m}+\int\limits_{P_i+m}+\int\limits_{P_{i+1}+m}\Big\}\varrho(t)\left|f(t)\right|dt\leqslant$$

$$\leqslant 3\left[\gamma\right]_1\int\limits_{\mathbf{R}}\varrho(t)\left|f(t)\right|\Big\{\sum_{P_i\in Q}\chi_{P_{i-1}+m}(t)+\sum_{P_i\in Q}\chi_{P_i+m}(t)+\sum_{P_i\in Q}\chi_{P_{i+1}+m}(t)\Big\}dt\leqslant$$

$$\leqslant 9\left[\gamma\right]_1\left\|f\right\|_{1,\varrho}.$$

Hence, for any $1\leqslant p<\infty$,

$$\left\|\mathbf{S}_{1,\tau}f\right\|_{p,\varrho}\leqslant 3\cdot 3^{\frac{2}{p}}\left[\varrho\right]_p^{1/p}\left\|f\right\|_{p,\varrho}.$$

□

Theorem 2. *Let $1\leqslant p<\infty$ and ϱ be a weight on $\mathbf{R}$. Then, for $f\in L^p(\varrho dx)$, we have*

$$\sup_{G\in L^{p'}(\varrho dx),\left\|G\right\|_{p',\varrho}\leqslant 1}\int\limits_{\mathbf{R}}\left|f(x)G(x)\right|\varrho(x)dx=\left\|f\right\|_{p,\varrho}. \tag{6}$$

In addition, the condition "$G\in L^{p'}(\varrho dx)$" in supremum can be replaced by the condition "$G\in L^{p'}(\varrho dx)\cap S_c$"

Proof. (6) is a consequence of Theorem 18.4 of [22]. On the other hand, methods given in Lemma 2.7.2 and Lemma 3.2.14 of [10] imply that the condition "$G\in L^{p'}(\varrho dx)$" in supremum can be replaced by "$G\in L^{p'}(\varrho dx)\cap S_c$" □

Theorem 3. *Let $1\leqslant p<\infty$, $\varrho\in A_{\mathbf{p}}$, and $f,g\in L^p(\varrho dx)$. In this case,*

(a) The function $F_f(\cdot)$ defined in (5) is bounded and uniformly continuous on $\mathbf{R}$.

(b) If $\left\|F_f\right\|_{C(\mathbf{R})}\leqslant c_1\left\|F_g\right\|_{C(\mathbf{R})}$ holds with an absolute constant $c_1>0$, then we have the weighted norm inequalities

$$\left\|f\right\|_{p,\varrho}\leqslant c_1\mathbb{C}_1\left\|g\right\|_{p,\varrho} \tag{7}$$

with $\mathbb{C}_1:=\mathbb{C}_1(p,\varrho):=6\cdot 3^{\frac{2}{p}}\left[\varrho\right]_p^{1/p}$.

Proof. (a) Since C_c is a dense subset of $L^p(\varrho dx)$, consider functions $H \in C_c$ and prove that $F_H(u) = \int_{\boldsymbol{R}} (\mathsf{S}_{1,u}H)(x)\,|G(x)|\,\varrho(x)\,dx$ is bounded and uniformly continuous on $\boldsymbol{R}$, where $G \in L^{p'}(\varrho dx) \cap S_c$ and $\|G\|_{p',\varrho} \leqslant 1$. Boundedness of $F_H(\cdot)$ is an easy consequence of the Hölder's inequality and Theorem 1. On the other hand, note that H is uniformly continuous on $\boldsymbol{R}$, see, e.g., Lemma 23.42 of [22, pp. 557–558]. Take $\varepsilon > 0$ and $u_1, u_2, x \in \boldsymbol{R}$. Then there exists a $\delta := \delta(\varepsilon) > 0$, such that

$$|H(x+u_1) - H(x+u_2)| \leqslant \frac{\varepsilon}{2\left(1 + \langle\varrho\rangle_{\mathrm{supp}(G)}\right)}$$

for $|u_1 - u_2| < \delta$. Then, for $|u_1 - u_2| < \delta$, $u_1, u_2 \in \boldsymbol{R}$, we have

$$|F_H(u_1) - F_H(u_2)| = \left|\int_{\boldsymbol{R}} (S_{1,u_1}H(x) - S_{1,u_2}H(x))\,|G(x)|\,\varrho(x)\,dx\right| \leqslant$$

$$\leqslant \frac{\varepsilon}{2\left(1+\langle\varrho\rangle_{\mathrm{supp}(G)}\right)} \int_{\boldsymbol{R}} |G(x)|\,\varrho(x)\,dx \leqslant \frac{\varepsilon\,\langle\varrho\rangle_{\mathrm{supp}(G)}}{\left(1+\langle\varrho\rangle_{\mathrm{supp}(G)}\right)} \|G\|_{p',\varrho} < \varepsilon.$$

Now the conclusion follows for the class C_c. For the general case $f \in L^p(\varrho dx)$ there exists an $H \in C_c(\boldsymbol{R})$ so that

$$\|f - H\|_{p,\varrho} < \xi/(12 \cdot 3^{\frac{2}{p}}\,[\varrho]_p^{1/p})$$

for any $\xi > 0$. Then, for this ξ,

$$|F_f(u_1) - F_f(u_2)| \leqslant \left|\int_{\boldsymbol{R}} \mathsf{S}_{1,u_1}(f - H)(x)\,|G(x)|\,\varrho(x)dx\right| +$$

$$+\left|\int_{\boldsymbol{R}} (\mathsf{S}_{1,u_1}H(x) - \mathsf{S}_{1,u_2}H(x))\,|G(x)|\,\varrho(x)\,dx\right| +$$

$$+\left|\int_{\boldsymbol{R}} \mathsf{S}_{1,u_2}(H - f)(x)\,|G(x)|\,\varrho(x)\,dx\right| \leqslant$$

$$\leqslant \|\mathsf{S}_{1,u_1}(f - H)\|_{p,\varrho} + \left|\int_{\boldsymbol{R}} \mathsf{S}_1(H(x+u_1) - H(x+u_2))\,|G(x)|\,\varrho(x)\,dx\right| +$$

$$+ \|\mathsf{S}_{1,u_2}(f - H)\|_{p,\varrho} \leqslant 6 \cdot 3^{\frac{2}{p}}\,[\varrho]_p^{1/p}\,\|f - H\|_{p,\varrho} + \xi/2 \leqslant \xi/2 + \xi/2 = \xi.$$

As a result, F_f is bounded, uniformly continuous function defined on $\boldsymbol{R}$.

(b) Let $0 \leqslant f,g \in L^p(\varrho dx)$. If $\|f\|_{p,\varrho} = \|g\|_{p,\varrho} = 0$, then results (7) are obvious. So, assume that $\|f\|_{p,\varrho}, \|g\|_{p,\varrho} \in (0,+\infty)$. Then

$$\left\|F_f\right\|_{\mathcal{C}(\mathbf{R})} \leqslant c_1 \left\|F_g\right\|_{\mathcal{C}(\mathbf{R})} = c_1 \left\| \int\limits_{\mathbf{R}} \mathsf{S}_{1,u}(g)(x) \left|G(x)\right| \varrho(x)\, dx \right\|_{\mathcal{C}(\mathbf{R})} =$$

$$= c_1 \sup_{u\in\mathbf{R}} \int\limits_{\mathbf{R}} \mathsf{S}_{1,u}(g)(x) \left|G(x)\right| \varrho(x)\, dx = c_1 \sup_{u\in\mathbf{R}} \left\|\mathsf{S}_{1,u}(g)\right\|_{p,\varrho} \left\|G\right\|_{p',\varrho} \leqslant$$

$$\leqslant 3c_1 3^{\frac{2}{p}} [\varrho]_p^{1/p} \|g\|_{p,\varrho}.$$

On the other hand, for any $\varepsilon \in (0, \|f\|_{p,\varrho}]$ we can choose $\bar{G}_\varepsilon \in L^{p'}(\varrho dx) \cap S_c$ with

$$\int\limits_{\mathbf{R}} h(x) \left|\bar{G}_\varepsilon(x)\right| \varrho(x)\, dx \geqslant \|h\|_{p,\varrho} - \varepsilon, \qquad \left\|\bar{G}_\varepsilon\right\|_{p',\varrho} \leqslant 1,$$

and one can find

$$\|F_f\|_{\mathcal{C}(\mathbf{R})} \geqslant |F_f(0)| \geqslant \int\limits_{\mathbf{R}} \mathsf{S}_{1,0} f(x) \left|G(x)\right| \varrho(x)\, dx =$$

$$= \mathsf{S}_{1,0}\Big(\int\limits_{\mathbf{R}} f(x) \left|G(x)\right| \varrho(x)\, dx \Big) \geqslant \mathsf{S}_{1,0}\left(\|f\|_{p,\varrho} - \varepsilon \right) = \|f\|_{p,\varrho} - \varepsilon.$$

In the last inequality, we let $\varepsilon \to 0+$ to obtain $\|F_f\|_{\mathcal{C}(\mathbf{R})} \geqslant \|f\|_{p,\varrho}$. Combining these inequalities, we get

$$\|f\|_{p,\varrho} \leqslant \|F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant c_1 \|F_g\|_{\mathcal{C}(\mathbf{R})} \leqslant 3c_1 3^{\frac{2}{p}} [\varrho]_p^{1/p} \|g\|_{p,\varrho}.$$

In the general case $f,g \in L^p(\varrho dx)$, we get

$$\|f\|_{p,\varrho} \leqslant 6c_1 3^{\frac{2}{p}} [\varrho]_p^{1/p} \|g\|_{p,\varrho}. \tag{8}$$

Then (7) holds. □

Averaging Operator and Mollifier.

Definition 4. *Let $B \subseteq \mathbf{R}$ be an open set, $\phi \in L_1(B)$ and $\int_B \phi(t)\, dt = 1$. For each $t > 0$, define $\phi_t(x) = \frac{1}{t}\phi\left(\frac{x}{t}\right)$. The sequence $\{\phi_t\}$ will be called the approximate identity. A function*

$$\tilde{\phi}(x) = \sup_{|y|\geqslant|x|} |\phi(y)|$$

will be called a radial majorant of ϕ. If $\tilde{\phi} \in L_1(B)$, then the sequence $\{\phi_t\}$ is called the potential-type approximate identity.

Definition 5.

(a) Let $U \subset \mathbb{R}$ be a measurable set and

$$A_U f := \frac{1}{|U|} \int\limits_U |f(t)|\, dt.$$

(b) For a family Q of open sets $U \subset \mathbb{R}$, define the averaging operator by

$$T_Q\colon L^1_{loc} \to L^0, \qquad T_Q f(x) := \sum_{U \in Q} \chi_U(x)\, A_U f, \quad x \in \mathbb{R},$$

where L^0 is the set of measurable functions on $\mathbb{R}$.

(c) For a measurable set $A \subset \mathbf{R}$, the symbol $|A|$ represents the Lebesgue measure of A.

Theorem 4. *Suppose that $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$ and $f \in L^p(\varrho dx)$. If Q is 1-finite family of open bounded subsets of $\mathbf{R}$, then the averaging operator T_Q is uniformly bounded in $L^p(\varrho dx)$:*

$$\|T_Q f\|_{p,\varrho} \leqslant [\varrho]_p^{1/p} \|f\|_{p,\varrho}. \tag{9}$$

Proof. Let Q be 1-finite family of open bounded subsets P_i of $\mathbf{R}$, such that $(\cup_i P_i) \cup A = \mathbf{R}$ for some null-set A. From Proposition 4.33 of [9], we have

$$\int\limits_{P_i} |T_Q f(x)|^p \varrho(x) dx \leqslant [\varrho]_p \int\limits_{P_i} |f(x)|^p \varrho(x) dx$$

(see also Part 5.2 of [10, p.150]). Then we have

$$\|T_Q f\|_{p,\varrho}^p = \sum_{P_i \in Q} \int\limits_{P_i} |T_Q f(x)|^p \varrho(x) dx \leqslant$$

$$\leqslant [\varrho]_p \sum_{P_i \in Q} \int\limits_{P_i} |f(x)|^p \varrho(x) dx = [\varrho]_p \|f\|_{p,\varrho}^p$$

and the desired result (9) follows. □

Theorem 5. *Suppose that $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $f \in L^p(\varrho dx)$, ϕ is a potential-type approximate identity. Then, for any $t > 0$,*

$$\|f * \phi_t\|_{p,\varrho} \leqslant 2 \left\|\tilde{\phi}\right\|_1 \mathbb{C}_1 \|f\|_{p,\varrho}$$

and

$$\lim_{t\to 0} \|f * \phi_t - f\|_{p,\varrho} = 0$$

holds.

Proof. We can use the transference result. Since $F_{f*\phi_t} = (F_f) * \phi_t$, we find

$$\|f * \phi_t\|_{p,\varrho} \leqslant \left\|F_{f*\phi_t}\right\|_{\mathcal{C}(\mathbf{R})} = \|(F_f) * \phi_t\|_{\mathcal{C}(\mathbf{R})} \leqslant \|\tilde{\phi}\|_1 \|F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant \mathbb{C}_1 \|\tilde{\phi}\|_1 \|f\|_{p,\varrho}.$$

□

3. Exponential approximation.

Definition 6. *Let $X := L^p(\mathbf{R})$ or $L^p(\varrho dx)$ or $\mathcal{C}(\mathbf{R})$.*

(i) We define $\mathcal{G}_\sigma(X)$ as the class of entire function of exponential type σ that belong to X. The quantity

$$A_\sigma(f)_X := \inf_g \{\|f - g\|_X : g \in \mathcal{G}_\sigma(X)\} \tag{10}$$

is called the deviation of the function $f \in X$ from $\mathcal{G}_\sigma(X)$.

(ii) Let W_X^r, $r \in \mathbb{N}$, be the class of functions $f \in X$, such that the derivatives $f^{(k)}$ exist for $k = 1, \ldots, r-1$, $f^{(r-1)}$ is absolutely continuous and, $f^{(r)} \in X$.

(iii) Define $\mathcal{G}_\sigma(p) := \mathcal{G}_\sigma(L^p(\mathbf{R}))$, $\mathcal{G}_\sigma(p,\varrho) := \mathcal{G}_\sigma(L^p(\varrho dx))$, $\mathcal{G}_\sigma(\infty) := \mathcal{G}_\sigma(\mathcal{C}(\mathbf{R}))$, $A_\sigma(f)_p := A_\sigma(f)_{L^p(\mathbf{R})}$, $A_\sigma(f)_{p,\varrho} := A_\sigma(f)_{L^p(\varrho dx)}$, $A_\sigma(f)_\infty := A_\sigma(f)_{\mathcal{C}(\mathbf{R})}$, $W_p^r := W_{L^p(\mathbf{R})}^r$, $W_{p,\varrho}^r := W_{L^p(\varrho dx)}^r$ and $W_\infty^r := W_{\mathcal{C}(\mathbf{R})}^r$.

In the following result of C. Bardaro, P. L. Butzer, R. L. Stens, and G. Vinti, the exponential approximation result of the de la Valèe Poussin operator in $L^p(\mathbf{R})$ $1 \leqslant p \leqslant \infty$ was proved.

Theorem 6. *[6] Let $\sigma > 0$, $1 \leqslant p \leqslant \infty$, $f \in L^p(\mathbf{R})$,*

$$\vartheta(x) := \frac{2}{\pi} \frac{\sin(x/2)\sin(3x/2)}{x^2}$$

and

$$J(f,\sigma) = \sigma \int_{\mathbf{R}} f(x-u)\,\vartheta(\sigma u)\,du$$

be the de la Valèe Poussin operator ([6, definition given in (5.3)]). It is known (see (5.4)-(5.5) of [6]) that if $f \in L^p(\mathbf{R})$, $1 \leqslant p \leqslant \infty$, *then*

(i) $J(f,\sigma) \in \mathcal{G}_{2\sigma}(p)$,

(ii) $J(g_\sigma,\sigma) = g_\sigma$ *for any* $g_\sigma \in \mathcal{G}_\sigma(p)$,

(iii) $\|J(f,\sigma)\|_{L_p(\mathbf{R})} \leqslant \frac{3}{2}\|f\|_{L_p(\mathbf{R})}$,

(iv) $(J(f,\sigma))^{(r)} = J\left(f^{(r)},\sigma\right)$ *for any* $r \in \mathbb{N}$ *and* $f \in W_p^r$,

(v) $\|J\left(f,\frac{\sigma}{2}\right) - f\|_{L_p(\mathbf{R})} \to 0$ *(as* $\sigma \to \infty$*) and, hence,*

$$\left\| \left(J\left(f,\frac{\sigma}{2}\right)\right)^{(k)} - f^{(k)} \right\|_{L_p(\mathbf{R})} \to 0 \text{ as } \sigma \to \infty,$$

for $f \in W_p^r$ *and* $1 \leqslant k \leqslant r$.

For $r \in \mathbb{N}$, we define $C^r(\mathbf{R})$ consisting of every member $f \in C(\mathbf{R})$, such that the derivative $f^{(k)}$ exists and is continuous on $\mathbf{R}$ for $k = 1, \ldots, r$.

Modulus of smoothness and the translated Steklov average. As a corollary of Theorem 3, we have the following two results.

Corollary 1. *Suppose that* $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $0 < \lambda < \infty$ *and* $\tau \in \mathbf{R}$. *Then:*

(i) $F_{\mathbf{S}_{\lambda,\tau}f} = \mathbf{S}_{\lambda,\tau}F_f$,

(ii) *the family of operators* $\{\mathbf{S}_{\lambda,\tau}f\}$*, defined by (4), is uniformly bounded (in* λ *and* τ*) in* $L^p(\varrho dx)$:

$$\|\mathbf{S}_{\lambda,\tau}f\|_{p,\varrho} \leqslant \mathbb{C}_1 \|f\|_{p,\varrho}$$

Proof of Corollary 1 is a consequence of the transference Theorem 3. Known results were proved under more restricted condition on λ and τ, such as $1 \leqslant \lambda < \infty$ and $|\tau| \leqslant \pi/\lfloor\lambda^\rho\rfloor$ for some ρ.

Proof. (i) follows from (4) and (5). For (ii), we will use (i) and Theorem 3:

$$\|\mathbf{S}_{\lambda,\tau}f\|_{p,\varrho} \leqslant \left\|F_{\mathbf{S}_{\lambda,\tau}f}\right\|_{\mathcal{C}(\mathbf{R})} = \|\mathbf{S}_{\lambda,\tau}F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant \|F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant \mathbb{C}_1 \|f\|_{p,\varrho}.$$

□

Corollary 2. *Let* $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $0 < \delta < \infty$, $f \in L^p(\varrho dx)$. *If* $\tau = \delta/2$ *then,*

$$\mathbf{S}_{\delta,\delta/2}f(x) = \frac{1}{\delta}\int_0^\delta f(x+t)\,dt = T_\delta f(x),$$

$$F_{T_\delta f} = T_\delta F_f \quad \text{and} \quad \|T_\delta f\|_{p,\varrho} \leqslant \mathbb{C}_1 \|f\|_{p,\varrho}. \tag{11}$$

Proof. Equality $F_{T_\delta f} = T_\delta F_f$ follows from (4) and (5). For the second inequality in (11), we estimate:

$$\|T_\delta f\|_{p,\varrho} \leqslant \|F_{T_\delta f}\|_{\mathcal{C}(\mathbf{R})} = \|T_\delta F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant \|F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant \mathbb{C}_1 \|f\|_{p,\varrho}.$$

□

For $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $f \in L^p(\varrho dx)$, $0 < \delta < \infty$, $r \in \mathbb{N}$, we define the modulus of smoothness

$$\Omega_r(f,\delta)_{p,\varrho} := \|(I - T_\delta)^r f\|_{p,\varrho}.$$

From the Transference Result:

$$\|(I - T_\delta)^r f\|_{p,\varrho} \leqslant 2^r \mathbb{C}_1 \|f\|_{p,\varrho}.$$

Lemma 3. $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$. *If* $r \in \mathbb{N}$ *and* $f \in W^r_{p,\varrho}$, *then* $\frac{d^k}{du^k} F_f(u)$ *exists and*

$$\frac{d^k}{du^k} F_f(u) = F_{f^{(k)}}(u) \text{ for } k \in \{1,\ldots,r\}, \text{ and } u \in \mathbf{R}. \tag{12}$$

Proof. We have the following equalities: for $u \in \mathbf{R}$,

$$\frac{d}{du}F_f(u) = \frac{d}{du}\int_{\mathbf{R}} \mathbf{S}_{1,u}f(x)\,|G(x)|\,\varrho(x)\,dx =$$

$$= \int_{\mathbf{R}} \int_{-1/2}^{1/2} \frac{d}{du}\left[f(x+u+t)\right] dt\,|G(x)|\,\varrho(x)\,dx =$$

$$= \int_{\mathbf{R}} \int_{-1/2}^{1/2} f'(x+u+t)\,dt\,|G(x)|\,\varrho(x)\,dx = F_{f'}(u).$$

Then (12) holds for $k = 1$. Using this procedure consecutively, we get

$$\frac{d^k}{du^k}F_f(u) = \frac{d^{k-1}}{du^{k-1}}\frac{d}{du}F_f(u) = \frac{d^{k-1}}{du^{k-1}}F_{f'}(u) = \ldots = F_{f^{(k)}}(u).$$

□

Lemma 4. *Let $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $r \in \mathbb{N}$, and $0 < \delta < \infty$. Then*

$$\left\| (I - T_\delta)^r f \right\|_{p,\varrho} \leqslant \mathbb{C}_1 \delta^r \left\| f^{(r)} \right\|_{p,\varrho}, \quad f \in W^r_{p,\varrho}$$

holds.

Proof. Note that the following inequality

$$\left\| (I - T_\delta) f \right\|_{p,\varrho} \leqslant 2^{-1} \mathbb{C}_1 \delta \left\| f' \right\|_{p,\varrho}, \quad \delta > 0 \tag{13}$$

holds for $f, f' \in L^p(\varrho dx)$. Then

$$\Omega_r (f, \delta)_{p,\varrho} = \left\| (I - T_\delta)^r f \right\|_{p,\varrho} \leqslant \cdots \leqslant 2^{-r} \mathbb{C}_1^r \delta^r \left\| f^{(r)} \right\|_{p,\varrho}, \quad \delta > 0$$

for $f \in W^r_{p,\varrho}$. □

K-functional.

Definition 7. *Let $X := L^p(\mathbf{R})$ or $L^p(\varrho dx)$ or $\mathcal{C}(\mathbf{R})$.*

(i) Let W^r_X, $r \in \mathbb{N}$, be the class of functions $f \in X$, such that derivatives $f^{(k)}$ exist for $k = 1, \ldots, r-1$, $f^{(r-1)}$ is absolutely continuous, and $f^{(r)} \in X$. In particular, we set $W^r_p := W^r_{L^p(\mathbf{R})}$, $W^r_{p,\varrho} := W^r_{L^p(\varrho dx)}$ and $W^r_\infty := W^r_{\mathcal{C}(\mathbf{R})}$.

(ii) We define Peetre's K-functional for the pair X and W^r_X as follows:

$$K_r(f, \delta, X) := \inf_{g \in W^r_X} \left\{ \|f - g\|_X + \delta^r \left\| g^{(r)} \right\|_X \right\}, \quad \delta > 0.$$

We use the notation $K_r(f, \delta, p, \varrho) := K_r(f, \delta, L^p(\varrho dx))$ for $r \in \mathbb{N}$, $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $\delta > 0$ and $f \in L^p(\varrho dx)$.

Also set $K_r(f, \delta, C) := K_r(f, \delta, \mathcal{C}(\mathbf{R}))$ for $r \in \mathbb{N}$, $\delta > 0$ and $f \in \mathcal{C}(\mathbf{R})$.

As a corollary of the Transference Result, we can obtain the following lemma:

Lemma 5. *Let $0 < h \leqslant \delta < \infty$, $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, and $\mathbf{f} \in L^p(\varrho dx)$. Then*

$$F_{(I - T_h)\mathbf{f}} = (I - T_h) F_{\mathbf{f}}, \tag{14}$$

$$\left\| (I - T_h) \mathbf{f} \right\|_{p,\varrho} \leqslant 72 \mathbb{C}_1 \left\| (I - T_\delta) \mathbf{f} \right\|_{p,\varrho} \tag{15}$$

hold.

Proof. Property (14) follows from definitions of $F_{\mathbf{f}}$ and T_h. First, we obtain the following inequalities:

$$\left\| \frac{d}{dx} T_\delta f(x) \right\|_{C(\mathbf{R})} \leqslant \frac{2}{\delta} \|f\|_{C(\mathbf{R})}, \tag{16}$$

$$\left\| \frac{d^2}{dx^2} T_\delta f(x) \right\|_{C(\mathbf{R})} \leqslant \frac{2}{\delta} \left\| \frac{d}{dx} T_\delta f \right\|_{C(\mathbf{R})}, \tag{17}$$

$$\left\| g(x) - T_\delta g(x) + \frac{\delta}{2} \frac{d}{dx} g(x) \right\|_{C(\mathbf{R})} \leqslant \frac{\delta^2}{6} \left\| \frac{d^2}{dx^2} g \right\|_{C(\mathbf{R})}, \tag{18}$$

$$(C_3(r))^{-1} K_r(f, \delta, C) \leqslant \left\| (I - T_\delta)^r f \right\|_{C(\mathbf{R})} \leqslant 2^r K_r(f, \delta, C); \tag{19}$$

they hold with $\mathbb{C}_2(1) = 36$, $\mathbb{C}_2(r) = 2^r(r^r + (34)^r)$ for $r > 1$, where $f \in C(\mathbf{R})$, $r \in \mathrm{N}$ and $g \in C^2(\mathbf{R})$. In fact, these inequalities are known from [3]. On the other hand, if $0 < h \leqslant \delta < \infty$ and $g \in C(\mathbf{R})$, then the inequality

$$\left\| F_{(I-T_h)g} \right\|_{C(\mathbf{R})} \leqslant 72 \left\| F_{(I-T_\delta)g} \right\|_{C(\mathbf{R})} \tag{20}$$

holds. To prove (20), we use (19) and obtain

$$\left\| F_{(I\text{-}T_h)g} \right\|_{C(\mathbf{R})} = \|(I\text{-}T_h) F_g\|_{C(\mathbf{R})} \leqslant 72 \|(I\text{-}T_\delta) F_g\|_{C(\mathbf{R})} = 72 \left\| F_{(I\text{-}T_\delta)g} \right\|_{C(\mathbf{R})}.$$

Now, the Transference Result, (20), and (14) give the result (15). □

Theorem 7. *Let $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, and $f \in L^p(\varrho dx)$. Then*

$$\frac{1}{2^r \mathbb{C}_1} \leqslant \frac{K_r(f, \delta, p, \varrho)}{\Omega_r(f, \delta)_{p,\varrho}} \leqslant \{(2r)^r + 2^r (34)^r\} \mathbb{C}_1.$$

Proof. For any $g \in W^r_{p,\varrho}$ we have $F_g \in W^r_\infty$, and $F_{(I-T_\delta)^r f} = (I - T_\delta)^r F_f$. Then, using the Transference Result:

$$\|(I - T_\delta)^r f\|_{p,\varrho} \leqslant \left\| F_{(I-T_\delta)^r f} \right\|_{\mathcal{C}(\mathbf{R})} = \|(I - T_\delta)^r F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant 2^r K_r(F_f, \delta, C) \leqslant 2^r \left\{ \|F_f - F_g\|_{\mathcal{C}(\mathbf{R})} + \delta^r \left\| \frac{d^r}{du^r} F_g \right\|_{\mathcal{C}(\mathbf{R})} \right\} \leqslant$$

$$\leqslant 2^r \{ \|F_{f-g}\|_{\mathcal{C}(\mathbf{R})} + \delta^r \|F_{g^{(r)}}\|_{\mathcal{C}(\mathbf{R})} \} \leqslant 2^r \mathbb{C}_1 \left\{ \|f - g\|_{p,\varrho} + \delta^r \left\| g^{(r)} \right\|_{p,\varrho} \right\}. \tag{21}$$

Taking infimum in (21) and considering definition of K-functional, one gets

$$\|(I - T_\delta)^r f\|_{p,\varrho} \leqslant 2^r \mathbb{C}_1 K_r(f, \delta, p, \varrho).$$

Now we consider the opposite direction of the last inequality. Defining

$$g(\cdot) = \sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} T_{\delta}^{2rl} f(\cdot),$$

using the Transference Result, we have for $g \in W_{p,\varrho}^{r}$

$$K_r(f, \delta, p, \varrho) \leqslant \|f - g\|_{p,\varrho} + \delta^r \left\| \frac{d^r}{dx^r} g \right\|_{p,\varrho} \leqslant$$

$$\leqslant \|F_{f-g}\|_{\mathcal{C}(\mathbf{R})} + \delta^r \left\| F_{g^{(r)}} \right\|_{\mathcal{C}(\mathbf{R})} = \|F_f - F_g\|_{\mathcal{C}(\mathbf{R})} + \delta^r \left\| \frac{d^r}{du^r} F_g \right\|_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant \left\| \left(I - T_{\delta}^{2r}\right)^r F_f \right\|_{\mathcal{C}(\mathbf{R})} + \delta^r \left\| \frac{d^r}{du^r} \sum_{l=1}^{r} (-1)^{l-1} \binom{r}{l} T_{\delta}^{2rl} F_f \right\|_{\mathcal{C}(\mathbf{R})} =$$

$$= \left\| \left(I - T_{\delta}^{2r}\right)^r F_f \right\|_{\mathcal{C}(\mathbf{R})} + \sum_{l=1}^{r} \left| \binom{r}{l} \right| \delta^r \left\| \frac{d^r}{du^r} T_{\delta}^{2rl} F_f \right\|_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant (2r)^r \left\| (I - T_{\delta})^r F_f \right\|_{\mathcal{C}(\mathbf{R})} + 2^r (34)^r \left\| (I - T_{\delta})^r F_f \right\|_{\mathcal{C}(\mathbf{R})} =$$

$$= \left[(2r)^r + 2^r (34)^r\right] \left\| F_{(I - T_{\delta})^r f} \right\|_{\mathcal{C}(\mathbf{R})} \leqslant \left\{(2r)^r + 2^r (34)^r\right\} \mathbb{C}_1 \left\| (I - T_{\delta})^r f \right\|_{p,\varrho}.$$

□

Theorem 8. *For $p \in [1, \infty)$, $\varrho \in A_{\mathbf{p}}$, $f, g \in L^p(\varrho dx)$ and $\delta > 0$, the following properties hold:*

1) $\Omega_r(f, \delta)_{p,\varrho}$ *is non-negative, non-decreasing function of δ;*
2) $\Omega_r(f, \delta)_{p,\varrho}$ *is sub-additive of f;*
3) *We have*

$$\lim_{\delta \to 0} K_r(f, \delta, p, \varrho) = 0. \tag{22}$$

As a result,

$$\lim_{\delta \to 0} \Omega_r(f, \delta)_{p,\varrho} = 0. \tag{23}$$

Proof. Properties 1) and 2) are clear from definition. Since

$$\lim_{\delta \to 0} K_r(f, \delta, p, \varrho) = 0,$$

we have, from Theorem 7, that (23) holds. □

Jackson type inequality.

Theorem 9. *Let $p\in[1,\infty)$, $\varrho\in A_{\mathbf{p}}$, $r\in\mathbb{N}$, $\sigma>0$ and $f\in L^p(\varrho dx)$. Then*

$$A_\sigma(f)_{p,\varrho}\leqslant 25\pi 8^{r-1}\mathbb{C}_2\mathbb{C}_1\left\|\left(I-T_{1/\sigma}\right)^r f\right\|_{p,\varrho}. \tag{24}$$

Proof. First we obtain

$$A_{2\sigma}(f)_{p,\varrho}\leqslant 25\pi 8^{r-1}\mathbb{C}_2\mathbb{C}_1\left\|\left(I-T_{1/(2\sigma)}\right)^r f\right\|_{p,\varrho} \tag{25}$$

and (24) follows from (25).

Let us take $g_\sigma\in\mathcal{G}_\sigma(\infty)$ with $\|F_f-g_\sigma\|_{\mathcal{C}(\mathbf{R})}=A_\sigma(F_f)_{\mathcal{C}(\mathbf{R})}$. Using $V_\sigma F_f=F_{V_\sigma f}$ and $V_\sigma g_\sigma=g_\sigma$, we get

$$A_{2\sigma}(f)_{p,\varrho}\leqslant\|f-V_\sigma f\|_{p,\varrho}\leqslant\|F_{f-V_\sigma f}\|_{\mathcal{C}(\mathbf{R})}=\|F_f-V_\sigma F_f\|_{\mathcal{C}(\mathbf{R})}\leqslant$$

$$\leqslant\|F_f-g_\sigma+g_\sigma-V_\sigma F_f\|_{\mathcal{C}(\mathbf{R})}=\|F_f-g_\sigma+V_\sigma g_\sigma-V_\sigma F_f\|_{\mathcal{C}(\mathbf{R})}\leqslant$$

$$\leqslant A_\sigma(F_f)_{\mathcal{C}(\mathbf{R})}+\frac{3}{2}A_\sigma(F_f)_{\mathcal{C}(\mathbf{R})}=\frac{5}{2}A_\sigma(F_f)_{\mathcal{C}(\mathbf{R})}.$$

For any $g\in W_\infty^r$, one gets

$$A_\sigma(u)_{\mathcal{C}(\mathbf{R})}\leqslant A_\sigma(u\text{-}g)_{\mathcal{C}(\mathbf{R})}+A_\sigma(g)_{\mathcal{C}(\mathbf{R})}\leqslant\|u\text{-}g\|_{\mathcal{C}(\mathbf{R})}+\frac{5\pi}{4}\frac{4^r}{\sigma^r}\left\|\frac{d^r}{dx^r}g\right\|_{\mathcal{C}(\mathbf{R})}\leqslant$$

$$\leqslant\frac{5\pi4^r}{4}K_r(u,\sigma^{-1},C)\leqslant\frac{5\pi8^r}{4}K_r\left(u,\frac{1}{2\sigma},C\right)\leqslant\frac{5\pi8^r}{4}\mathbb{C}_2\left\|\left(I\text{-}T_{(2\sigma)^{-1}}\right)^r u\right\|_{\mathcal{C}(\mathbf{R})}$$

where $\mathbb{C}_2(1)=36$ and $\mathbb{C}_2(r)=2^r(r^r+(34)^r)$ for $r>1$. Therefore,

$$A_{2\sigma}(f)_{p,\varrho}\leqslant\frac{5}{2}A_\sigma(F_f)_{\mathcal{C}(\mathbf{R})}\leqslant 25\pi8^{r-1}\mathbb{C}_2\left\|(I-T_{\frac{1}{2\sigma}})^r F_f\right\|_{\mathcal{C}(\mathbf{R})}=$$

$$=25\pi8^{r-1}\mathbb{C}_2\left\|F_{\left(I-T_{1/(2\sigma)}\right)^r f}\right\|_{\mathcal{C}(\mathbf{R})}\leqslant 25\pi8^{r-1}\mathbb{C}_2\mathbb{C}_1\left\|\left(I-T_{1/(2\sigma)}\right)^r f\right\|_{p,\varrho}.$$

□

Inverse theorem.

Theorem 10. *Let $p\in[1,\infty)$, $\varrho\in A_{\mathbf{p}}$, $r\in\mathbb{N}$, $\delta\in(0,\infty)$ and $f\in L^p(\varrho dx)$. Then*

$$\Omega_r(f,\delta)_{p,\varrho}\leqslant\mathbb{C}_3\delta^r\left(A_0(f)_{p,\varrho}+\int_{1/2}^{1/\delta}u^{r-1}A_{u/2}(f)_{p,\varrho}\,du\right)$$

holds with $\mathbb{C}_3 := \mathbb{C}_1 (1 + 3\mathbb{C}_1) 2^{r+1} (1 + 2^{2r-1})$.

Proof. We use the transference result to obtain

$$\Omega_r (f, \delta)_{p,\varrho} = \|(I - T_\delta)^r f\|_{p,\varrho} \leqslant \left\|F_{(I-T_\delta)^r f}\right\|_{\mathcal{C}(\mathbf{R})} = \|(I - T_\delta)^r F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant 2^r \left(1 + 2^{2r-1}\right) \delta^r \left(A_0 (F_f)_{\mathcal{C}(\mathbf{R})} + \int_{1/2}^{1/\delta} u^{r-1} A_u (F_f)_{\mathcal{C}(\mathbf{R})}\, du\right) \leqslant$$

$$\leqslant \mathbb{C}_1 (1 + 3\mathbb{C}_1) 2^r \left(1 + 2^{2r-1}\right) \delta^r \left(A_0 (f)_{p,\varrho} + \int_{1/2}^{1/\delta} u^{r-1} A_{u/2} (f)_{p,\varrho}\, du\right)$$

because, taking g_σ as $\|f - g_\sigma\|_{p,\varrho} = A_\sigma(f)_{p,\varrho}$,

$$A_{2\sigma} (F_f)_{\mathcal{C}(\mathbf{R})} \leqslant \|F_f - V_\sigma F_f\|_{\mathcal{C}(\mathbf{R})} = \|F_{f-V_\sigma f}\|_{\mathcal{C}(\mathbf{R})} \leqslant \mathbb{C}_1 \|f - V_\sigma f\|_{p,\varrho} =$$

$$= \mathbb{C}_1 \|f - g_\sigma + g_\sigma - V_\sigma f\|_{p,\varrho} \leqslant \mathbb{C}_1 \left(\|f - g_\sigma\|_{p,\varrho} + \|V_\sigma g_\sigma - V_\sigma f\|_{p,\varrho}\right) \leqslant$$

$$\leqslant \mathbb{C}_1 \left(\|f - g_\sigma\|_{p,\varrho} + 3\mathbb{C}_1 \|g_\sigma - f\|_{p,\varrho}\right) = \mathbb{C}_1 (1 + 3\mathbb{C}_1) A_\sigma (f)_{p,\varrho}.$$

□

Marchaud inequality.

Theorem 11. *Let* $r, k \in \mathbb{N}$, $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $f \in L^p (\varrho dx)$ *and* $t \in (0, 1/2)$. *Then*

$$\Omega_r (f, t)_{p,\varrho} \leqslant \mathbb{C}_4 t^r \int_t^1 \frac{\Omega_{r+k} (f, u)_{p,\varrho}}{u^{r+1}} du$$

holds with $\mathbb{C}_4 := 20\pi\mathbb{C}_1 (1 + 2^{2r-1}) 2^{2r+3k}\mathbb{C}_2(r + k)$ *where* $\mathbb{C}_2 (1) := 36$, *and* $\mathbb{C}_2 (r) := 2^r (r^r + (34)^r)$ *for* $r > 1$.

Proof. Let $\sigma > 0$ and g_σ be an exponential-type entire function of degree $\leqslant \sigma$, belonging to $L^p (\varrho dx)$, as the best approximation of $f \in L^p (\varrho dx)$. Then

$$\Omega_r (f, t)_{p,\varrho} = \|(I - T_t)^r f\|_{p,\varrho} \leqslant \left\|F_{(I-T_t)^r f}\right\|_{\mathcal{C}(\mathbf{R})} = \|(I - T_t)^r F_f\|_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant (\mathbb{C}_4\mathbb{C}_1)t^r \int_t^1 \frac{\left\| (I-T_u)^{r+k} F_f \right\|_{\mathcal{C}(\mathbf{R})}}{u^{r+1}} du = (\mathbb{C}_4\mathbb{C}_1)t^r \int_t^1 \frac{\left\| F_{(I-T_u)^{r+k} f} \right\|_{\mathcal{C}(\mathbf{R})}}{u^{r+1}} du \leqslant$$

$$\leqslant \mathbb{C}_4 t^r \int_t^1 \frac{\left\| (I-T_u)^{r+k} f \right\|_{p,\varrho}}{u^{r+1}} du = \mathbb{C}_4 t^r \int_t^1 \frac{\Omega_{r+k}(f,u)_{p,\varrho}}{u^{r+1}} du.$$

□

Inverse theorem for derivatives.

Set $\lfloor \sigma \rfloor := \max\{n \in \mathbb{Z} : n \leqslant \sigma\}$.

Theorem 12. *Let $1 \leqslant p < \infty$, $\varrho \in A_{\mathbf{p}}$, $r \in \mathbb{N}$ and $f \in L^p(\varrho dx)$. If*

$$\sum_{\nu=0}^{\infty} \nu^{k-1} A_{\nu/2}(f)_{p,\varrho} < \infty$$

holds for some $k \in \mathbb{N}$, then $f^{(k)} \in L^p(\varrho dx)$ and

$$\Omega_r\left(f^{(k)}, \frac{1}{\sigma}\right)_{p,\varrho} \leqslant \mathbb{C}_5\left(\frac{1}{\sigma^r}\sum_{\nu=0}^{\lfloor\sigma\rfloor}(\nu+1)^{r+k-1}A_{\nu/2}(f)_{p,\varrho} + \sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty}\nu^{k-1}A_{\nu/2}(f)_{p,\varrho}\right) \tag{26}$$

with $\mathbb{C}_5 = 2^{2k+r+1}\mathbb{C}_1$.

Proof. Let g_σ be an exponential-type entire function of degree $\leqslant \sigma$, belonging to $\mathcal{C}(\mathbf{R})$, as the best approximation of $f \in \mathcal{C}(\mathbf{R})$. For natural numbers $p \leqslant k$, consider the series $g_1^{(p)} + \sum_{\nu=0}^{\infty}\{g_{2^{\nu+1}}^{(p)} - g_{2^\nu}^{(p)}\}$. Using Bernstein's inequality and assuming $2^\nu \leqslant \sigma < 2^{\nu+1}$, we get

$$\|g_{2^{(\nu+1)}}^{(p)}\text{-}g_{2^\nu}^{(p)}\|_{\mathcal{C}(\mathbf{R})} \leqslant 2^{(\nu+1)p}\ \|g_{2^{\nu+1}}\text{-}g_{2^\nu}\|_{\mathcal{C}(\mathbf{R})} \leqslant 2^{(\nu+1)p+1}\ A_{2^\nu}(f)_{\mathcal{C}(\mathbf{R})}.$$

Now, by the following estimation:

$$2^{(\nu+1)p}A_{2^\nu}(f)_{\mathcal{C}(\mathbf{R})} \leqslant 2^{2p}\sum_{\mu=2^{\nu-1}+1}^{2^\nu}\mu^{p-1}A_\mu(f)_{\mathcal{C}(\mathbf{R})},$$

we have

$$\|g_1^{(p)} + \sum_{\nu=0}^{\infty}\{g_{2^{\nu+1}}^{(p)} - g_{2^\nu}^{(p)}\}\|_{\mathcal{C}(\mathbf{R})} \leqslant \|g_1^{(p)}\|_{\mathcal{C}(\mathbf{R})} + \sum_{\nu=0}^{\infty}\|g_{2^{\nu+1}}^{(p)} - g_{2^\nu}^{(p)}\|_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant \|g_1^{(p)}\|_{\mathcal{C}(\mathbf{R})} + 2\sum_{\nu=0}^{\infty} 2^{(\nu+1)p} \ A_{2^\nu}(f)_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant \|g_1^{(p)}\|_{\mathcal{C}(\mathbf{R})} + 2^{p+1}A_1(f)_{\mathcal{C}(\mathbf{R})} + 2^{2p+1} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{p-1}A_\mu(f)_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant \|g_1^{(p)}\|_{\mathcal{C}(\mathbf{R})} + 2^{p+1}A_1(f)_{\mathcal{C}(\mathbf{R})} + 2^{2p+1} \sum_{\mu=2}^{\infty} \mu^{p-1}A_\mu(f)_{\mathcal{C}(\mathbf{R})} < \infty.$$

Denote the partial sum of the above series by $S_n^{(p)}$ for $p = 0, 1, 2, \ldots, k$; then the sequence of $S_n^{(p)}$ converge in the norm of $\mathcal{C}(\mathbf{R})$. So, for $p = k$, one can write

$$\Omega_r\Big(f^{(k)}, \frac{1}{\sigma}\Big)_{\mathcal{C}(\mathbf{R})} \leqslant \Omega_r\Big(f^{(k)} - S_n^{(k)}, \frac{1}{\sigma}\Big)_{\mathcal{C}(\mathbf{R})} + \Omega_r\Big(S_n^{(k)}, \frac{1}{\sigma}\Big)_{\mathcal{C}(\mathbf{R})} = I_1 + I_2.$$

Let us deal with the first item I_1. By boundedness of T_h and the Bernstein inequality, we obtain

$$\Omega_r\Big(f^{(k)} - S_n^{(k)}, \frac{1}{\sigma}\Big)_{\mathcal{C}(\mathbf{R})} \leqslant 2^r\|f^{(k)} - S_n^{(k)}\|_{\mathcal{C}(\mathbf{R})} = 2^r\Big\|\sum_{\nu=n+1}^{\infty}\{g_{2^{\nu+1}}^{(k)} - g_{2^\nu}^{(k)}\}\Big\|_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant 2^{r+1}\sum_{\nu=n+1}^{\infty} 2^{(\nu+1)k} \ A_{2^\nu}(f)_{\mathcal{C}(\mathbf{R})} \leqslant 2^{r+1}\sum_{\nu=n+1}^{\infty}\Big\{2^{2k}\sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{k-1}A_\mu(f)_{\mathcal{C}(\mathbf{R})}\Big\} \leqslant$$

$$\leqslant 2^{2k+r+1}\sum_{\mu=2^n+1}^{\infty} \mu^{k-1}A_\mu(f)_{\mathcal{C}(\mathbf{R})} \leqslant 2^{2k+r+1}\sum_{\mu=\lfloor\sigma\rfloor+1}^{\infty} \mu^{k-1}A_\mu(f)_{\mathcal{C}(\mathbf{R})}.$$

Next, let us estimate I_2.

$$\Omega_r\Big(S_n^{(k)}, \frac{1}{\sigma}\Big)_{\mathcal{C}(\mathbf{R})} \leqslant \Omega_r\Big(g_1^{(k)}, \frac{1}{\sigma}\Big)_{\mathcal{C}(\mathbf{R})} + \sum_{\nu=0}^{n}\Omega_r\Big(g_{2^{\nu+1}}^{(k)} - g_{2^\nu}^{(k)}, \frac{1}{\sigma}\Big)_{\mathcal{C}(\mathbf{R})}.$$

Hence,

$$\Omega_r\Big(S_n^{(k)}, \frac{1}{\sigma}\Big)_{\mathcal{C}(\mathbf{R})} \leqslant \frac{1}{\sigma^r}\|g_1^{(k+r)} - g_0^{(k+r)}\|_{\mathcal{C}(\mathbf{R})} + \frac{1}{\sigma^r}\sum_{\nu=0}^{n}\|g_{2^{\nu+1}}^{(k+r)} - g_{2^\nu}^{(k+r)}\|_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant \frac{1}{\sigma^r}\Big\{2A_0(f)_{\mathcal{C}(\mathbf{R})} + A_1(f)_{\mathcal{C}(\mathbf{R})} + \sum_{\nu=1}^{n} 2^{2(k+r)}\sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^k A_\mu(f)_{\mathcal{C}(\mathbf{R})}\Big\} \leqslant$$

$$\leqslant \frac{2}{\sigma^r}\Big\{\sum_{\mu=0}^{2^n}(\mu+1)^k A_\mu(f)_{\mathcal{C}(\mathbf{R})}\Big\} \leqslant \frac{2}{\sigma^r}\Big\{\sum_{\mu=0}^{\lfloor\sigma\rfloor}(\mu+1)^k A_\mu(f)_{\mathcal{C}(\mathbf{R})}\Big\}.$$

Since $2^n \leqslant \sigma < 2^{n+1}$, the inequalities imply

$$\Omega_r\left(f^{(k)}, \frac{1}{\sigma}\right)_{\mathcal{C}(\mathbf{R})} \leqslant$$

$$\leqslant 2^{2k+r+1}\Big(\frac{1}{\sigma^r}\sum_{\nu=0}^{\lfloor\sigma\rfloor}(\nu+1)^{r+k-1}A_\nu(f)_{\mathcal{C}(\mathbf{R})} + \sum_{\nu=\lfloor\sigma\rfloor+1}^{\infty}\nu^{k-1}A_\nu(f)_{\mathcal{C}(\mathbf{R})}\Big).$$

Now we use $A_\nu\left(F_f\right)_{\mathcal{C}(\mathbf{R})} \leqslant A_{\nu/2}\left(f\right)_{p,\varrho}$ and Theorem 3 to obtain (26). □

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