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B. Selmi, N. Yu. Svetova

ON THE MUTUAL MULTIFRACTAL ANALYSIS FOR SOME NON-REGULAR MORAN MEASURES

Abstract. In this paper, we study the mutual multifractal Hausdorff dimension and the packing dimension of level sets $K(\alpha, \beta)$ for some non-regular Moran measures satisfying the so-called Strong Separation Condition. We obtain sufficient conditions for the valid multifractal formalisms of such measures and discuss examples.

Key words: *fractal/multifractal dimensions, Moran sets, non-regular Moran measures*

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1. Introduction. Recently, the issue of using the methods of fractal geometry to compare the distributions of various probability measures has been actively discussed (see for example [17], [18], [21], [23]). However, in practical applications, the comparison of distributions of measures can be difficult. Often, different distributions of measures can give subtle or indistinguishable differences in the spectra. To solve this problem, various new methods for direct comparison of distributions are proposed. One such method is the mixed or mutual multifractal analysis [17], [18], [21], [23], which allows to better understand the local geometry of fractal measures and the simultaneous scale behavior of multiple measures.

The mixed multifractal analysis is a natural extension of the multifractal analysis of single objects, such as measures, functions, statistical data, distributions, etc. It has been developed quite recently from a purely mathematical point of view. In physics, and statistics, it was appearing in different forms, but not really, and strongly linked to the mathematical theory (see [12]). In some applications, such as clustering topics, each attribute in a data sample may be described by more than one type of measure. This leads researchers to apply measures well adapted for mixed-type data as [12]. Mixed multifractal analysis has been applied in explaining joint movements in volatility for asset markets, such as joint multifractal Markov-switching models.

The mixed multifractal analysis is not really new in financial series processing. It has been, in contrast, merged under the name of multivariate multifractal analysis, where many situations in financial markets and their volatility have been described. Multivariate models have been also applied for long memory with mixture distributions [14], [15]. Multifractal analysis of measures for the so-called mixed logical dynamical models to the classification of signals, especially network traffic, is developed in [13]. These models are widely applied in the control of hybrid systems, such as multiserver ones.

Many authors were interested in studying the properties of mutual multifractal dimensions and spectra and establishing connections between them. More about the use of mixed multifractal analysis/formalism of measures, as well as functions and time series or images is developed in [1], [3], [4], [5], [6], [8], [9], [22], [24], [25], [26] and the references therein. The inverse problem of the mixed multifractal formalism and interesting examples are developed in [19].

From a mathematical point of view, Moran sets are generalizations of the classical self-similar objects, characterized by arbitrary basic sets in each step of construction. Moreover, the associated similarities at each step have also different rations from their predecessors. Moran's construction permits zero value for the lower limit of the contraction ratios (see [28]). Moran structures are also met in geography. Spatial autocorrelation in geographic information systems is based on the degree to which one object is similar to other nearby objects. Geographers call this concept Moran Index measures for spatial autocorrelation. Although the concept is defined independently in geography, it uses the similar idea of subdividing maps into Moran sets based on the axiom stating that everything is related to everything else, but near things are more related than distant things, see for example [2] and the references therein. More backgrounds and information on the applications may be found in [11], [20].

For given two compactly supported Borel probability measures μ and ν on $\mathbb{R}^n$ and for $\alpha, \beta \geqslant 0$, we consider the set

$$K(\alpha,\beta) = \left\{ x \in E\,;\, \lim_{r\to 0} \frac{\log\big(\mu(B_r(x))\big)}{\log r} = \alpha \text{ and } \lim_{r\to 0} \frac{\log\big(\nu(B_r(x))\big)}{\log r} = \beta \right\},$$

where $E = \operatorname{supp}\mu \cap \operatorname{supp}\nu$ and $B_r(x)$ is the closed ball with center x and radius r. That is, we will be interested in the set of points for which the local dimensions of $\mu(B_r(x))$ and $\nu(B_r(x))$ simultaneously describe the power-law behavior of the measures at a small radius r. The main problem is to estimate the size of this set. For some α, β, the mutual Hausdorff and packing dimensions of these sets $K(\alpha, \beta)$ have a close connection with the Legendre transform of some function $\tau(q, t)$ associated with measures μ and ν. The purpose of this work is to obtain conditions for the valid and non-valid multifractal formalism for some non-regular Moran measures.

2. Preliminaries. Before detailing our results, let us recall the mutual multifractal formalism introduced by Svetova [25]. Let μ and ν be two Borel probability measures on $\mathbb{R}^n$ with the same compact supports $\operatorname{supp}\mu = \operatorname{supp}\nu$. For $(q, t, s) \in \mathbb{R}^3$ and $\delta > 0$, we introduce

$$\mathcal{H}^{q,t,s}_{\mu,\nu,\delta}(E) = \inf\Big\{ \sum_i \mu(B_{r_i}(x_i))^q \nu(B_{r_i}(x_i))^t (2r_i)^s \Big\},$$

where the infimum is taken over all centered δ-coverings of $E \subset \mathbb{R}^n$. The mutual Hausdorff measure is defined as follows:

$$\mathcal{H}^{q,t,s}_{\mu,\nu,0}(E) = \sup_{\delta>0} \mathcal{H}^{q,t,s}_{\mu,\nu,\delta}(E), \quad \text{and} \quad \mathcal{H}^{q,t,s}_{\mu,\nu}(E) = \sup_{F\subset E} \mathcal{H}^{q,t,s}_{\mu,\nu,0}(F).$$

We make the dual definitions

$$\mathcal{P}^{q,t,s}_{\mu,\nu,\delta}(E) = \sup\{ \sum_i \mu(B_{r_i}(x_i))^q \nu(B_{r_i}(x_i))^t (2r_i)^s \},$$

where the supremum is taken over all the centered δ-packings of $E \subset \mathbb{R}^n$. We define the mutual packing as follows:

$$\mathcal{P}^{q,t,s}_{\mu,\nu,0}(E) = \inf_{\delta>0} \mathcal{P}^{q,t,s}_{\mu,\nu,\delta}(E), \quad \text{and} \quad \mathcal{P}^{q,t,s}_{\mu,\nu}(E) = \inf_{E\subset \bigcup_i E_i} \mathcal{P}^{q,t,s}_{\mu,\nu,0}(E_i).$$

It holds, as for the case of the multifractal analysis of a single measure, that each of the measures $\mathcal{H}^{q,t,s}_{\mu,\nu}$ and $\mathcal{P}^{q,t,s}_{\mu,\nu}$ assigns a multifractal dimension to each subset E of $\mathbb{R}^n$. They are respectively denoted by

$$\dim^{q,t}_{\mu,\nu}(E) = \sup\left\{s\colon \mathcal{H}^{q,t,s}_{\mu,\nu}(E) = \infty\right\} = \inf\left\{s\colon \mathcal{H}^{q,t,s}_{\mu,\nu}(E) = 0\right\},$$

and

$$\operatorname{Dim}^{q,t}_{\mu,\nu}(E) = \sup\left\{s\colon \mathcal{P}^{q,t,s}_{\mu,\nu}(E) = \infty\right\} = \inf\left\{s\colon \mathcal{P}^{q,t,s}_{\mu,\nu}(E) = 0\right\}.$$

Now, we define the multifractal function $b_{\mu,\nu}$, $B_{\mu,\nu}\colon \mathbb{R}^2 \to [-\infty,+\infty]$ as follows:

$$b_{\mu,\nu}(q,t) = \dim_{\mu,\nu}^{q,t}(\operatorname{supp}\mu), \quad \text{and} \quad B_{\mu,\nu}(q,t) = \operatorname{Dim}_{\mu,\nu}^{q,t}(\operatorname{supp}\mu).$$

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of Borel probability measures on $\mathbb{R}^n$. A measure $\mu \in \mathcal{P}(\mathbb{R}^n)$ with support $\operatorname{supp}\mu$ is said to satisfy the doubling condition if

$$P(a,\mu) = \limsup_{r\searrow 0}\left(\sup_{x\in\operatorname{supp}\mu}\frac{\mu\big(B_{ar}(x)\big)}{\mu\big(B_r(x)\big)}\right) < \infty,$$

for all $a > 1$. Denote by $\mathcal{P}_D(\mathbb{R}^n)$ the family of Borel probability measures on $\mathbb{R}^n$ that satisfy the doubling condition.

Let $\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers. Define $D_0 = \varnothing$, for any integer $k \geqslant 1$, set $D_{m,k} = \{(i_m i_{m+1}\dots i_k); 1 \leqslant i_j \leqslant n_j,\ m \leqslant j \leqslant k\}$, and $D_k = D_{1,k}$. Define $D = \bigcup\limits_{k\geqslant 0} D_k$. If $\sigma = (\sigma_1\dots\sigma_k) \in D_k$, and $\tau = (\tau_1\dots\tau_m) \in D_m$, then $\sigma * \tau = (\sigma_1\dots\sigma_k\tau_1\dots\tau_m) \in D_{k+m}$. If $l \leqslant k$, then $\sigma|l = (\sigma_1\dots\sigma_l)$. Suppose J is a closed interval of length 1.

Definition 1. *[30], [31] The collection $\Omega = \{J_\sigma, \sigma \in D\}$ of closed subintervals of J is called having the Moran structure, if it satisfies the following conditions:*

(i) $J_\varnothing = J$;

(ii) For all $k \geqslant 0$ and $\sigma \in D_k$, $J_{\sigma*1},\dots,J_{\sigma*n_{k+1}}$ are subintervals of J_σ, and satisfy $\operatorname{int}(J_{\sigma*i}) \cap \operatorname{int}(J_{\sigma*j}) = \varnothing$ for $i \neq j$, where $\operatorname{int}(A)$ denotes the interior of the set A;

(iii) For any $k \geqslant 1$ and $\sigma \in D_{k-1}$, $1 \leqslant j \leqslant n_k$, $\dfrac{|J_{\sigma*j}|}{|J_\sigma|} = c_{kj}$, where $|A|$ denotes the diameter of A.

Suppose that Ω is a collection of closed subintervals of J having the Moran structure, set

$$E_k = \bigcup_{\sigma\in D_k} J_\sigma, \quad \text{and} \quad E = \bigcap_{k\geqslant 0} E_k.$$

It is clear that E is a nonempty set. The set $E = E(\Omega)$ i s called the Moran set associated with the collection Ω.

Let $\Omega_k = \{J_\sigma; \sigma \in D_k\}$; obviously, $\Omega = \bigcup\limits_{k\geqslant 0}\Omega_k$. The elements of Ω_k are called the basic elements of order k of the Moran set E, and the elements of Ω are called the basic elements of the Moran set E. Further we will assume that $\lim\limits_{k\to\infty} \sup\limits_{\sigma\in D_k} |J_\sigma| = 0$.

Suppose, for any integer $k \geqslant 1$, any $\sigma \in D_k$, and for $1 \leqslant j \leqslant n_{k+1}$, the $(k+1)$-order basic element $J_{\sigma * j} \subset J_\sigma$. Let

$$\frac{\operatorname{dist}(J_{\sigma * i}, J_{\sigma * j})}{|J_\sigma|} \geqslant \Delta_k,$$

for all $i \neq j$, where $\operatorname{dist}(A, B) := \inf\limits_{x\in A,\, y\in B} \operatorname{dist}(x, y)$ for any two sets A and B, and $\operatorname{dist}(x, y)$ is the Euclidean distance between the points x and y. Denote $\Delta := \inf\limits_{k\geqslant 1} \Delta_k$.

Definition 2. *We say that the Moran set E satisfies the Strong Separation Condition (SSC) if $\Delta > 0$.*

Now, we define two probability measures on the Moran set E. For $k \geqslant 1$, let $p = \{p_{kj}\}_{j=1}^{n_k}$, $\tilde{p} = \{\tilde{p}_{kj}\}_{j=1}^{n_k}$ be two positive probability vectors, i. e.,

$$p_{kj} > 0, \qquad \tilde{p}_{kj} > 0, \qquad \sum_{i=1}^{n_k} p_{kj} = 1, \quad \text{and} \quad \sum_{i=1}^{n_k} \tilde{p}_{kj} = 1.$$

For $\sigma \in D_k$, $k \geqslant 1$, we define $\mu(J_\sigma) = p_{1\sigma_1} p_{2\sigma_2} \ldots p_{k\sigma_k}$ and $\nu(J_\sigma) = \tilde{p}_{1\sigma_1} \tilde{p}_{2\sigma_2} \ldots \tilde{p}_{k\sigma_k}$. It is obvious that $\operatorname{supp}\mu = \operatorname{supp}\nu$. The measures μ and ν are defined on one set $E = \operatorname{supp}\mu = \operatorname{supp}\nu$. Denote

$$p_{\min} = \min\{p_{ij}\}, \quad p_{\max} = \max\{p_{ij}\} \quad \text{for } 1 \leqslant j \leqslant n_k, 1 \leqslant i \leqslant k,$$

$$\tilde{p}_{\min} = \min\{\tilde{p}_{ij}\}, \quad \tilde{p}_{\max} = \max\{\tilde{p}_{ij}\} \quad \text{for } 1 \leqslant j \leqslant n_k, 1 \leqslant i \leqslant k,$$

$$c_{\min} = \min\{c_{ij}\}, \quad c_{\max} = \max\{c_{ij}\} \quad \text{for } 1 \leqslant j \leqslant n_k, 1 \leqslant i \leqslant k.$$

3. The main result. Let μ, ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. For $\alpha, \beta \geqslant 0$, let

$$K(\alpha, \beta) = \left\{x\, ;\ \lim_{r\to 0} \frac{\log\big(\mu(B_r(x))\big)}{\log r} = \alpha, \ \text{and} \ \lim_{r\to 0} \frac{\log\big(\nu(B_r(x))\big)}{\log r} = \beta\right\}.$$

We are interested in the estimation of the Hausdorff and packing dimension of $K(\alpha, \beta)$. Let us mention that in the last decade there has been a great interest for the multifractal analysis and positive results have been obtained for various situations (see, for example, [16], [29], [31]). The authors in [8], [9] prove the result of Theorem 3 in [25] under less restrictive assumptions, as follows:

Theorem 1. *Let μ, ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. Suppose that $B_{\mu,\nu}$ is differentiable at (q,t) and set $\alpha = -\frac{\partial B_{\mu,\nu}(q,t)}{\partial q}$ and $\beta = -\frac{\partial B_{\mu,\nu}(q,t)}{\partial t}$. Assume that*

$$\mathcal{H}_{\mu,\nu}^{q,t,B_{\mu,\nu}(q,t)}\big(\operatorname{supp}\mu \cap \operatorname{supp}\nu\big) > 0.$$

Then we have

$$\dim_H(K(\alpha,\beta)) = \dim_P(K(\alpha,\beta)) = B^*_{\mu,\nu}(\alpha,\beta) = b^*_{\mu,\nu}(\alpha,\beta),$$

where $f^*(\alpha,\beta) = \inf_{q,t}\big(\alpha q + \beta t + f(\alpha,\beta)\big)$ denotes the Legendre transform of the function f. Here $\dim_H$ and $\dim_P$ denote the Hausdorff and packing dimensions (see [16] for the definitions), and in this case we say that the mutual multifractal formalism is valid.

Let us define the function $\tau_k(q,t)$ as the only solution to the equation

$$\sum_{\sigma\in D_k} \mu(J_\sigma)^q \cdot \nu(J_\sigma)^t \cdot |J_\sigma|^{\tau_k(q,t)} = 1. \tag{1}$$

Using (1), the theorem of implicit differentiation shows that τ_k is partially differentiable with respect to all variables. It is clear that if $\mu(J_\sigma) \neq |J_\sigma|^{s_k}$ and $\nu(J_\sigma) \neq |J_\sigma|^{s_k}$, where s_k satisfies $\sum_{\sigma\in D_k} c_\sigma^{s_k} = 1$, then $\tau_k(q,t)$ is strictly convex for $(q,t) \in \mathbb{R}^2$. Define now the following functions:

$$\underline{\tau}(q,t) = \liminf_{k\to\infty} \tau_k(q,t), \quad \overline{\tau}(q,t) = \limsup_{k\to\infty} \tau_k(q,t), \quad \tau(q,t) = \lim_{k\to\infty} \tau_k(q,t).$$

We can now state the main result of the paper. Explicit formulas for the mutual multifractal dimensions of the level sets $K(\alpha,\beta)$, for which the classical formalism does not hold, are given by obtaining some new sufficient conditions (which are different from the ones in Theorem 1) for the non-validity of the mutual multifractal formalism of non-regular Moran measures, i.e., the case for which the multifractal functions $b_{\mu,\nu}$ and $B_{\mu,\nu}$ do not necessarily coincide (see Figure 2). Our main theorem generalizes the main results of [7], [29], [30], [31] (by taking ($q = 0$ or $t = 0$) and $q = t = 0$).

Theorem 2. *Assume that $\Delta > 0$, and let $(q,t) \in \mathbb{R}^2$. Let μ, ν be two non-regular Moran measures on the Moran set E.*

1) *Suppose that* $\left(\frac{\partial \underline{\tau}(q,t)}{\partial q}, \frac{\partial \underline{\tau}(q,t)}{\partial t}\right)$ *exists, and*

$$\liminf_{k\to\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \nu\left(J_\sigma\right)^t \left|J_\sigma\right|^{\underline{\tau}(q,t)} > 0, \quad \textit{for all } (q,t)\in\mathbb{R}^2.$$

If $(\alpha,\beta)=-\left(\frac{\partial \underline{\tau}(q,t)}{\partial q}, \frac{\partial \underline{\tau}(q,t)}{\partial t}\right)$, *then* $\dim_H(K(\alpha,\beta))=b^*_{\mu,\nu}(\alpha,\beta)=\underline{\tau}^*(\alpha,\beta)$.

2) *Suppose that* $\left(\frac{\partial \overline{\tau}(q,t)}{\partial q}, \frac{\partial \overline{\tau}(q,t)}{\partial t}\right)$ *exists, and*

$$\limsup_{k\to\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \nu\left(J_\sigma\right)^t \left|J_\sigma\right|^{\overline{\tau}(q,t)} > 0, \quad \textit{for all } (q,t)\in\mathbb{R}^2.$$

If $(\alpha,\beta)=-\left(\frac{\partial \overline{\tau}(q,t)}{\partial q}, \frac{\partial \overline{\tau}(q,t)}{\partial t}\right)$, *then,* $\dim_P(K(\alpha,\beta))=B^*_{\mu,\nu}(\alpha,\beta)=\overline{\tau}^*(\alpha,\beta)$.

3) *Suppose that* $\tau(q,t)$ *and* $\left(\frac{\partial \tau(q,t)}{\partial q}, \frac{\partial \tau(q,t)}{\partial t}\right)$ *exist, and inequality* $\tau(q,t) < \frac{c}{k} + \tau_k(q,t)$ *is true for some constant* $c>0$, *and all* $k\geqslant 1$. *If* $(\alpha,\beta) = -\left(\frac{\partial \tau(q,t)}{\partial q}, \frac{\partial \tau(q,t)}{\partial t}\right)$, *then,*

$$\dim_H(K(\alpha,\beta)) = \dim_P(K(\alpha,\beta)) = b^*_{\mu,\nu}(\alpha,\beta) = B^*_{\mu,\nu}(\alpha,\beta) = \tau^*(\alpha,\beta).$$

Remark.

1) *Our results hold naturally when replacing the interval* J *by a compact subset (denoted also* J*) of* $\mathbb{R}^n$ *with* $\overline{int(J)} = J$.

2) *If there exists a family of similitudes* $\{S_{k,j}\colon k\geqslant 1, 1\leqslant j\leqslant n_k\}$ *with*

$$J_\sigma = S_\sigma(J) = S_{1,\sigma_1}\circ S_{2,\sigma_2}\circ\cdots\circ S_{k,\sigma_k}(J), \qquad \textit{for } \sigma=(\sigma_1,\ldots,\sigma_k)\in D_k,$$

then the corresponding Moran set $E = \bigcap_k \bigcup_\sigma J_\sigma$ *is called a generalized self-similar set, which is a generalization of the self-similar sets, and the Moran measure is an extension of the self-similar measure. This implies that our main results hold for the self-similar sets and measures.*

3) *For* $\gamma\geqslant 0$, *we consider the following fractal set:*

$$\mathcal{E}(\gamma) = \left\{x\in \operatorname{supp}\mu\cap\operatorname{supp}\nu;\ \lim_{r\to 0}\frac{\log\big(\mu(B_r(x))\big)}{\log\big(\nu(B_r(x)\big)} = \gamma\right\}.$$

Denote $\mathbb{R}_+ \times \mathbb{R}_+^* := [0, +\infty[\times]0, +\infty[$. *It is clear that*

$$\bigcup_{\substack{(\alpha,\beta)\in\mathbb{R}_+\times\mathbb{R}_+^*,\\ \frac{\alpha}{\beta}=\gamma}} K(\alpha, \beta) \subset \mathcal{E}(\gamma).$$

The union is composed of an uncountable number of pairwise disjoint nonempty sets. Theorems 1 and 2 show that, surprisingly, the Hausdorff and packing dimensions of $\mathcal{E}(\gamma)$ *are fully carried by some subset* $K(\alpha, \beta)$. *Also, our main results give an optimal lower bound of the Hausdorff and packing dimensions of* $\mathcal{E}(\gamma)$.

4. Examples. In this section, we illustrate our main results with two examples.

Example 1. Let $I = [0,1]$, $n_k = 2$, and $c_k = \frac{1}{5}$, for all $k \geqslant 1$. The set E is the middle-$\frac{1}{5}$ Cantor set, and μ and ν are two Bernoulli measures, such that $\mu = \nu$, with $p := P_1^1 = P_1^2$ and $\widetilde{p} := P_2^1 = P_2^2$. Then, for $(q, t) \in \mathbb{R}^2$, we have:

$$b_{\mu,\nu}(q, t) = B_{\mu,\nu}(q, t) = \tau(q, t) = \frac{\log\left(p^{q+t} + \widetilde{p}^{q+t}\right)}{\log 5},$$

and

$$\frac{\partial \tau(q,t)}{\partial q} = \frac{\partial \tau(q, t)}{\partial t} = \frac{p^{q+t}\log p + \widetilde{p}^{q+t}\log \widetilde{p}}{(p^{q+t} + \widetilde{p}^{q+t})\log 5}.$$

Now, it follows from Theorem 2 that

$$\dim_H(K(\alpha, \beta)) = \dim_P(K(\alpha, \beta)) = \tau^*(\alpha, \beta),$$

for $(\alpha, \beta) = -\left(\frac{\partial \tau(q,t)}{\partial q}, \frac{\partial \tau(q,t)}{\partial t}\right)$. Figure 1 shows the plots of the multifractal functions τ and τ^*.

Example 2. Let $(S_k)_k$ be a sequence of integers, such that

$$S_1 = 1,\ \ S_2 = 3,\ \ \text{and}\ \ S_{k+1} = 2S_k,\ \ \forall\, k \geqslant 2.$$

Define the family of parameters n_i, c_i, and $p_{i,m}$ as follows:

$$n_1 = 2,\ \ n_i = \begin{cases} 3, \text{ if } \ S_{2k-1} \leqslant i < S_{2k}, \\ 2, \text{ if } \ S_{2k} \leqslant i < S_{2k+1}, \end{cases}$$

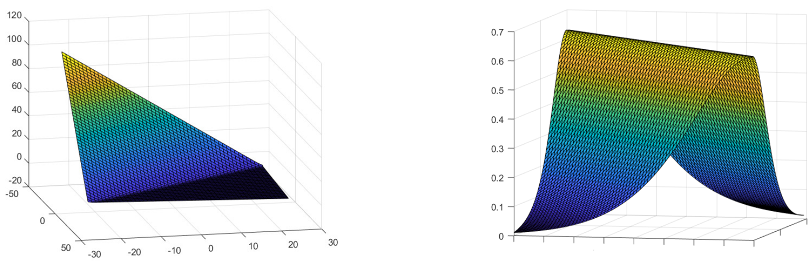

Figure 1: The plots of the functions τ and τ^*.

and

$$c_1 = \frac{1}{5}, \quad c_i = \begin{cases} \frac{1}{7}, \text{ if } \ S_{2k-1} \leqslant i < S_{2k}, \\ \frac{1}{5}, \text{ if } \ S_{2k} \leqslant i < S_{2k+1}. \end{cases}$$

Let $(p_{a,m})_{m=1}^{2}$ and $(p_{b,m})_{m=1}^{3}$ be two probability vectors. We define

$$p_{1,m} = p_{a,m}, \text{ for all } 1 \leqslant m \leqslant 2,$$

and

$$p_{i,m} = \begin{cases} p_{b,m}, \text{ if } \ S_{2k-1} \leqslant i < S_{2k}, \ \ 1 \leqslant m \leqslant 3, \\ p_{a,m}, \text{ if } \ S_{2k} \leqslant i < S_{2k+1}, \ \ 1 \leqslant m \leqslant 2. \end{cases}$$

Let N_k be the number of integers $i \leqslant k$, such that $p_{i,m} = p_{a,m}$; then

$$\liminf_{k\to+\infty} \frac{N_k}{k} = \frac{1}{3} \quad \text{and} \quad \limsup_{k\to+\infty} \frac{N_k}{k} = \frac{2}{3}.$$

Now, let μ and ν be two Moran measures, with $\mu = \nu$, which are generated by $(p_{i,m})$. For $(q,t) \in \mathbb{R}^2$, we get

$$\tau_k(q,t) = \frac{\frac{N_k}{k} \log\left(\sum\limits_{m=1}^{2} p_{a,m}^{q+t}\right) + \left(1 - \frac{N_k}{k}\right) \log\left(\sum\limits_{m=1}^{3} p_{b,m}^{q+t}\right)}{\frac{N_k}{k} \log 5 + \left(1 - \frac{N_k}{k}\right) \log 7}.$$

Let

$$\phi(q,t) = \frac{\frac{1}{3} \log\left(\sum\limits_{m=1}^{2} p_{a,m}^{q+t}\right) + \frac{2}{3} \log\left(\sum\limits_{m=1}^{3} p_{b,m}^{q+t}\right)}{\frac{1}{3} \log 5 + \frac{2}{3} \log 7},$$

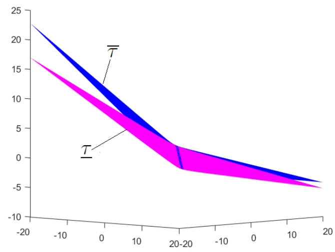

Figure 2: The plots of the functions $\underline{\tau}$ and $\overline{\tau}$.

and

$$\widetilde{\phi}(q,t) = \frac{\frac{2}{3}\log\left(\sum_{m=1}^{2} p_{a,m}^{q+t}\right) + \frac{1}{3}\log\left(\sum_{m=1}^{3} p_{b,m}^{q+t}\right)}{\frac{2}{3}\log 5 + \frac{1}{3}\log 7}.$$

Observing that the assumptions of Theorem 2 are satisfied, we obtain

$$\underline{\tau}(q,t) = \min\left\{\phi(q,t),\ \widetilde{\phi}(q,t)\right\} \quad \text{and} \quad \overline{\tau}(q,t) = \max\left\{\phi(q,t),\ \widetilde{\phi}(q,t)\right\}.$$

Figure 2 shows the plots of the functions $\underline{\tau}$ and $\overline{\tau}$.

4. Proof of the main result. Suppose that the set E is a Moran set associated with the collection Ω. Let μ and ν be two non-regular Moran measures supported by the set E. We start with the following interesting intermediate results.

Proposition 1. *Let $\Delta > 0$, $0 < r < \Delta$, and $a \geqslant 1$. For $x \in J_\sigma$, $\sigma \in D$, we find $k, l \in \mathbb{N}$, such that $|J_{\sigma|k}| \leqslant r < |J_{\sigma|k-1}|$ and $\Delta|J_{\sigma|l}| \leqslant ar < \Delta|J_{\sigma|l-1}|$. Then $0 \leqslant k - l \leqslant M$, where M is a positive constant.*

Proof. Since $0 < \Delta \leqslant 1$ and $a \geqslant 1$, we have: $\frac{|J_{\sigma|k}|}{\Delta|J_{\sigma|l}|} < \frac{1}{a}$. Then $\frac{|J_{\sigma|k}|}{|J_{\sigma|l}|} < \frac{\Delta}{a} \leqslant 1$. Therefore $|J_{\sigma|k}| \leqslant |J_{\sigma|l}|$, which implies $k \geqslant l$. As $k \geqslant l$, the inequality $r < |J_{\sigma|k-1}|$ can be rewritten as

$$r < \left|J_{\sigma|l-1\,*(l,l+1,\ldots,k-1)}\right| \leqslant \left|J_{\sigma|l-1}\right| \cdot c_{\max}^{k-l-1}.$$

We have

$$\frac{r}{ar} \leqslant \frac{|J_{\sigma|k-1}|}{\Delta|J_{\sigma|l}|} \leqslant \frac{|J_{\sigma|l}| \cdot c_{max}^{k-l-1}}{\Delta|J_{\sigma|l-1}|} \leqslant \frac{c_{max}^{k-l-1}}{\Delta}.$$

Then, $\dfrac{1}{a} \leqslant \dfrac{c_{\max}^{k-l-1}}{\Delta}$, and $k-l \leqslant \dfrac{\log \frac{\Delta}{a}}{\log c_{\max}} + 1 = M$. $\square$

Proposition 2. [10] *If $\Delta > 0$, then $\mu, \nu \in \mathcal{P}_D(E)$.*

Proposition 3. *For any $x \in E$, and small $r > 0$, we can find $\sigma \in D_n$, $k, l \in \mathbb{N}$. There are some constants $A_1(q,t)$, $A_2(q,t)$, $B_1(q,t)$, and $B_2(q,t)$ for $(q,t) \in \mathbb{R}^2$, such that*

$$A_1(q,t)\mu(J_{\sigma|l})^q \nu(J_{\sigma|l})^t \leqslant \mu(B_r(x))^q \nu(B_r(x))^t \leqslant A_2(q,t)\mu(J_{\sigma|l})^q \nu(J_{\sigma|l})^t,$$

and

$$B_1(q,t)\mu(J_{\sigma|k})^q \nu(J_{\sigma|k})^t \leqslant \mu(B_r(x))^q \nu(B_r(x))^t \leqslant B_2(q,t)\mu(J_{\sigma|k})^q \nu(J_{\sigma|k})^t.$$

Proof. Fix $x \in E$ and $r > 0$. We can find $\sigma \in D_n$, such that $x \in J_\sigma$, and integers k, $l \in \mathbb{N}$, such that $|J_{\sigma|k}| \leqslant r < |J_{\sigma|k-1}|$, and $\Delta|J_{\sigma|l}| \leqslant r < \Delta|J_{\sigma|l-1}|$. Notice that $J_{\sigma|k} \subseteq B_r(x)$ and $E \cap B_r(x) \subseteq J_{\sigma|l}$.

Let us estimate $\mu(B_r(x))^q \nu(B_r(x))^t$ for different q and t.

1) Suppose $q, t < 0$. By Proposition 1, we have, for $M \geqslant k-l$:

$$\begin{aligned}
\mu(J_{\sigma|l})^q \nu(J_{\sigma|l})^t \leqslant \mu(B_r(x))^q \nu(B_r(x))^t \leqslant \mu(J_{\sigma|k})^q \nu(J_{\sigma|k})^t = \\
= (p_{1\sigma_1} p_{2\sigma_2} \dots p_{k\sigma_k})^q (\tilde{p}_{1\sigma_1} \tilde{p}_{2\sigma_2} \dots \tilde{p}_{k\sigma_k})^t = \\
= (p_{1\sigma_1} p_{2\sigma_2} \dots p_{l\sigma_l} p_{l+1\sigma_{l+1}} \dots p_{k\sigma_k})^q (\tilde{p}_{1\sigma_1} \tilde{p}_{2\sigma_2} \dots \tilde{p}_{l\sigma_l} \tilde{p}_{l+1\sigma_{l+1}} \dots \tilde{p}_{k\sigma_k})^t \leqslant \\
\leqslant p_{\min}^{(k-l)q} \tilde{p}_{\min}^{(k-l)t} \mu(J_{\sigma|l})^q \nu(J_{\sigma|l})^t \leqslant p_{\min}^{Mq} \tilde{p}_{\min}^{Mt} \mu(J_{\sigma|l})^q \nu(J_{\sigma|l})^t.
\end{aligned}$$

2) For $q, t \geqslant 0$:

$$\begin{aligned}
\mu(J_{\sigma)|l})^q \nu(J_{\sigma|l})^t = \frac{\mu(J_{\sigma|l})^q}{\mu(B_r(x))^q} \mu(B_r(x))^q \frac{\nu(J_{\sigma|l})^t}{\nu(B_r(x))^t} \nu(B_r(x))^t \leqslant \\
\leqslant \frac{\mu(J_{\sigma|l})^q}{\mu(J_{\sigma|k})^q} \frac{\nu(J_{\sigma|l})^t}{\nu(J_{\sigma|k})^t} \mu(B_r(x))^q \nu(B_r(x))^t \leqslant \\
\leqslant \frac{1}{p_{\min}^{(k-l)q} \tilde{p}_{\min}^{(k-l)t}} \mu(B_r(x))^q \nu(B_r(x))^t \leqslant \\
\leqslant \frac{1}{p_{\min}^{Mq} \tilde{p}_{\min}^{Mt}} \mu(B_r(x))^q \nu(B_r(x))^t \leqslant \frac{1}{p_{\min}^{Mq} \tilde{p}_{\min}^{Mt}} \mu(J_{\sigma|l})^q \nu(J_{\sigma|l})^t.
\end{aligned}$$

3) For $q < 0$, $t \geqslant 0$:

$$\mu(J_{\sigma|l})^q\nu(J_{\sigma|l})^t \leqslant \mu(B_r(x))^q \frac{\nu(J_{\sigma|l})^t}{\nu(B_r(x))^t}\nu(B_r(x))^t \leqslant$$
$$\leqslant \frac{\nu(J_{\sigma|l})^t}{\nu(J_{\sigma|k})^t}\mu(B_r(x))^q\nu(B_r(x))^t \leqslant \frac{1}{\tilde{p}_{\min}^{Mt}}\mu(B_r(x))^q\nu(B_r(x))^t \leqslant$$
$$\leqslant \frac{p_{\min}^{Mq}}{\tilde{p}_{\min}^{Mt}}\mu(J_{\sigma|l})^q\nu(B_r(x))^t \leqslant \frac{p_{\min}^{Mq}}{\tilde{p}_{\min}^{Mt}}\mu(J_{\sigma|l})^q\nu(J_{\sigma|l})^t.$$

4) For $q \geqslant 0$, $t < 0$, similarly to case 3, we have

$$\mu(J_{\sigma|l})^q\nu(J_{\sigma|l})^t \leqslant \frac{1}{p_{\min}^{Mq}}\mu(B_r(x))^q\nu(B_r(x))^t \leqslant \frac{\tilde{p}_{\min}^{Mt}}{p_{\min}^{Mq}}\mu(J_{\sigma|l})^q\nu(J_{\sigma|l})^t.$$

Combining all these cases, we get

$$A_1(q,t)\mu(J_{\sigma|l})^q\nu(J_{\sigma|l})^t \leqslant \mu(B_r(x))^q\nu(B_r(x))^t \leqslant A_2(q,t)\mu(J_{\sigma|l})^q\nu(J_{\sigma|l})^t,$$

where

$$A_1(q,t) = \begin{cases} 1, & q,t<0, \\ p_{\min}^{Mq}\,\tilde{p}_{\min}^{Mt}, & q,t \geqslant 0, \\ \tilde{p}_{\min}^{Mt}, & q<0, t\geqslant 0, \\ p_{\min}^{Mq}, & q \geqslant 0, t<0, \end{cases} \quad A_2(q,t) = \begin{cases} p_{\min}^{Mq}\,\tilde{p}_{\min}^{Mt}, & q,t<0, \\ 1, & q,t \geqslant 0, \\ p_{\min}^{Mq}, & q<0, t\geqslant 0, \\ \tilde{p}_{\min}^{Mt}, & q \geqslant 0, t<0. \end{cases}$$

Similar arguments give

$$B_1(q,t)\mu(J_{\sigma|k})^q\nu(J_{\sigma|k})^t \leqslant \mu(B_r(x))^q\nu(B_r(x))^t \leqslant B_2(q,t)\mu(J_{\sigma|k})^q\nu(J_{\sigma|k})^t,$$

where

$$B_1(q,t) = \begin{cases} \dfrac{1}{p_{\min}^{Mq}\tilde{p}_{\min}^{Mt}}, & q,t<0, \\ 1, & q,t \geqslant 0, \\ \dfrac{1}{p_{\min}^{Mq}}, & q<0, t\geqslant 0, \\ \dfrac{1}{\tilde{p}_{\min}^{Mt}}, & q \geqslant 0, t<0, \end{cases} \quad B_2(q,t) = \begin{cases} 1, & q,t<0, \\ \dfrac{1}{p_{\min}^{Mq}\tilde{p}_{\min}^{Mt}}, & q,t \geqslant 0, \\ \dfrac{1}{\tilde{p}_{\min}^{Mt}}, & q<0, t\geqslant 0, \\ \dfrac{1}{p_{\min}^{Mq}}, & q \geqslant 0, t<0. \end{cases}$$

□

For $(q,t,s)\in\mathbb{R}^3$, we denote

$$\underline{h}=\liminf_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^q\nu(J_\sigma)^t|J_\sigma|^s,\quad\text{and}\quad\overline{h}=\limsup_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^q\nu(J_\sigma)^t|J_\sigma|^s.$$

Proposition 4. *If $\Delta>0$, there exists a constant $c>0$, such that, for any $(q,t,s)\in\mathbb{R}^3$,*

$$c\cdot\underline{h}\leqslant\mathcal{H}^{q,t,s}_{\mu,\nu}(E)\leqslant\underline{h}.$$

Proof. It is not hard to see that $\mathcal{H}^{q,t,s}_{\mu,\nu}(E)\leqslant\underline{h}$. If $\underline{h}\in(0,+\infty)$, there is a sufficiently large number n, such that

$$\sum_{\sigma\in D_n}\mu(J_\sigma)^q\nu(J_\sigma)^t|J_\sigma|^s>\frac{\underline{h}}{2}.$$

Let $\delta>0$ and $\{B_{r_i}(x_i)\}$ be a centered δ-covering of E. For any $i\in\mathbb{N}$, we choose $\sigma(i)\in D_n$, $n\geqslant 1$, such that $x_i\in J_{\sigma(i)}$. For any $i\in\mathbb{N}$, let k_i,$l_i\in\mathbb{N}$ be such that

$$|J_{\sigma(i)|k_i}|\leqslant r_i<|J_{\sigma(i)|k_i-1}|,\quad\text{and}\quad\Delta|J_{\sigma(i)|l_i}|\leqslant r_i<\Delta|J_{\sigma(i)|l_i-1}|.$$

If $s\geqslant 0$, then $|J_{\sigma(i)|l_i}|^s\leqslant\left(\frac{r_i}{\Delta}\right)^s$.
Let $s<0$. Since $|J_{\sigma(i)|l_i}|\geqslant c_{\min}J_{\sigma(i)|l_i-1}\geqslant c_{\min}\frac{r_i}{\Delta}$, then $|J_{\sigma(i)|l_i}|^s\leqslant c^s_{min}\left(\frac{r_i}{\Delta}\right)^s$. Thus,

$$|J_{\sigma(i)|l_i}|^s\leqslant K_0\left(\frac{r_i}{\Delta}\right)^s,\quad\text{where}\quad K_0=\begin{cases}1, & s\geqslant 0,\\ c^s_{\min}, & s<0.\end{cases}$$

There exists a probability measure $\chi_{q,t,s}$ supported on E, such that

$$\chi_{q,t,s}(J_{\sigma(i)})=\frac{\mu(J_{\sigma(i)})^q\nu(J_{\sigma(i)})^t|J_{\sigma(i)}|^s}{\sum\limits_{\sigma\in D_{|\sigma(i)|}}\mu(J_{\sigma(i)})^q\nu(J_{\sigma(i)})^t|J_{\sigma(i)}|^s}.$$

Obviously, $\chi_{q,t,s}(E)=\sum\limits_i\chi_{q,t,s}(J_{\sigma(i)})=1$.

For a Moran set E and $(q,t,s)\in\mathbb{R}^3$, we have, by using Proposition 3,

$$\chi_{q,t,s}(E)\leqslant\sum_i\chi_{q,t,s}(B_{r_i}(x_i))\leqslant\sum_i\chi_{q,t,s}(J_{\sigma(i)|l_i})=$$

$$= \sum_i \frac{\mu(J_{\sigma(i)|l_i})^q \nu(J_{\sigma(i)|l_i})^t |J_{\sigma(i)|l_i}|^s}{\sum\limits_{\sigma\in D_{l_i}} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s} <$$

$$< \frac{2}{\underline{h}} \sum_i \mu(J_{\sigma(i)|l_i})^q \nu(J_{\sigma(i)|l_i})^t |J_{\sigma(i)|l_i}|^s \leqslant$$

$$\leqslant \frac{2K_0}{A_1(q,t)\underline{h}} \sum_i \mu(B_{r_i}(x_i))^q \nu(B_{r_i}(x_i))^t \frac{(2r_i)^s}{(2\Delta)^s} =$$

$$= \frac{K}{\underline{h}} \sum_i \mu(B_{r_i}(x_i))^q \nu(B_{r_i}(x_i))^t (2r_i)^s,$$

where $K = \dfrac{2K_0}{(2\Delta)^s A_1(q,t)}$. Let $c = \dfrac{1}{K}$, hence,

$$c\underline{h} = c\underline{h}\chi_{q,t,s}(E) \leqslant \mathcal{H}^{q,t,s}_{\mu,\nu,\delta}(E) \leqslant \mathcal{H}^{q,t,s}_{\mu,\nu,0}(E) \leqslant \mathcal{H}^{q,t,s}_{\mu,\nu}(E).$$

Let $\underline{h} = +\infty$. For any $\varepsilon > 0$, there is a sufficiently large n, such that

$$\sum_{\sigma\in D_n} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s > \frac{1}{\varepsilon}.$$

Then, for $\delta = \delta(n) > 0$ and a centered δ-covering $\{B_{r_i}(x_i)\}$ of the Moran set E, we have

$$\chi_{q,t,s}(E) \leqslant \sum_i \chi_{q,t,s}(B_{r_i}(x_i)) \leqslant \sum_{i,|\sigma(i)|l_i|=n} \chi_{q,t,s}(J_{\sigma(i)|l_i}) =$$

$$= \sum_i \frac{\mu(J_{\sigma(i)|l_i})^q \nu(J_{\sigma(i)|l_i})^t |J_{\sigma(i)|l_i}|^s}{\sum\limits_{\sigma\in D_n} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s} \leqslant \frac{\varepsilon K}{2} \sum_i \mu(B_{r_i}(x_i))^q \nu(B_{r_i}(x_i))^t (2r_i)^s.$$

Then

$$\mathcal{H}^{q,t,s}_{\mu,\nu}(E) \geqslant \mathcal{H}^{q,t,s}_{\mu,\nu,0}(E) \geqslant \mathcal{H}^{q,t,s}_{\mu,\nu,\delta}(E) \geqslant \frac{2}{\varepsilon K}\chi_{q,t,s}(E) = \frac{2}{\varepsilon K}.$$

Therefore, $\mathcal{H}^{q,t,s}_{\mu,\nu}(E) = +\infty$. $\square$

Proposition 5. *Let* $\Delta > 0$.

1) *If* $0 < \overline{h} < +\infty$, *there are some constants* A, B, *such that* $A\overline{h} \leqslant \mathcal{P}^{q,t,s}_{\mu,\nu}(E) \leqslant B\overline{h}$.

2) *If* $\overline{h} = 0$, *then,* $\mathcal{P}_{\mu,\nu}^{q,t,s}(E) = 0$.

3) *If* $\overline{h} = +\infty$, *then,* $\mathcal{P}_{\mu,\nu}^{q,t,s}(E) = +\infty$.

Proof.

1) Since $\overline{h} > 0$, we can find n, such that

$$\sum_{\sigma \in D_n} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s > \frac{\bar{h}}{2}.$$

Let $\delta > 0$, for $\sigma \in D_n$, take $\tau \in D_m$, such that $|J_{\sigma * \tau}| < \delta$, and $J_{\sigma * \tau} \subset J_\sigma$. Let $x \in E \cap J_{\sigma * \tau}$. Let $0 < r < 1$, such that J_σ contains a ball $B_{r*}(x)$ of radius $r^* = r|J_{\sigma * \tau}|/2$. The collection of balls $\{B_{r*}(x)\}$ is a δ-packing of E. Thus,

$$\begin{cases} (c_{\min})^m |J_\sigma| \leqslant |J_{\sigma * \tau}| \leqslant (c_{\max})^m |J_\sigma|, \\ (p_{\min})^m \mu(J_\sigma) \leqslant \mu(J_{\sigma * \tau}) \leqslant (p_{\max})^m \mu(J_\sigma), \\ (\tilde{p}_{\min})^m \nu(J_\sigma) \leqslant \nu(J_{\sigma * \tau}) \leqslant (\tilde{p}_{\max})^m \nu(J_\sigma). \end{cases} \tag{2}$$

For any $(q,t,s) \in \mathbb{R}^3$, we get

$$\mu(J_{\sigma * \tau})^q \geqslant k_1 \mu(J_\sigma)^q, \nu(J_{\sigma * \tau})^t \geqslant k_2 \nu(J_\sigma)^t, |J_{\sigma * \tau}|^s \geqslant k_3 |J_\sigma|^s.$$

Then

$$\sum_{\sigma \in D_n} \mu(J_{\sigma * \tau})^q \nu(J_{\sigma * \tau})^t |J_{\sigma * \tau}|^s > k_1 k_2 k_3 \sum_{\sigma \in D_n} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s \geqslant A\overline{h},$$

where $A = k_1 k_2 k_3 / 2$. This implies that $\mathcal{P}_{\mu,\nu}^{q,t,s}(E) \geqslant A\overline{h}$. On the other hand, since $\overline{h} > 0$, there is a sufficiently large n, such that

$$\sum_{\sigma \in D_n} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s < 2\overline{h}.$$

Let $\delta > 0$, and $\{B_{r_i}(x_i)\}$ be a centered δ-packing of the Moran set E. For $i \in \mathbb{N}$, we can choose $\sigma(i) \in D_n$, such that $x_i \in J_{\sigma_i}$. We can found $k_i, l_i \in \mathbb{N}$, such that

$$|J_{\sigma(i)|k_i}| \leqslant r_i < |J_{\sigma(i)|k_i - 1}|, \quad \text{and} \quad |J_{\sigma(i)|l_i}| \leqslant r_i < |J_{\sigma(i)|l_i - 1}|.$$

If $s \leqslant 0$, then $(2r_i)^s \leqslant 2^s |J_{\sigma(i)|k_i}|^s$, if $s > 0$. Hence,

$$(2r_i)^s \leqslant 2^s |J_{\sigma(i)|k_i - 1}|^s \leqslant \frac{2^s}{c_{\min}^s} |J_{\sigma(i)|k_i}|^s.$$

Therefore,

$$(2r_i)^s \leqslant C(s)|J_{\sigma(i)|k_i}|^s, \text{ where } C(s) = \begin{cases} 2^s, & s \leqslant 0, \\ \left(\frac{2}{c_{\min}}\right)^s, & s > 0. \end{cases} \tag{3}$$

There exists a probability measure $\chi_{q,t,s}$ supported on E, for which,

$$\chi_{q,t,s}(J_{\sigma(i)}) = \frac{\mu(J_{\sigma(i)})^q \nu(J_{\sigma(i)})^t |J_{\sigma(i)}|^s}{\sum\limits_{\sigma \in D_{|\sigma(i)|}} \mu(J_{\sigma(i)})^q \nu(J_{\sigma(i)})^t |J_{\sigma(i)}|^s}.$$

By Proposition 3 and (3), for $(q,t,s) \in \mathbb{R}^3$, we get

$$\begin{aligned}
&\sum_i \mu(B_{r_i}(x_i)^q \nu(B_{r_i}(x_i)^t (2r_i)^s \leqslant \\
\leqslant\ & B_2(q,t)C(s) \sum_{\sigma \in D_n} \mu(J_{\sigma(i)|k_i})^q \nu(J_{\sigma(i)|k_i})^t |J_{\sigma(i)|k_i}|^s = \\
=\ & B_2(q,t)C(s) \sum_i \frac{\mu(J_{\sigma(i)|k_i})^q \nu(J_{\sigma(i)|k_i})^t |J_{\sigma(i)|k_i}|^s}{\sum\limits_{\sigma \in D_{k_i}} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s} \sum_{\sigma \in D_{k_i}} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s < \\
<\ & 2B_2(q,t)C(s)\overline{h} \sum_i \chi_{q,t,s}(J_{\sigma(i)|k_i}) \leqslant 2B_2(q,t)C(s)\overline{h} \sum_i \chi_{q,t,s}(B_{r_i}(x_i)) = \\
=\ & 2B_2(q,t)C(s)\overline{h} \chi_{q,t,s}\Big(\bigcup_i B_{r_i}(x_i)\Big) \leqslant 2B_2(q,t)C(s)\overline{h}.
\end{aligned}$$

Hence, it follows that $\mathcal{P}^{q,t,s}_{\mu,\nu}(E) \leqslant \mathcal{P}^{q,t,s}_{\mu,\nu,0}(E) \leqslant B\overline{h}$, where $B = 2B_2(q,t)C(s)$.
2) Let $\overline{h} = 0$. Then, for any $\varepsilon > 0$, there is a sufficiently large n, such that

$$\sum_{\sigma \in D_n} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s < \varepsilon.$$

Let $\delta > 0$, and $\{B_{r_i}(x_i)\}$ be a centered δ-packing of E. Then, for $i \in \mathbb{N}$, we can choose $\sigma(i) \in D_n$, $x_i \in J_{\sigma(i)}$, and $k_i, l_i \in \mathbb{N}$, such that

$$|J_{\sigma(i)|k_i}| \leqslant r_i < |J_{\sigma(i)|k_i - 1}|, \quad \text{and} \quad \Delta|J_{\sigma(i)|l_i}| \leqslant r_i < \Delta|J_{\sigma(i)|l_i - 1}|.$$

Then, $J_{\sigma(i)|k_i} \subseteq B_{r_i}(x_i)$, and $E \cap B_{r_i}(x_i) \subseteq J_{\sigma(i)|l_i}$. If $s \leqslant 0$, then $(2r_i)^s \leqslant 2^s|J_{\sigma(i)|k_i}|^s$. If $s > 0$, then

$$(2r_i)^s \leqslant 2^s|J_{\sigma(i)|k_i - 1}|^s \leqslant \frac{2^s}{c^s_{\min}}|J_{\sigma(i)|k_i}|^s.$$

Therefore, $(2r_i)^s \leqslant C(s)|J_{\sigma(i)|k_i}|^s$.
By Proposition 3,

$$\mu(B_{r_i}(x_i))^q \nu(B_{r_i}(x_i))^t \leqslant B_2(q,t)\mu(J_{\sigma(i)|k_i})^q \nu(J_{\sigma(i)|k_i})^t.$$

Then,

$$\begin{aligned}
&\sum_i \mu(B_{r_i}(x_i))^q \nu(B_{r_i}(x_i))^t (2r_i)^s \leqslant \\
&\leqslant C(s)B_2(q,t)\sum_i \mu(J_{\sigma(i)|k_i})^q \nu(J_{\sigma(i)|k_i})^t |J_{\sigma(i)|k_i}|^s = \\
&= C(s)B_2(q,t)\sum_i \frac{\mu(J_{\sigma(i)|k_i})^q \nu(J_{\sigma(i)|k_i})^t |J_{\sigma(i)|k_i}|^s}{\sum\limits_{\sigma\in D_{k_i}} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s} \sum_{\sigma\in D_{k_i}} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s \leqslant \\
&\leqslant \varepsilon C(s)B_2(q,t)\sum_i \chi_{q,t}(J_{\sigma(i)|k_i}) \leqslant \varepsilon C(s)B_2(q,t)\sum_i \chi_{q,t}(B_{r_i}(x_i)) \leqslant \\
&\leqslant \varepsilon C(s)B_2(q,t)\chi_{q,t}\Big(\bigcup_i B_{r_i}(x_i)\Big) \leqslant \varepsilon C(s)B_2(q,t).
\end{aligned}$$

Since ε is small enough, then $\mathcal{P}^{q,t,s}_{\mu,\nu}(E) = 0$.
3) Since $\overline{h} = +\infty$, then, for any $\varepsilon > 0$, we can find an infinite number of integers n with

$$\sum_{\sigma\in D_n} \mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s > \frac{1}{\varepsilon}. \tag{4}$$

Now, take any open set U intersecting E; it contains a basic interval, say J_{σ_0} for $\sigma_0 \in D_n$. For any $\sigma \in D_k$, $\tau \in D_{k+1,n}$, it follows from the definitions of μ and ν that

$$\frac{\mu(J_{\sigma*\tau})}{\mu(J_\sigma)} = \frac{\mu(J_{\sigma_0*\tau})}{\mu(J_{\sigma_0})}, \quad \frac{\nu(J_{\sigma*\tau})}{\nu(J_\sigma)} = \frac{\nu(J_{\sigma_0*\tau})}{\nu(J_{\sigma_0})}, \quad \text{and} \quad \frac{|J_{\sigma*\tau}|}{|J_\sigma|} = \frac{|J_{\sigma_0*\tau}|}{|J_{\sigma_0}|}.$$

Combining with (4), we get

$$\begin{aligned}
&\sum_{\sigma\in D_{k+1,n}} \mu(J_{\sigma_0*\tau})^q \nu(J_{\sigma_0*\tau})^t |J_{\sigma_0*\tau}|^s = \\
&= \sum_{\sigma\in D_{k+1,n}} \frac{\mu(J_{\sigma_0})^q \nu(J_{\sigma_0})^t \mu(J_{\sigma*\tau})^q \nu(J_{\sigma*\tau})^t |J_{\sigma_0}|^s |J_{\sigma*\tau}|^s}{\mu(J_\sigma)^q \nu(J_\sigma)^t |J_\sigma|^s} \geqslant
\end{aligned}$$

$$\geqslant \frac{1}{\varepsilon}\frac{\mu\left(J_{\sigma_0}\right)^q\nu\left(J_{\sigma_0}\right)^t\left|J_{\sigma_0}\right|^s}{\mu\left(J_{\sigma}\right)^q\nu\left(J_{\sigma}\right)^t\left|J_{\sigma}\right|^s} \geqslant \frac{1}{\varepsilon}\frac{\mu\left(J_{\sigma_0}\right)^q\nu\left(J_{\sigma_0}\right)^t\left|J_{\sigma_0}\right|^s}{\sum\limits_{\sigma\in D_k}\mu\left(J_{\sigma}\right)^q\nu\left(J_{\sigma}\right)^t\left|J_{\sigma}\right|^s}.$$

By using a similar argument as in the proof of the assertion above, and (2), we learn that there are three positive constants k_1, k_2 and k_3, such that, as $\varepsilon \to 0$

$$\mathcal{P}^{q,t,s}_{\mu,\nu,0}(E\cap I_{\sigma_0}) \geqslant \frac{k_1k_2k_3\mu\left(J_{\sigma_0}\right)^q\nu\left(J_{\sigma_0}\right)^t\left|J_{\sigma_0}\right|^s}{2\varepsilon\sum\limits_{\sigma\in D_k}\mu\left(J_{\sigma}\right)^q\nu\left(J_{\sigma}\right)^t\left|J_{\sigma}\right|^s} = \frac{k_1k_2k_3}{2\varepsilon}\chi_{q,t,s}\left(J_{\sigma_0}\right) \longrightarrow +\infty.$$

This gives:

$$\mathcal{P}^{q,t,s}_{\mu,\nu,0}(E\cap U) \geqslant \mathcal{P}^{q,t,s}_{\mu,\nu,0}(E\cap I_{\sigma_0}) = +\infty.$$

Using techniques as in [27, Theorem 2], we obtain $\mathcal{P}^{q,t,s}_{\mu,\nu}(E) = +\infty$. □

Proposition 6. *Let* $\Delta > 0$. *For* $(q,t,s) \in \mathbb{R}^3$, *we have:*

$$\mathcal{H}^{q,t,s}_{\mu,\nu}(E) \leqslant \mathcal{P}^{q,t,s}_{\mu,\nu}(E).$$

Proof. By Proposition 2, the measures μ and ν satisfy the doubling condition. Therefore, the required inequality follows from [23, Theorem 1]. □

Proposition 7. *Assume that* $\Delta > 0$. *We have:*

$$b_{\mu,\nu}(q,t) = \underline{\tau}(q,t), \quad \text{and} \quad B_{\mu,\nu}(q,t) = \overline{\tau}(q,t).$$

Proof. We will prove the first assertion. The proof of the second assertion is very similar and is therefore omitted. Let $s < \underline{\tau}(q,t)$. We can find $k_0 \in \mathbb{N}$, such that for any $k \geqslant k_0$ we get $s < \tau_k(q,t)$. This gives:

$$\sum_{\sigma\in D_k}\mu\left(J_{\sigma}\right)^q\nu\left(J_{\sigma}\right)^t\left|J_{\sigma}\right|^s \geqslant \sum_{\sigma\in D_k}\mu\left(J_{\sigma}\right)^q\nu\left(J_{\sigma}\right)^t\left|J_{\sigma}\right|^{\tau_k(q,t)} = 1.$$

Then

$$\liminf_{k\to+\infty}\sum_{\sigma\in D_k}\mu\left(J_{\sigma}\right)^q\nu\left(J_{\sigma}\right)^t\left|J_{\sigma}\right|^s > 0.$$

By using Proposition 4, we have $\mathcal{H}^{q,t,s}_{\mu,\nu}(E) > 0$, for all $s < \underline{\tau}(q,t)$, which implies that $b_{\mu,\nu}(q,t) \geqslant \underline{\tau}(q,t)$.

Now fix $s > \underline{\tau}(q,t)$. We can find a sequence $(k_i)_i$, such that $\tau_{k_i}(q,t) < s$, for any sufficiently large i. Let $F \subseteq E$, and $\sigma \in D_{k_i}$, such that $F \cap J_\sigma \neq \varnothing$;

then we can choose $x_i \in F \cap J_\sigma(x_i)$. This shows that $\big(B_{|J_\sigma(x_i)|}(x_i)\big)_i$ is a centered $(c_{\max})^{k_i}$-covering of F. By Proposition 2, and $B_{\Delta|J_{\sigma|v_i}|}(x_i) \cap E \subset J_\sigma$, for all $\sigma \in D_{k_i}$, we can choose positive constants $\mathcal{P}_0(\Delta) > 0$ and $\mathcal{P}_1(\Delta) > 0$, such that

$$\frac{\mu\big(B_{|J_\sigma(x_i)|}(x_i)\big)}{\mu\big(J_\sigma(x_i)\big)} \leqslant \frac{\mu\big(B_{|J_\sigma(x_i)|}(x_i)\big)}{\mu\big(B_{\Delta|J_\sigma(x_i)|}(x_i)\big)} \leqslant \mathcal{P}_0(\Delta),$$

and

$$\frac{\nu\big(B_{|J_\sigma(x_i)|}(x_i)\big)}{\nu\big(J_\sigma(x_i)\big)} \leqslant \frac{\nu\big(B_{|J_\sigma(x_i)|}(x_i)\big)}{\nu\big(B_{\Delta|J_\sigma(x_i)|}(x_i)\big)} \leqslant \mathcal{P}_1(\Delta).$$

Then there is a constant $C(q,t,s) > 0$ with

$$\begin{aligned}
\sum_i \mu\big(B_{|J_\sigma(x_i)|}(x_i)\big)^q \nu\big(B_{|J_\sigma(x_i)|}(x_i)\big)^t \big(2|J_\sigma(x_i)|\big)^s \leqslant \\
\leqslant C(q,t,s) \sum_i \mu\big(J_\sigma(x_i)\big)^q \nu\big(J_\sigma(x_i)\big)^t |J_\sigma(x_i)|^s \leqslant \\
\leqslant C(q,t,s) \sum_{\sigma\in D_{k_i}} \mu\left(J_\sigma\right)^q \nu\left(J_\sigma\right)^t \left|J_\sigma\right|^s \leqslant \\
\leqslant C(q,t,s) \sum_{\sigma\in D_{k_i}} \mu\left(J_\sigma\right)^q \nu\left(J_\sigma\right)^t \left|J_\sigma\right|^{\tau_{k_i}(q,t)} = C(q,t,s).
\end{aligned}$$

We now deduce that $\mathcal{H}^{q,t,s}_{\mu,\nu,0}(F) \leqslant C(q,t,s)$, and $\mathcal{H}^{q,t,s}_{\mu,\nu}(E) < +\infty$, for all $s > \underline{\tau}(q,t)$. Finally, we conclude that $b_{\mu,\nu}(q,t) \leqslant \underline{\tau}(q,t)$. □

Proposition 8. *Assume that* $\Delta > 0$, $\left(\frac{\partial\underline{\tau}(q,t)}{\partial q}, \frac{\partial\underline{\tau}(q,t)}{\partial t}\right)$ *exists, and that*

$$\liminf_{k\to\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \nu\left(J_\sigma\right)^t \left|J_\sigma\right|^{\underline{\tau}(q,t)} > 0, \quad \textit{for all } (q,t) \in \mathbb{R}^2.$$

If $(\alpha,\beta) = -\left(\frac{\partial\underline{\tau}(q,t)}{\partial q}, \frac{\partial\underline{\tau}(q,t)}{\partial t}\right)$, *then* $\mathcal{H}^{q,t,\underline{\tau}(q,t)}_{\mu,\nu}\left(K(\alpha,\beta)\right) > 0$.

Proof. By using Proposition 4, we have

$$\mathcal{H}^{q,t,\underline{\tau}(q,t)}_{\mu,\nu}(E) \geqslant A \liminf_{k\to\infty} \sum_{\sigma\in D_k} \mu\left(J_\sigma\right)^q \nu\left(J_\sigma\right)^t \left|J_\sigma\right|^{\underline{\tau}(q,t)} > 0.$$

Now, for $\boldsymbol{\alpha} = (\alpha,\beta) \in \mathbb{R}^2$, consider the following sets:

$$F_{\boldsymbol{\alpha}} = \left\{x;\ \limsup_{r\to 0} \frac{\log\big(\mu(B_r(x))\big)}{\log r} > \alpha, \text{ or } \limsup_{r\to 0} \frac{\log\big(\nu(B_r(x))\big)}{\log r} > \beta\right\},$$

$$F_{\boldsymbol{\alpha}}^1 = \Big\{x;\ \liminf_{r\to 0} \frac{\log\big(\mu(B_r(x))\big)}{\log r} < \alpha, \text{ or } \liminf_{r\to 0} \frac{\log\big(\nu(B_r(x))\big)}{\log r} < \beta\Big\},$$

$$F_{\boldsymbol{\alpha}}^2 = \Big\{x;\ \limsup_{r\to 0} \frac{\log\big(\mu(B_r(x))\big)}{\log r} > \alpha, \text{ or } \liminf_{r\to 0} \frac{\log\big(\nu(B_r(x))\big)}{\log r} < \beta\Big\},$$

and

$$F_{\boldsymbol{\alpha}}^3 = \Big\{x;\ \liminf_{r\to 0} \frac{\log\big(\mu(B_r(x))\big)}{\log r} < \alpha, \text{ or } \limsup_{r\to 0} \frac{\log\big(\nu(B_r(x))\big)}{\log r} > \beta\Big\}.$$

The σ-subadditivity of the mutual Hausdorff measure allows this:

$$\mathcal{H}_{\mu,\nu}^{q,t,\underline{\tau}(q,t)}(F_{\boldsymbol{\alpha}}) = 0, \text{ for every } \alpha > -\frac{\partial\underline{\tau}(q,t)}{\partial q}, \text{ and } \beta > -\frac{\partial\underline{\tau}(q,t)}{\partial t}, \tag{5}$$

$$\mathcal{H}_{\mu,\nu}^{q,t,\underline{\tau}(q,t)}(F_{\boldsymbol{\alpha}}^1) = 0, \text{ for every } \alpha < -\frac{\partial\underline{\tau}(q,t)}{\partial q}, \text{ and } \beta < -\frac{\partial\underline{\tau}(q,t)}{\partial t}, \tag{6}$$

$$\mathcal{H}_{\mu,\nu}^{q,t,\underline{\tau}(q,t)}(F_{\boldsymbol{\alpha}}^2) = 0, \text{ for every } \alpha > -\frac{\partial\underline{\tau}(q,t)}{\partial q}, \text{ and } \beta < -\frac{\partial\underline{\tau}(q,t)}{\partial t}, \tag{7}$$

and

$$\mathcal{H}_{\mu,\nu}^{q,t,\underline{\tau}(q,t)}(F_{\boldsymbol{\alpha}}^3) = 0, \text{ for every } \alpha < -\frac{\partial\underline{\tau}(q,t)}{\partial q}, \text{ and } \beta > -\frac{\partial\underline{\tau}(q,t)}{\partial t}. \tag{8}$$

The proof of (6)–(8) follows by using the same ideas as in the proof of (5). So, we will prove (5). Let $\alpha > -\frac{\partial\underline{\tau}(q,t)}{\partial q}$ and $\beta > -\frac{\partial\underline{\tau}(q,t)}{\partial t}$; then, there is $h > 0$ with

$$\underline{\tau}(q-h,t) < \underline{\tau}(q,t) + \alpha h, \text{ and } \underline{\tau}(q,t-h) < \underline{\tau}(q,t) + \beta h.$$

It follows from Proposition 4 that

$$\begin{aligned}
&\liminf_{n\to+\infty} \sum_{\sigma\in D_n} \mu\left(J_\sigma\right)^{q-h} \nu\left(J_\sigma\right)^{t} \left|J_\sigma\right|^{\underline{\tau}(q,t)+\alpha h} = \\
&= \liminf_{n\to+\infty} \sum_{\sigma\in D_n} \mu\left(J_\sigma\right)^{q} \nu\left(J_\sigma\right)^{t-h} \left|J_\sigma\right|^{\underline{\tau}(q,t)+\beta h} = 0.
\end{aligned}$$

Let $\varepsilon > 0$; there are a sequence $(n(i))_i$ and $i_0 \in \mathbb{N}$, such that if $i \geqslant i_0$, we get

$$\sum_{\sigma\in D_{n(i)}} \mu\left(J_\sigma\right)^{q-h} \nu\left(J_\sigma\right)^{t} \left|J_\sigma\right|^{\underline{\tau}(q,t)+\alpha h} < \varepsilon, \tag{9}$$

and
$$\sum_{\sigma\in D_{n(i)}} \mu\left(J_\sigma\right)^q \nu\left(J_\sigma\right)^{t-h} \left|J_\sigma\right|^{\underline{\tau}(q,t)+\beta h} < \varepsilon. \tag{10}$$

We need to show that $\mathcal{H}_{\mu,\nu,0}^{q,t,\underline{\tau}(q,t)}(F) = 0$, for any $F \subset F_{\boldsymbol{\alpha}}$. Let $x \in F \subseteq F_{\boldsymbol{\alpha}}$, then there exists $r_x > 0$ with $\mu(B_{r_x}(x)) \leqslant r_x^\alpha$ or $\nu(B_{r_x}(x)) \leqslant r_x^\beta$. We have

$$\mu(B_{r_x}(x))^q \nu(B_{r_x}(x))^t (2r_x)^{\underline{\tau}(q,t)} \leqslant \begin{cases} \mu(B_{r_x}(x))^{q-h} \nu(B_{r_x}(x))^t (2r_x)^{\underline{\tau}(q,t)+\alpha h}, \\ \text{or} \\ \mu(B_{r_x}(x))^q \nu(B_{r_x}(x))^{t-h} (2r_x)^{\underline{\tau}(q,t)+\beta h}. \end{cases} \tag{11}$$

If $(B_{r_i^*}(x_i^*))_i$ and $(B_{r_i^{**}}(x_i^{**}))_i$ are, respectively, a centered δ-covering of F and $E\backslash F$, then $(B_{r_i}(x_i))_i = \Big(B_{r_i^*}(x_i^*) \cup B_{r_i^{**}}(x_i^{**})\Big)_i$ is a centered δ-covering of E. It follows from Proposition 3 and (3) that for all $(q,t,s) \in \mathbb{R}^3$:

$$\begin{aligned} \sum_i \mu(B_{r_i^*}(x_i^*))^q \nu(B_{r_i^*}(x_i^*))^t (2r_i^*)^s &\leqslant \sum_i \mu(B_{r_i}(x_i))^q \nu\left(B_{r_i}(x_i)\right)^t \left(2r_i\right)^s \leqslant \\ &\leqslant k_1k_2k_3 \sum_i \mu\left(J_{\sigma(i)|n(i)}(x_i)\right)^q \nu\left(J_{\sigma(i)|n(i)}(x_i)\right)^t \left|J_{\sigma(i)|n(i)}(x_i)\right|^s, \end{aligned}$$

where k_1,k_2 and k_3 are suitable constants. Let $K = k_1k_2k_3$; using (9) and (10), we have:

$$\sum_i \mu(B_{r_i^*}(x_i^*))^{q-h} \nu(B_{r_i^*}(x_i^*))^t (2r_i^*)^{\underline{\tau}(q,t)+\alpha h} \leqslant \varepsilon K \sum_i \chi_{q-h,t,\underline{\tau}(q,t)+\alpha h}(B_{r_i}(x_i)),$$

or

$$\sum_i \mu(B_{r_i^*}(x_i^*)^q \nu(B_{r_i^*}(x_i^*))^{t-h} (2r_i^*)^{\underline{\tau}(q,t)+\beta h} \leqslant \varepsilon K \sum_i \chi_{q,t-h,\underline{\tau}(q,t)+\beta h}(B_{r_i}(x_i)).$$

It follows from (11) that

$$\begin{aligned} \mathcal{H}_{\mu,\nu,\delta}^{q,t,\underline{\tau}(q,t)}(F) &\leqslant \sum_i \mu\left(B_{r_i^*}(x_i^*)\right)^q \nu\left(B_{r_i^*}(x_i^*)\right)^t (2r_i^*)^{\underline{\tau}(q,t)} \leqslant \\ &\leqslant \varepsilon K \sum_i \max\left(\chi_{q-h,t,\underline{\tau}(q,t)+\alpha h}\left(B_{r_i}(x_i)\right), \chi_{q,t-h,\underline{\tau}(q,t)+\beta h}\left(B_{r_i}(x_i)\right)\right). \end{aligned}$$

We deduce that

$$\mathcal{H}_{\mu,\nu,0}^{q,t,\underline{\tau}(q,t)}(F) \leqslant \varepsilon K \sum_i \max(\chi_{q-h,t,\underline{\tau}(q,t)+\alpha h}(B_{r_i}(x_i)), \chi_{q,t-h,\underline{\tau}(q,t)+\beta h}\left(B_{r_i}(x_i)\right)).$$

Since $(B_{r_i}(x_i))_i$ is a centered δ-covering of the set E, then by using Besicovitch's covering theorem, there exists $\xi = \xi(n)$ finite sub-families $(B_{r_{1j}}(x_{1j}))_j, \ldots, (B_{r_{\xi j}}(x_{\xi j}))_j$ fulfilling the following: for each $i \in \{1, 2, \ldots, \xi\}$, the family $(B_{r_{ij}}(x_{ij}))_j$ is a δ-packing of E, and $E \subset \bigcup_{i=1}^{\xi} \bigcup_j B_{r_{ij}}(x_{ij})$. Which implies that

$$\begin{aligned}
&\mathcal{H}^{q,t,\underline{\tau}(q,t)}_{\mu,\nu,0}(F) \leqslant \\
&\leqslant \varepsilon K \sum_{i=1}^{\xi} \sum_j \max\left(\chi_{q-h,t,\underline{\tau}(q,t)+\alpha h}\left(B_{r_{ij}}(x_{ij})\right), \chi_{q,t-h,\underline{\tau}(q,t)+\beta h}\Big(B_{r_{ij}}(x_{ij})\Big)\right) = \\
&= \varepsilon K \sum_{i=1}^{\xi} \max\Big(\chi_{q-h,t,\underline{\tau}(q,t)+\alpha h}\Big(\bigcup_j B_{r_{ij}}(x_{ij})\Big), \chi_{q,t-h,\underline{\tau}(q,t)+\beta h}\Big(\bigcup_j B_{r_{ij}}(x_{ij})\Big)\Big) \leqslant \\
&\qquad\qquad \leqslant \varepsilon K \xi \longrightarrow 0 \ \text{ as } \ \varepsilon \to 0,
\end{aligned}$$

which implies that $\mathcal{H}^{q,t,\underline{\tau}(q,t)}_{\mu,\nu,0}(F) = 0$, then, $\mathcal{H}^{q,t,\underline{\tau}(q,t)}_{\mu,\nu}(F_{\boldsymbol{\alpha}}) = 0$. $\square$

Proof of Theorem 2. By using the convexity and differentiability of the function $\underline{\tau}$, we have

i) The function $\underline{\tau}^*$ is concave.

ii)

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 \mid \underline{\tau}^*(\alpha,\beta) > -\infty\right\} = \overline{\left(-\left(\frac{\partial \underline{\tau}(q,t)}{\partial q}, \frac{\partial \underline{\tau}(q,t)}{\partial t}\right)\right)(\mathbb{R}^2)}.$$

iii)

$$\begin{aligned}
&\underline{\tau}^*(\alpha,\beta) = \\
&= \begin{cases} \alpha q + \beta t + \underline{\tau}(q,t), \ \text{for } (q,t) \in \mathbb{R}^2, (\alpha,\beta) \in \overline{\left(\left(\frac{\partial \underline{\tau}(q,t)}{\partial q}, \frac{\partial \underline{\tau}(q,t)}{\partial t}\right)\right)(\mathbb{R}^2)}, \\ -\infty, \ \text{for } (\alpha,\beta) \notin \overline{\left(-\left(\frac{\partial \underline{\tau}(q,t)}{\partial q}, \frac{\partial \underline{\tau}(q,t)}{\partial t}\right)\right)(\mathbb{R}^2)}. \end{cases}
\end{aligned} \tag{12}$$

It follows now from [23], [24] that

$$\mathcal{H}^{q,t,\underline{\tau}(q,t)}_{\mu,\nu}\left(K(\alpha,\beta)\right) \leqslant \mathcal{H}^{q\alpha+t\beta+\underline{\tau}(q,t)-\delta}\left(K(\alpha,\beta)\right),$$

for any $0 < \delta \leqslant q\alpha + t\beta + \underline{\tau}(q,t)$. By Proposition 8, we get

$$\mathcal{H}^{q\alpha+t\beta+\underline{\tau}(q,t)-\delta}\left(K(\alpha,\beta)\right) > 0,$$

and this gives

$$\dim_H(K(\alpha,\beta)) \geqslant q\alpha + t\beta + \underline{\tau}(q,t) - \delta, \text{ for any } 0 < \delta \leqslant q\alpha + t\beta + \underline{\tau}(q,t).$$

Letting $\delta \to 0$ yields that

$$\dim_H(K(\alpha,\beta)) \geqslant q\alpha + t\beta + \underline{\tau}(q,t). \tag{13}$$

Now, by using [23], [24], and Proposition 7, we obtain

$$\dim_H(K(\alpha,\beta)) \leqslant q\alpha + t\beta + \underline{\tau}(q,t). \tag{14}$$

It follows from (12)–(14) that

$$\dim_H(K(\alpha,\beta)) = q\alpha + t\beta + \underline{\tau}(q,t) = \underline{\tau}^*(\alpha,\beta).$$

The desired result follows immediately from Proposition 7.

The proof of the second statement is very identical to the proof of the first. The third follows from the first and the second statements.

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Bilel Selmi
Analysis, Probability and Fractals Laboratory LR18ES17
Faculty of sciences of Monastir
Department of mathematics
5000-Monastir-Tunisia
E-mail: bilel.selmi@fsm.rnu.tn, bilel.selmi@isetgb.rnu.tn

Nina Yu. Svetova
Petrozavodsk State University
33 Lenina pr., Petrozavodsk 185910, Russia
E-mail: nsvetova@petrsu.ru