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TANGENT APPROXIMATION BY SOLUTIONS OF THE CONVOLUTION EQUATION

Abstract. The article for the first time studies the approximation of a function together with its derivatives on the real line by solutions of a multidimensional convolution equation of the form

$$g * T = 0,$$

where T is a given compactly supported radial distribution other than a constant multiplied by the Dirac delta function at zero. The following analogue of the well-known Carleman's theorem on tangent approximation by entire functions is obtained: if $f \in C^k(\mathbb{R})$ for some $k \in \mathbb{Z}_+$ and for any $\nu \in \{0, \dots, k\}$ there is a finite limit

$$\lim_{t \to +0} \left(\frac{d}{dt}\right)^{\nu} f(\ln t),$$

then for an arbitrary positive continuous function $\varepsilon(t)$ on $\mathbb{R}$ there exists an entire function $\Phi\colon \mathbb{C}^n \to \mathbb{C}$, such that $\Phi\big|_{\mathbb{R}^n} * T = 0$ and

$$\left| f^{(\nu)}(t) - \left(\frac{\partial}{\partial x_1}\right)^{\nu} \Phi(t, 0, \dots, 0)\right| < \varepsilon(e^t)$$

for all $t \in \mathbb{R}$, $\nu \in \{0, \dots, k\}$ (Theorem 1). It is shown that when approximating a function on subsets of the real line that do not contain a neighborhood of the point $-\infty$, the condition for the existence of a limit in Theorem 1 can be omitted. In addition, the method of proving Theorem 1 allows one to obtain new results that are of interest for the theory of convolution equations of the indicated type. These are results about the growth of solutions (Corollary 3), on the distribution of values (Theorem 2), and also on the solvability of the interpolation problem for solutions of the convolution equation (Corollary 4).

Key words: *convolution equation, mean periodicity, Carleman's approximation theorem*

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1. Introduction. Famous Carleman's tangent approximation theorem derived in 1927 states that for every function $f \in C(\mathbb{R})$ and every positive function $\varepsilon \in C(\mathbb{R})$ there exists an entire function $g\colon \mathbb{C} \to \mathbb{C}$, such that

$$|f(t) - g(t)| < \varepsilon(t)$$

for all $t \in \mathbb{R}$ (see, for example, [10, Chap. 4, Sect. 3]). Everywhere below any positive function $\varepsilon \in C(\mathbb{R})$ will be called *an error function.* Carleman's theorem has been further developed and refined in many papers (see bibliography in [8] and [10]). Carleman himself had already generalized his result by replacing $\mathbb{R}$ by more general curves and systems of curves in the complex plane. Many authors have studied, in connection with Carleman's theorem, approximation in combination with interpolation, as well as tangent approximation of smooth functions together with their derivatives (see [11], [14], [16], [19]– [22], [24]). In articles [1]–[3], [5], [6], [9], [17], [18], [26], approximation with a certain rate of decrease of the error function was considered. Questions related to tangent and uniform approximation under restrictions on the growth of the approximating function were also studied (see [4], [17]). We also note the multidimensional analog of Carleman's theorem obtained in [23]. Carleman's theorem and its generalizations play an important role in the study of boundary properties of analytic functions and in the study of the distribution of their values (see [10, Chap. 4, Sect. 5]).

The class of entire functions $g\colon \mathbb{C} \to \mathbb{C}$ coincides with the set of solutions of the differential equation

$$\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) g = 0, \quad (x_1, x_2) \in \mathbb{R}^2.$$

In this regard, it is of interest to obtain analogues of Carleman's theorem in which the approximation is made by solutions of other linear partial differential equations in $\mathbb{R}^n$, $n \geqslant 2$, with constant coefficients. For the solutions of most of these equations, many important and useful properties of the class of entire functions are not fulfilled (for example, they as a rule do not form an algebra), which prevents them from obtaining analogues of Carleman's theorem by the known methods. The simplest example is the class of eigenfunctions of the Laplace operator in $\mathbb{R}^2$, that is, the set of solutions of the equation

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)g+\lambda g=0, \quad (x_1,x_2)\in\mathbb{R}^2,$$

for $\lambda \neq 0$.

In this paper, for the first time, we study the approximation of a function together with its derivatives on the real line by solutions of a multidimensional convolution equation of the form

$$g * T = 0, \tag{1}$$

where T is a given radial distribution with compact support in $\mathbb{R}^n$, $n \geqslant 2$ (see Sect. 2). The theory of equations (1) originates in the work of the famous Romanian mathematician D. Pompeiu who considered the case when T is the indicator of a ball in $\mathbb{R}^n$ (see, e. g., [30]). Equation (1) as well as its various analogues and generalizations have been intensively studied over the past fifty years by F. John, J. Delsarte, J. D. Smith, L. Zalcman, C. A. Berenstein, and others (see the overviews in [7], [30], [31] and monographs [27]–citel23, which provide extensive bibliographies). We note that with an appropriate choice of T they characterize such important classes of functions as functions with zero spherical (or ball) means, functions with the property of mean values from the theory of harmonic functions, and also solutions of elliptic differential equations of the form

$$p(\Delta)g = 0, \tag{2}$$

where Δ is the Laplace operator in $\mathbb{R}^n$, and p is an arbitrary algebraic polynomial other than the identical constant.

The main results of this paper are formulated in Sect. 2. Among them note Theorem 1 on the approximation of a function and its derivatives on $\mathbb{R}$ by solutions of equation (1), as well as a number of new results that are of interest for the theory of equations of the form (1). These are results about the growth of solutions (Corollary 3), on the distribution of values (Theorem 2), and also on the solvability of the interpolation problem for solutions of equation (1) (Corollary 4). Section 3 contains the necessary auxiliary assertions. The proofs of Theorems 1 and 2, and Corollaries 1–3 are contained in Sects. 4 and 5.

2. Statements of the main results. Everywhere in what follows, $\mathbb{R}^n$ is a Euclidean space of dimension $n \geqslant 2$. Denote by $\mathcal{D}'(\mathbb{R}^n)$ (respectively, $\mathcal{E}'(\mathbb{R}^n)$) the space of distributions (respectively, distributions with compact supports) in $\mathbb{R}^n$; $\mathcal{D}(\mathbb{R}^n)$ is the space of finite infinitely differentiable functions in $\mathbb{R}^n$; $\mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$.

Let $T \in \mathcal{E}'(\mathbb{R}^n)$, $T \neq 0$, and let $\operatorname{supp} T$ be the support of T. For every $f \in \mathcal{D}'(\mathbb{R}^n)$, the convolution $f * T$ is defined by the equality

$$\langle f * T, \varphi \rangle = \langle f_y, \langle T_x, \varphi(x+y) \rangle \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n),$$

as a distribution in $\mathcal{D}(\mathbb{R}^n)$ (the index at the bottom of the distributions f and T means the action on the specified variable). A distribution of the class

$$\mathcal{D}'_T(\mathbb{R}^n) = \{f \in \mathcal{D}'(\mathbb{R}^n) \colon f * T = 0\}$$

is called *mean periodic* with respect to T. Denote by $E_T(\mathbb{R}^n)$ the set of all entire functions $h \colon \mathbb{C}^n \to \mathbb{C}$ whose restrictions to $\mathbb{R}^n$ are mean periodic with respect to T.

Let $SO(n)$ be the rotation group of $\mathbb{R}^n$. A distribution $T \in \mathcal{E}'(\mathbb{R}^n)$ is called *radial* if it is invariant under the group $SO(n)$, i. e.,

$$\langle T, \varphi(\tau x) \rangle = \langle T, \varphi(x) \rangle \quad \text{for all} \quad \varphi \in \mathcal{E}(\mathbb{R}^n), \quad \tau \in SO(n).$$

Denote by $\mathcal{E}'_\natural(\mathbb{R}^n)$ the set of all radial distributions $T \in \mathcal{E}'(\mathbb{R}^n)$. The simplest example of distribution in the class $\mathcal{E}'_\natural(\mathbb{R}^n)$ is the Dirac delta function δ_0 with support at zero, i. e.,

$$\langle \delta_0, \varphi \rangle = \varphi(0), \quad \varphi \in \mathcal{E}(\mathbb{R}^n).$$

In the study of equation (1), the Fourier-Laplace transform of the distribution T plays an important role, which for radial T is expressed in terms of the so-called *spherical transform* $\widetilde{T}$ (see [15, Ch. 7, Sect. 7.3; Ch. 16], [25, Ch. 4, Theorem 3.3] and Lemma 1 below). To define $\widetilde{T}$, we consider an even entire function

$$\eta(z) = 2^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n}{2}-1}(z)}{z^{\frac{n}{2}-1}}, \quad z \in \mathbb{C},$$

where, as usual, Γ is the gamma function and $J_{\frac{n}{2}-1}$ is the Bessel function of the first kind of order $\dfrac{n}{2} - 1$. Note that for any $z \in \mathbb{C}$, the function

$$\eta(z|x|), \quad x \in \mathbb{R}^n$$

is a spherical function on $\mathbb{R}^n$, that is, an eigenfunction of the operator Δ in $\mathbb{R}^n$ satisfying the condition $\eta(0) = 1$ (see [13, Ch. 4, Sect. 2]). The spherical transform $\widetilde{T}$ of the distribution $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$ is defined by the equality

$$\widetilde{T}(z) = \langle T_x, \eta(z|x|)\rangle \tag{3}$$

and is an even entire function of the variable $z \in \mathbb{C}$.

Example 1. Let $T = p(\Delta)\delta_0$ for some polynomial p. Then $\widetilde{T}(z) = = p(-z^2)$, $z \in \mathbb{C}$. In this case, equation (1) is equivalent to equation (2).

Example 2. Let $T = \chi_r$ be the indicator of a ball of radius r centered at zero in $\mathbb{R}^n$. Then (see [29, Part II, Ch. 3, formula (3.90)])

$$\widetilde{T}(z) = \left(\frac{2\pi r}{z}\right)^{\frac{n}{2}} J_{\frac{n}{2}}(rz).$$

A detailed exposition of the theory of spherical functions and spherical transformations on symmetric spaces is contained in [13, Ch. 4, Sect. 2] (see also [28], [29]).

Let

$$\mathcal{Z}(\widetilde{T}) = \{z \in \mathbb{C} : \widetilde{T}(z) = 0\}.$$

The set $\mathcal{Z}(\widetilde{T})$ plays an important role in the study of the equation (1). The necessary information about the properties of this set is contained in Lemma 2 and Corollaries 6 and 7 below. As shown in these statements, $\mathcal{Z}(\widetilde{T}) \neq \varnothing$ if and only if $T \neq c\delta_0$, $c \in \mathbb{C}\backslash\{0\}$. In addition, equation (1) has a nonzero solution if and only if $\mathcal{Z}(\widetilde{T}) \neq \varnothing$.

Let us proceed to the formulations of the main results of this paper.

Theorem 1. *Let $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$, $\mathcal{Z}(\widetilde{T}) \neq \varnothing$, let $f \in C^k(\mathbb{R})$ for some $k \in \mathbb{Z}_+$, and assume that for any $\nu \in \{0, \dots, k\}$ there exists a finite limit*

$$\lim_{t\to+0} \left(\frac{d}{dt}\right)^{\nu} f(\ln t). \tag{4}$$

Then for any error function $\varepsilon(t)$ there exists $\Phi \in E_T(\mathbb{R}^n)$, such that

$$\left| f^{(\nu)}(t) - \left(\frac{\partial}{\partial x_1}\right)^{\nu} \Phi(t, 0, \dots, 0)\right| < \varepsilon(e^t) \tag{5}$$

for all $t \in \mathbb{R}$, $\nu \in \{0, \dots, k\}$.

Due to arbitrariness of the function $\varepsilon(t)$, inequality (5) provides uniform approximation of the functions $f^{(\nu)}$ on $\mathbb{R}$ and tangent approximation of these functions in a neighborhood of the point $+\infty$.

According to what has been said above, the condition $\mathcal{Z}(\widetilde{T}) \neq \varnothing$ in Theorem 1 (and all subsequent assertions of this section) is necessary. The question of the necessity of the condition for the existence of limits (4) in Theorem 1 remains open. However, as it is easy to see, when approximating a function on subsets of the real line that do not contain a neighborhood of the point $-\infty$, this condition can be omitted. Let us note some consequences of Theorem 1 in which this condition is absent.

Corollary 1. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $\mathcal{Z}(\widetilde{T}) \neq \varnothing$, $a \in \mathbb{R}$, and assume that $f \in C^k([a, +\infty))$ for some $k \in \mathbb{Z}_+$. Then, for any error function $\varepsilon(t)$, there exists a function $\Phi \in E_T(\mathbb{R}^n)$, such that*

$$\left| f^{(\nu)}(t) - \left(\frac{\partial}{\partial x_1}\right)^{\nu} \Phi(t, 0, \ldots, 0) \right| < \varepsilon(t)$$

for all $t \in [a, +\infty)$, $\nu \in \{0, \ldots, k\}$.

Corollary 2. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $\mathcal{Z}(\widetilde{T}) \neq \varnothing$, and assume that $f \in C^k(\mathbb{R})$ for some $k \in \mathbb{Z}_+$. Then there exists a sequence Φ_m, $m = 1, 2, \ldots$ of functions in the class $E_T(\mathbb{R}^n)$, such that for any $\nu \in \{0, \ldots, k\}$ the sequence of functions*

$$\left(\frac{\partial}{\partial x_1}\right)^{\nu} \Phi_m(t, 0, \ldots, 0)$$

converges to $f^{(\nu)}(t)$ uniformly on $[a, +\infty)$ for every $a \in \mathbb{R}$.

Let us now consider some applications of Theorem 1 to the study of properties of solutions to the equation (1).

The following result shows that for $\mathcal{Z}(\widetilde{T}) \neq \varnothing$ equation (1) has smooth solutions of arbitrarily fast growth at infinity.

Corollary 3. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $\mathcal{Z}(\widetilde{T}) \neq \varnothing$. Then, for any function $\psi \in C(\mathbb{R})$, there exists a function $\Phi \in E_T(\mathbb{R}^n)$, such that*

$$|\Phi(t, 0, \ldots, 0)| > \psi(t) \tag{6}$$

for all $t > 0$.

The following assertion about the distribution of values of some functions of the class $E_T(\mathbb{R}^n)$ is also fulfilled:

Theorem 2. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $\mathcal{Z}(\widetilde{T}) \neq \varnothing$, and let $m \in \mathbb{Z}_+$. Then, for any unbounded sequence $\{t_k\}_{k=1}^{\infty}$ of real numbers, there exists a function $\Phi \in E_T(\mathbb{R}^n)$, such that the set of points*

$$\Big\{(\zeta_{0,k}, \zeta_{1,k}, \ldots, \zeta_{m,k}) \in \mathbb{C}^{m+1} : \\ \zeta_{\nu,k} = \Big(\frac{\partial}{\partial x_1}\Big)^{\nu} \Phi(t_k, 0, \ldots, 0), k \in \mathbb{N}, \nu \in \{0, \ldots, m\}\Big\}$$

is dense in $\mathbb{C}^{m+1}$. In particular, there exists a function $\Psi \in E_T(\mathbb{R}^n)$, for which the set of values of the function $\Psi(t, 0, \ldots, 0)$ on the ray $[0, +\infty)$ is dense in $\mathbb{C}$.

It can be seen, from the proof of Theorem 2 for the case $m = 0$, that the following result about the solvability of the interpolation problem for solutions of equation (1) with nodes on an arbitrary ray in $\mathbb{R}^n$ holds:

Corollary 4. *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of pairwise distinct points lying on an arbitrary ray in $\mathbb{R}^n$ and satisfying the condition*

$$\lim_{k\to\infty} a_k = \infty. \tag{7}$$

Let also $T \in \mathcal{E}'_{\natural}(\mathbb{R}^n)$ and $\mathcal{Z}(\widetilde{T}) \neq \varnothing$. Then, for any sequence of complex numbers $\{b_k\}_{k=1}^{\infty}$ there exists a function $\Phi \in E_T(\mathbb{R}^n)$, such that

$$\Phi(a_k) = b_k \tag{8}$$

for all $k \in \mathbb{N}$.

It is easy to see that condition (7) in Corollary 4 is necessary. Indeed, if it is not satisfied, then the sequence $\{a_k\}_{k=1}^{\infty}$ has a finite limit point. In this case, for functions Φ, continuous on $\mathbb{R}^n$, condition (8) cannot be satisfied for any sequence $\{b_k\}_{k=1}^{\infty}$.

To conclude this section, we note that the analogues of all the above statements for distributions $T \in \mathcal{E}'(\mathbb{R}^n)$ without the radiality condition are, generally speaking, false. Corresponding counterexamples can be constructed by considering, for example, the distribution $T = \dfrac{\partial}{\partial x_1}\delta_0$.

3. Auxiliary results. For an arbitrary $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$, we put

$$(z, \zeta) = \sum_{j=1}^{n} z_j \zeta_j.$$

Let $S = \{z \in \mathbb{C}^n \colon\ (z, z) = 1\}$, $\mathbb{S}^{n-1} = S \cap \mathbb{R}^n$. For any $T \in \mathcal{E}'(\mathbb{R}^n)$ denote by $\widehat{T}$ the Fourier-Laplace transform of the distribution T, i. e.,

$$\widehat{T}(z) = \left\langle T_x, e^{-i(z,x)} \right\rangle, \quad z \in \mathbb{C}^n. \tag{9}$$

Lemma 1. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$. Then*

$$\widetilde{T}(\lambda) = \widehat{T}(\lambda\xi) \tag{10}$$

for all $\lambda \in \mathbb{C}$, $\xi \in S$.

Proof. Let $\lambda \in \mathbb{C}$, $\xi \in S$, and let

$$w_\xi(x) = e^{i\lambda(x,\,\xi)}, \quad x \in \mathbb{R}^n. \tag{11}$$

Since T is radial, we have

$$\langle T, w_\xi(-x)\rangle = \langle T, w_\xi(-\tau x)\rangle \tag{12}$$

for all $\tau \in SO(n)$. Let $d\tau$ be a Haar measure on $SO(n)$ for which

$$\int\limits_{SO(n)} d\tau = 1.$$

From (12), we find

$$\langle T, w_\xi(-x)\rangle = \int\limits_{SO(n)} \langle T, w_\xi(-\tau x)\rangle\, d\tau = \left\langle T, \int\limits_{SO(n)} w_\xi(-\tau x) d\tau \right\rangle. \tag{13}$$

Taking [13, Introduction, Sect. 3.1, formula (9)] into account, we obtain

$$\int\limits_{SO(n)} w_\xi(-\tau x) d\tau = \frac{1}{\omega_{n-1}} \int\limits_{\mathbb{S}^{n-1}} e^{-i\lambda(|x|\eta,\,\xi)} d\omega(\eta), \tag{14}$$

where $d\omega$ is the surface measure on $\mathbb{S}^{n-1}$ and

$$\omega_{n-1} = \int\limits_{\mathbb{S}^{n-1}} d\omega(\eta) = n\pi^{\frac{n}{2}}/\Gamma\Big(1+\frac{n}{2}\Big).$$

By (14) and [13, Introduction, Sect. 3.2, Lemma 3.6] we have:

$$\int\limits_{SO(n)} w_\xi(-\tau x) d\tau = 2^{\frac{n}{2}-1}\Gamma\Big(\frac{n}{2}\Big) \frac{J_{\frac{n}{2}-1}(\lambda|x|)}{(\lambda|x|)^{\frac{n}{2}-1}}.$$

Using now equalities (9), (8) and (3), we arrive at the relation (10). □

Corollary 5. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, $\lambda \in \mathcal{Z}(\widetilde{T})$, and let $\xi \in S$. Then the function w_ξ given by the equality (11) belongs to the class $E_T(\mathbb{R}^n)$.*

Proof. For every $y \in \mathbb{R}^n$, we have

$$(w_\xi * T)(y) = \langle T, w_\xi(-x)\rangle\, w_\xi(y) \tag{15}$$

However, according to (10),

$$\langle T, w_\xi(-x)\rangle = \widehat{T}(\lambda\xi) = \widetilde{T}(\lambda) = 0.$$

Hence, by (15) the required assertion follows. □

Lemma 2. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$. Then the following statements are equivalent:*

(i) $\operatorname{supp} T = \{0\}$*;*

(ii) $\widetilde{T}(z) = p(-z^2)$ *for some nonzero polynomial* p*;*

(iii) $T = p(\Delta)\delta_0$*;*

(iv) the set $\mathcal{Z}(\widetilde{T})$ *is finite or empty.*

Proof. (i)→(ii). It follows from the condition $\operatorname{supp} T = \{0\}$, Lemma 1, and the Paley-Wiener-Schwartz theorem (see [15, Theorem 7.3.1]) that there exists $N > 0$, such that

$$|\widetilde{T}(z)| \leqslant (2 + |z|)^N, \quad z \in \mathbb{C}.$$

Then, by Liouville's theorem, the function $\widetilde{T}$ is an even polynomial in the variable z. This means that (ii) is satisfied.

(ii)→(iii). Since

$$\widetilde{p(\Delta)\delta_0}(z) = p(-z^2),$$

the spherical transform of the distribution $T - p(\Delta)\delta_0$ is equal to zero identically. Hence, from (10) we obtain that $T = p(\Delta)\delta_0$.

Implications (iii)→(i) and (ii)→(iv) are obvious. Therefore, to complete the proof of the lemma, it suffices to prove the implication (iv)→(ii). Since $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$, it follows from the equality (10) and the Paley-Wiener-Schwartz theorem (see [15, Theorem 7.3.1]) that the function $\widetilde{T}$ is an entire function of exponential type. Hence, if the set $\mathcal{Z}(T)$ is finite or empty, then, from the Hadamard factorization theorem for entire functions, we have the equality

$$\widetilde{T}(z) = e^{cz}q(z), \quad z \in \mathbb{C}, \tag{16}$$

where $c \in \mathbb{C}$ and q is some non-zero polynomial. Considering that the function $\widetilde{T}$ is even, from (16) we find

$$e^{2cz}q(z) = q(-z), \quad z \in \mathbb{C}.$$

Consequently, $c = 0$ and the polynomial q is even. Thus, $q(z) = p(-z^2)$ for some nonzero polynomial p, and Lemma 2 is completely proved. □

From Lemma 2 and Corollary 5, we obtain the following assertions:

Corollary 6. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$. Then $\mathcal{Z}(\widetilde{T}) = \varnothing$ if and only if $T = c\delta_0$ for some nonzero constant $c \in \mathbb{C}$.*

Corollary 7. *Let $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$. Then, in order for equation (1) to have a nonzero solution, it is necessary and sufficient that $\mathcal{Z}(\widetilde{T}) \neq \varnothing$.*

Lemma 3. *Let $\nu \in \mathbb{Z}_+$. Then there are numbers $\gamma_{j,\nu} \in \mathbb{Z}_+$, $j \in \{0, \dots, \nu\}$, such that $\gamma_{\nu,\nu} = 1$ and*

$$\left(\frac{d}{dt}\right)^\nu (h(e^t)) = \sum_{j=0}^{\nu} \gamma_{j,\nu}\, h^{(j)}(e^t)e^{jt}, \quad t \in \mathbb{R} \tag{17}$$

for any function $h \in C^\nu((0, +\infty))$.

The proof of Lemma 3 is easily obtained by induction on m.

Lemma 4. *Let $m, \lambda \in \mathbb{Z}_+$, and let*

$$g(z) = \frac{\Gamma(m + \lambda + 2)}{\Gamma(m + 1)\Gamma(\lambda + 1)} \int_0^z (1 - \zeta)^m \zeta^\lambda d\zeta, \quad z \in \mathbb{C}. \tag{18}$$

Then

$$g(1) = 1, \quad g^{(s)}(1) = 0 \quad \text{for} \quad 1 \leqslant s \leqslant m, \tag{19}$$

and

$$|g(z)| \leqslant (m + \lambda + 1)2^{m+\lambda}(1 + |z|)^m |z|^{\lambda+1}, \quad z \in \mathbb{C}. \tag{20}$$

Proof. The equalities (19) are obvious. Next,

$$\frac{\Gamma(m + \lambda + 2)}{\Gamma(m + 1)\Gamma(\lambda + 1)} \leqslant (m + \lambda + 1) \sum_{j=0}^{m+\lambda} \binom{m + \lambda}{j} \leqslant (m + \lambda + 1)2^{m+\lambda}.$$

Estimate (20) now follows from (18). □

4. Proof of Theorem 1 and its corollaries. Let $k \in \mathbb{Z}_+$, $f \in C^k(\mathbb{R})$, and assume that for any $\nu \in \{0, \dots, k\}$ there exists a finite limit (4). We put

$$g(t) = f(\ln t) \quad \text{for} \quad t > 0. \tag{21}$$

It follows from the existence of limits (4) that the function g can be extended to the whole real line up to a function of the class $C^k(\mathbb{R})$. Consider one of these extensions, for which we retain the notation g. Let $\varepsilon(t)$ be an arbitrary error function. Denote

$$M = \max_{0 \leqslant \nu \leqslant k} \sum_{j=0}^{\nu} \gamma_{j,\nu}, \tag{22}$$

where the numbers $\gamma_{j,\nu} \in \mathbb{N}$ are determined by the equality (17) in Lemma 3. Let also

$$\varphi(t) = \begin{cases} 1, & \text{if} \;\; t < 1, \\ t^k, & \text{if} \;\; t \geqslant 1. \end{cases}$$

Then there exists an entire function $u : \mathbb{C} \to \mathbb{C}$, such that

$$\left| g^{(\nu)}(t) - u^{(\nu)}(t) \right| < \frac{\varepsilon(t)}{M\varphi(t)}, \quad t \in \mathbb{R}, \quad \nu \in \{0, \dots, k\}. \tag{23}$$

(see [10, Chap. 4, Remark to Sect. 3]). For $\nu = 0$, from here and from (21), we have

$$\left| f(t) - u(e^t) \right| = \left| g(e^t) - u(e^t) \right| < \varepsilon(e^t), \quad t \in \mathbb{R}. \tag{24}$$

If $k > 0$, then for any $\nu \in \{0, \dots, k\}$ we get

$$\left| \sum_{j=0}^{\nu} \gamma_{j,\nu} \, g^{(j)}(e^t) e^{jt} - \sum_{j=0}^{\nu} \gamma_{j,\nu} \, u^{(j)}(e^t) e^{jt} \right| \leqslant$$

$$\leqslant \sum_{j=0}^{\nu} \gamma_{j,\nu} \, e^{jt} \left| g^{(j)}(e^t) - u^{(j)}(e^t) \right|, \quad t \in \mathbb{R}.$$

Taking into account (21), (22), and (23), from this estimate and Lemma 3 we find

$$\left| f^{(\nu)}(t) - \left(\frac{d}{dt}\right)^{\nu} (u(e^t)) \right| = \left| \left(\frac{d}{dt}\right)^{\nu} (g(e^t)) - \left(\frac{d}{dt}\right)^{\nu} (u(e^t)) \right| \leqslant$$

$$\leqslant \varphi(e^t) \sum_{j=0}^{\nu} \gamma_{j,\nu} \left| g^{(j)}(e^t) - u^{(j)}(e^t) \right| < \varepsilon(e^t) \tag{25}$$

for any $t \in \mathbb{R}$. To go further, let the Taylor expansion of the function u have the form

$$u(z) = \sum_{p=0}^{\infty} c_p z^p, \quad z \in \mathbb{C}. \tag{26}$$

Then

$$\sum_{p=0}^{\infty} |c_p| R^p < +\infty \tag{27}$$

for any $R > 0$.

Suppose first that the set $\mathcal{Z}(\widetilde{T})$ contains a point $\lambda \neq 0$. For any $p \in \mathbb{Z}_+$, we define a function

$$\Phi_p(z) = e^{i\lambda(z,\xi)}, \quad z \in \mathbb{C}^n,$$

where $\xi = (\xi_1, \dots, \xi_n) \in S$ and $\xi_1 = -ip/\lambda$. Let also

$$\Phi(z) = \sum_{p=0}^{\infty} c_p \Phi_p(z), \quad z \in \mathbb{C}^n. \tag{28}$$

Condition (27) implies that the series on the right-hand side of equality (28) converges locally uniformly in $\mathbb{C}^n$ and the function Φ is entire. In addition, this series converges in $\mathcal{D}'(\mathbb{R}^n)$. Hence, from Corollary 5 and the continuity of the convolution operator in $\mathcal{D}'(\mathbb{R}^n)$, we have

$$\Phi * T = \sum_{p=0}^{\infty} c_p \Phi_p * T = 0.$$

Thus, $\Phi \in E_T(\mathbb{R}^n)$.

From (28) and (26), we have the equality

$$\Phi(t, 0, \dots, 0) = u(e^t), \quad t \in \mathbb{R}.$$

Taking into account estimates (24) and (25), we obtain that the function Φ satisfies all the requirements of the theorem.

It remains to consider the case when $\mathcal{Z}(\widetilde{T}) = \{0\}$. In this case, it follows from Lemma 2 that

$$T = c\Delta^m \delta_0$$

for some constant $c \in \mathbb{C}$ and some $m \in \mathbb{N}$. Then, for any entire function $g\colon \mathbb{C} \to \mathbb{C}$, the function

$$G(x) = g(x_1 + ix_2), \quad x \in \mathbb{R}^n$$

belongs to the class $E_T(\mathbb{R}^n)$ and

$$G(t, 0, \ldots, 0) = g(t), \quad t \in \mathbb{R}.$$

Therefore, the assertion of Theorem 1 in this case follows from Carleman's theorem. □

Let us proceed to the proof of Corollaries 1–3 of Theorem 1. Let $a \in \mathbb{R}$, and let $f \in C^k([a, +\infty))$ for some $k \in \mathbb{Z}_+$. According to the Hermite interpolation formula (see, for example, [12, Chap. 1, Sect. 4, Subsect. 3]) there exists a polynomial P that satisfies the conditions

$$P^{(\nu)}(a-1) = 0, \quad P^{(\nu)}(a) = f^{(\nu)}(a)$$

for all $\nu \in \{0, \ldots, k\}$. Then the function

$$F(t) = \begin{cases} 0, & \text{if } \ t < a-1; \\ P(t), & \text{if } \ t \in [a-1, a]; \\ f(t), & \text{if } \ t > a \end{cases}$$

belongs to the class $C^k(\mathbb{R})$ and satisfies condition (5). Applying Theorem 1 to this function, we obtain Corollary 1. Since $a \in \mathbb{R}$ is arbitrary, we also have Corollary 2.

Next, let ε be some error function and a function $f \in C(\mathbb{R})$ satisfy condition (3) for $\nu = 0$. Let also

$$f(t) > |\psi(t)| + \varepsilon(e^t) \quad \text{for} \quad t > 0. \tag{29}$$

Applying Theorem 1 to f for $k = 0$, from inequalities (29) and (5) we obtain (6). This proves Corollary 3.

5. Proof of Theorem 2. Without loss of generality, we can assume that $\{t_k\}_{k=1}^{\infty}$ is a strictly increasing unbounded sequence of positive numbers. We put

$$\tau_k = \exp t_k, \quad k \in \mathbb{N};$$

then

$$\lim_{k\to\infty} \tau_k = +\infty. \tag{30}$$

For a given $m \in \mathbb{Z}_+$, consider an arbitrary set

$$A_m = \{(\zeta_{0,k}, \dots, \zeta_{m,k}) \in \mathbb{C}^{m+1}, \quad k \in \mathbb{N}\},$$

that is dense in $\mathbb{C}^{m+1}$. It suffices to prove that there exists a function $\Phi \in E_T(\mathbb{R}^n)$, such that

$$\left(\frac{\partial}{\partial x_1}\right)^{\nu} \Phi(t_k, 0, \dots, 0) = \zeta_{\nu,k}, \quad k \in \mathbb{N}, \quad \nu \in \{0, \dots, m\}. \tag{31}$$

By virtue of the classical Hadamard theorem, there exists an entire function $H\colon \mathbb{C} \to \mathbb{C}$, such that

$$\begin{cases} H^{(j)}(\tau_k) = 0 & \text{for} \quad j \in \{0, \dots, m\}, \quad k \in \mathbb{N}, \\ H^{(j)}(\tau_k) \neq 0 & \text{for} \quad j = m+1, \quad k \in \mathbb{N}. \end{cases} \tag{32}$$

Consider the functions

$$H_{k,l}(z) = \frac{H(z)}{(z - \tau_k)^l}, \quad k \in \mathbb{N}, \quad l \in \{1, \dots, m+1\}. \tag{33}$$

Condition (32) implies that the functions $H_{k,l}$ are entire and

$$\begin{cases} H_{k,l}^{(j)}(\tau_k) = 0, & \text{if} \quad 0 \leqslant j \leqslant m - l, \\ H_{k,l}^{(j)}(\tau_k) \neq 0, & \text{if} \quad j = m - l + 1. \end{cases} \tag{34}$$

Let $\gamma_{j,\nu}$ be the numbers defined in Lemma 3. Then for any $k \in \mathbb{N}$, there are constants $\alpha_{k,j} \in \mathbb{C}$, $j \in \{0, \dots, m\}$ that satisfy the condition

$$\sum_{j=0}^{\nu} \gamma_{j,\nu} \tau_k^j \alpha_{k,j} = \zeta_{\nu,k}, \quad \nu = 0, \dots, m. \tag{35}$$

Next, from (34) it is clear that there are numbers $\beta_{k,l} \in \mathbb{C}$, $l \in \{1, \dots, m+1\}$, such that the functions

$$H_k(z) = \sum_{l=1}^{m+1} \beta_{k,l} H_{k,l}(z) \tag{36}$$

satisfy the conditions

$$H_k^{(j)}(\tau_k) = \alpha_{k,j}, \quad j \in \{0, \dots, m\}. \tag{37}$$

Relations (32), (33) and (36) also imply that

$$H_k^{(j)}(\tau_p) = 0, \quad \text{if} \quad p \in \mathbb{N}, \quad p \neq k, \quad j \in \{0, \dots, m\}. \tag{38}$$

Next, let

$$M_k = \max_{|z| \leqslant \tau_k/4} |H_k(z)|,$$

$$\lambda_k \in \mathbb{N} \quad \text{and} \quad \lambda_k > 2m + k + M_k \tag{39}$$

for all $k \in \mathbb{N}$. Consider the function

$$w(z) = \sum_{k=1}^{\infty} g_k\left(\frac{z}{\tau_k}\right) H_k(z), \quad z \in \mathbb{C}, \tag{40}$$

where

$$g_k(z) = \frac{\Gamma(m + \lambda_k + 2)}{\Gamma(m+1)\Gamma(\lambda_k + 1)} \int_0^z (1 - \zeta)^m \zeta^{\lambda k} d\zeta, \quad z \in \mathbb{C}.$$

Let us show that the series on the right-hand side of equality (40) converges locally uniformly in $\mathbb{C}$. Let $R > 0$, $|z| \leqslant R$ and k be chosen so large that $\tau_k > 4R$ (see (30)). Then

$$|H_k(z)| \leqslant \max_{|z| \leqslant R} |H_k(z)| \leqslant \max_{|z| \leqslant \tau_k/4} |H_k(z)| = M_k.$$

Hence, according to Lemma 4, we have

$$\left| g_k\left(\frac{z}{\tau_k}\right) H_k(z) \right| \leqslant (m + \lambda_k + 1) 2^{m+\lambda_k} \left(\frac{R}{\tau_k}\right)^{\lambda_k+1} \left(1 + \frac{R}{\tau_k}\right)^m M_k.$$

Since

$$\frac{R}{\tau_k} \leqslant \frac{1}{4}, \quad m < \frac{\lambda_k}{2} \quad \text{and} \quad M_k < \lambda_k$$

(see (39)), we arrive at the inequality

$$\left| g_k\left(\frac{z}{\tau_k}\right) H_k(z) \right| \leqslant \left(\frac{3\lambda_k}{2} + 1\right) \lambda_k \left(\frac{5}{8}\right)^{\lambda_k/2}.$$

Bearing in mind that $\lambda_k > k$ (see (39)), we get from here that the series in (40) converges uniformly in the circle $|z| \leqslant R$. Because of the arbitrariness of $R > 0$, this means that this series converges locally uniformly in

$\mathbb{C}$ and the function w is entire. Moreover, relations (37), (38) and (19) imply that

$$w^{(j)}(\tau_k) = \alpha_{k,j}, \quad k \in \mathbb{N}, \quad j \in \{0, \dots, m\}.$$

Hence, from Lemma 3 and equality (35) we conclude that

$$\left(\frac{d}{dt}\right)^{\nu} (w(e^t))\Big|_{t=t_k} = \zeta_{k,\nu} \tag{41}$$

for all $k \in \mathbb{N}$, $\nu \in \{0, \dots, m\}$. Now repeating the arguments from the proof of Theorem 1 for the function w instead of u, we infer that there exists a function $\Phi \in E_T(\mathbb{R}^n)$ satisfying the condition

$$\Phi(t, 0, \dots, 0) = w(e^t), \quad t \in \mathbb{R}.$$

From here and from (41) follows equality (31). Thus, Theorem 2 is completely proved. □

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