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STABILITY-PRESERVING PERTURBATION OF THE MAXIMAL TERMS OF DIRICHLET SERIES

Abstract. We study stability of the maximal term of the Dirichlet series with positive exponents, the sum of which is an entire function. This problem is of interest, because the Leont'ev formulas for coefficients calculated using a biorthogonal system of functions play the key role in obtaining asymptotic estimates for entire Dirichlet series on various continua going to infinity (for example, curves). This fact naturally leads to the need to study the behavior of the logarithm of the maximum term also for the Hadamard composition of the corresponding Dirichlet series. For the wide class of entire Dirichlet series determined by a convex growth majorant, we establish a criterion for the equivalence of the logarithms of the moduli of the original series and a modified Dirichlet series outside some exceptional set.

Key words: *Dirichlet series, Hadamard composition, stability of the maximal term, Borel–Nevanlinna lemma, convex function.*

2020 Mathematical Subject Classification: *30D40*

1. Introduction. The equivalence problem of logarithms of maximal terms of entire Dirichlet series $\sum\limits_n a_n e^{\lambda_n s}$ and $\sum\limits_n a_n b_n e^{\lambda_n s}$ $(0 < \lambda_n \uparrow \infty)$ was first studied in [2]. This important property, called the stability of the maximal term, turned out to be very useful for obtaining asymptotic estimates of the sum of the Dirichlet series on curves going to infinity, namely, for proving the well-known Polya conjecture (1929). Similar research was later carried out for Dirichlet series of a given growth, for example, of a finite Ritt order [3]. The key role in such research is played by Borel–Nevanlinna type lemmas (see, for example, [2], [1]). However, in the mentioned works [2], [3], a rather strong, although natural for the main problems considered there, restriction was required on the exponents λ_n of the Dirichlet series

$$\varlimsup_{n\to\infty} \frac{\ln n}{\ln \lambda_n} < \infty. \tag{1}$$

In other words, λ_n are zeros of some entire function of finite order.

Studying stability of the maximal term is also of interest by itself. In this situation, as shown in [5], it is enough to assume that

$$\sum_{n=1}^{\infty} \frac{1}{n\lambda_n} < \infty \tag{2}$$

or (this is the same):

$$\int_1^{\infty} \frac{\ln n(t)}{t^2}\, dt < \infty, \quad n(t) = \sum_{\lambda_n \leqslant t} 1. \tag{3}$$

It is clear that condition (2) is weaker than (1).

In this paper, we consider the entire Dirichlet series in the class $\underline{D}(\Phi)$ defined by some convex majorant Φ (we considered the dual class $D(\Phi)$ in [1]). As in works [2]– [5], the stability criterion of the maximal term of the Dirichlet series of class $\underline{D}(\Phi)$ is proved in terms of the sequence $\{b_n\}$. In the situation we are considering here, the convergence of the series (2) (of the integral (3)) is not assumed at all.

2. The Main Result. Let $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ be a sequence, such that

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n} = 0. \tag{4}$$

We denote by $D(\Lambda)$ the class of all functions F that can be represented by the Dirichlet series in the whole plane:

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \quad (s = \sigma + it). \tag{5}$$

By (4), if the series (5) converges in the whole plane, then it absolutely converges in the plane and its sum F is an entire function [4]. Denote by L the class of all continuous infinitely increasing positive functions on $\mathbb{R}_+ = [0, \infty)$. Let Φ be a convex function of class L, and let

$$\underline{D}_m(\Phi) = \{F \in D(\Lambda)\colon \exists\, \{\sigma_n\}, 0 < \sigma_n \uparrow \infty,\ \ln M(\sigma_n) \leqslant \Phi(m\sigma_n)\}\,, m \geqslant 1,$$

where $M(\sigma)=\sup\limits_{|t|<\infty}|F(\sigma+it)|$. Set $\underline{D}(\Phi)=\bigcup\limits_{m=1}^{\infty}\underline{D}_m(\Phi)$.

Note that in [1] the class $D(\Phi)=\bigcup\limits_{m=1}^{\infty}D_m(\Phi)$, is considered, where

$$D_m(\Phi)=\{F\in D(\Lambda)\colon \ \ln M(\sigma)\leqslant\Phi(m\sigma)\}, \quad m\geqslant 1.$$

Together with the series (5), we consider the series

$$F_b^*(s)=\sum_{n=1}^{\infty}a_nb_ne^{\lambda_n s}, \tag{6}$$

where the sequence $b=\{b_n\}$ of complex numbers b_n ($b_n\neq 0$ for $n\geqslant N$) satisfies the condition

$$\varlimsup_{n\to\infty}\frac{|\ln|b_n||}{\lambda_n}<\infty. \tag{7}$$

In this case, $F_b^*\in\underline{D}(\Phi)$ if and only if $F\in\underline{D}(\Phi)$ (see below).

Let $E\subset[0,\infty)$ be a Lebesgue measurable set. By the upper DE and lower dE densities of the set E we mean the quantities

$$DE=\varlimsup_{\sigma\to\infty}\frac{\mathrm{mes}(E\cap[0,\sigma])}{\sigma}, \quad dE=\varliminf_{\sigma\to\infty}\frac{\mathrm{mes}(E\cap[0,\sigma])}{\sigma}.$$

In what follows, we assume that all exceptional sets $E\subset[0,\infty)$ outside of which asymptotic estimates will be obtained, are represented by the unions of segments of the form $[a_n,a_n']$, where

$$0<a_1<a_1'\leqslant a_2<a_2'\leqslant\ldots\leqslant a_n<a_n'\leqslant\ldots\,.$$

Let φ be the inverse of Φ, such that

$$\varlimsup_{x\to\infty}\frac{\varphi(x^2)}{\varphi(x)}<\infty. \tag{8}$$

From (8) it follows that $\Phi\in M$, where M is the class of convex functions Φ, such that $x\Phi(x)<\Phi(Kx)$ for $x\geqslant x_0$, where K is some constant. Indeed, due to (8) there exists a constant $K>0$, such that $\varphi(t^2)\leqslant K\varphi(t)$, and, hence, $t^2\leqslant\Phi(K\varphi(t))$, $t\geqslant 0$. Denoting $x=\varphi(t)$ and using the fact that $x<\Phi(x)$ for $x\geqslant x_0$, we have: $x\Phi(x)<\Phi^2(x)\leqslant\Phi(Kx)$. So $\Phi\in M$.

We introduce the class of functions

$$\underline{W}(\varphi)=\left\{w\in L\colon \sqrt{x}\leqslant w(x),\ \lim_{x\to\infty}\frac{w(x)}{x\varphi(x)}=0,\ \varliminf_{x\to\infty}\frac{1}{\varphi(x)}\int_1^x\frac{w(t)}{t^2}dt=0\right\}.$$

We note that for any $\varphi \in L$ the function $w(x) = \sqrt{x}$ belongs to the class

$$W(\varphi) = \left\{ w \in L\colon \sqrt{x} \leqslant w(x),\ \lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{w(t)}{t^2}\, dt = 0 \right\}.$$

Denote by $\mu(\sigma)$ and $\mu_b^*(\sigma)$ the maximal terms of the series (5) and (6), respectively:

$$\mu(\sigma) = \max_{n\geqslant 1} \left\{ |a_n|\, e^{\lambda_n \sigma} \right\},\ \mu_b^*(\sigma) = \max_{n\geqslant 1} \left\{ |a_n|\, |b_n|\, e^{\lambda_n \sigma} \right\}.$$

Let $n(t) = \sum_{\lambda_n \leqslant t} 1$ be the counting functions of the sequence Λ, and let $n_l(t)$ be the least concave majorant of $\ln n(t)$. It is well defined because of the condition (4).

We formulate the main result of the paper.

Theorem 1. *Let $\{b_n\}$ be a sequence of complex numbers ($b_n \neq 0$, $n \geqslant N$), satisfying (7), and let Φ be a convex function of class L. We assume that the inverse φ of Φ satisfies (8) and the function $n_l(t)$ satisfies*

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{n_l(t)}{t^2} dt = 0.$$

For any $F \in \underline{D}(\Phi)$ for $\sigma \to \infty$, for the asymptotic equality

$$\ln \mu(\sigma) = (1 + o(1)) \ln \mu_b^*(\sigma), \tag{9}$$

to be valid outside some set $E \subset [0,\infty)$ of zero lower density, it is sufficient and necessary that there exists a function $w \in W(\varphi)$ such that

$$|\ln |b_n|| \leqslant w(\lambda_n) \qquad (n \geqslant N). \tag{10}$$

In this theorem, the function $n_l(t)$ can be replaced by $\ln n(t)$. Proving this statement requires a slightly different approach. Therefore, this case will be considered in another paper.

3. Proof of Theorem 1.

The proof of theorem 1 is based on the following accessory statement:

Theorem 2. *Let $\Phi \in L$, and the inverse φ of Φ satisfy (8). Let $u(\sigma)$ be a nondecreasing positive continuous function on $[r_0, \infty)$, moreover*

$$\lim_{\sigma\to\infty} u(\sigma) = \infty, \qquad \varliminf_{\sigma\to\infty} \frac{u(\sigma)}{\ln\Phi(\sigma)} < \infty. \tag{11}$$

Let $\{x_n\}$ be a sequence chosen so that

$$u(x_n) \leqslant C \ln \Phi(x_n), \quad 0 < C < \infty.$$

Suppose that $w \in W(\varphi)$. If $v = v(\sigma)$ is a solution to the equation

$$w(v) = e^{u(\sigma)}, \tag{12}$$

then, for the same sequence $\{x_n\}$ for $\sigma \to \infty$ outside a set $E \subset [0, \infty)$,

$$\operatorname{mes}(E \cap [0, x_n]) = o(\varphi(v(x_n))) + 4\int\limits_{v(x_1)}^{v(x_n)} \frac{w^*(t)}{t^2}\,dt = o(\varphi(v(x_n))),\ x_n \to \infty,$$

the following estimate holds:

$$u\Big(\sigma + \frac{w(v(\sigma))}{v(\sigma)}\Big) < u(\sigma) + o(1).$$

Here function w^* is some function of class $W(\varphi)$, having a form $w^*(t) = \beta(t)w(t) \quad (\beta \in L)$.

The theorem 2 is proved in the same way as the corresponding Borel-Nevanlinna-type theorem from [1]. The difference is only in the choice of the sequence $\{x_n\}$. However, the theorem 2 needs proof.

Proof. There exists a function $w^*(t) = \beta(t)w(t)$, $(0 < \beta(t) \uparrow \infty,\ t \to \infty)$ of class $W(\varphi)$, satisfying the condition of theorem.

We show that

$$u\Big(\sigma + \frac{w(v(\sigma))}{v(\sigma)}\Big) < u(\sigma) + \frac{1}{\beta(v(\sigma))}$$

outside a set $E \subset [0, \infty)$ of zero lower density.

Indeed, let $E \subset [0, \infty)$ be a set on which

$$u\Big(\sigma + \frac{w(v(\sigma))}{v(\sigma)}\Big) \geqslant u(\sigma) + \frac{1}{\beta(v(\sigma))}.$$

Then, in the same way as in [1], the sequences $\{\sigma_n\}$ and $\{\sigma'_n\}$ are constructed, such that $E \subset \bigcup\limits_{n=1}^{\infty} [\sigma_n, \sigma'_n]$, and

$$0 < \sigma'_n - \sigma_n \leqslant \frac{w(v(\sigma_n))}{v(\sigma_n)}, \quad u(\sigma_{n+1}) - u(\sigma_n) \geqslant \frac{1}{\beta(v(\sigma_n))}. \tag{13}$$

Let $v_n = v(\sigma_n)$, $\delta_n = \frac{w(v_n)}{v_n}$ $(n \geqslant 1)$. If $2v_n \leqslant v_{n+1}$, then

$$\delta_n \leqslant 2 \int\limits_{v_n}^{v_{n+1}} \frac{w(t)}{t^2}\, dt < 2 \int\limits_{v_n}^{v_{n+1}} \frac{w^*(t)}{t^2}\, dt. \tag{14}$$

If $2v_n > v_{n+1}$, then we have, from (12), (13), and the monotonicity of $w = w(t)$ and $\beta = \beta(t)$:

$$\delta_n \leqslant \frac{w^*(v_n)}{v_n} [u(\sigma_{n+1}) - u(\sigma_n)] \leqslant 2 \int\limits_{v_n}^{v_{n+1}} \frac{w^*(t)}{t}\, d\ln w^*(t) =$$

$$= 2\left[\frac{w^*(v_{n+1})}{v_{n+1}} - \frac{w^*(v_n)}{v_n} + \int\limits_{v_n}^{v_{n+1}} \frac{w^*(t)}{t^2}\, dt\right]. \tag{15}$$

Since

$$\int\limits_{v_n}^{v_{n+1}} \frac{dw^*(t)}{t} \geqslant 0,$$

then, obviously,

$$\int\limits_{v_n}^{v_{n+1}} \frac{dw^*(t)}{t^2} \leqslant \frac{w^*(v_{n+1})}{v_{n+1}} - \frac{w^*(v_n)}{v_n} + 2 \int\limits_{v_n}^{v_{n+1}} \frac{w^*(t)}{t^2}\, dt.$$

Consequently, from (14), (15) we conclude that

$$\delta_n \leqslant 2\left[\frac{w^*(v_{n+1})}{v_{n+1}} - \frac{w^*(v_n)}{v_n}\right] + 4 \int\limits_{v_n}^{v_{n+1}} \frac{w^*(t)}{t^2}\, dt.$$

It follows from the condition of the theorem that there exists a sequence $\{x_n\}(0 < x_n \uparrow \infty)$, such that $u(x_n) \leqslant C\Phi(x_n)$ $(0 < C < \infty)$. It is clear that for any $n \geqslant 0$ there exists $k \geqslant 0$, such that $\sigma_{k-1} \leqslant x_n < \sigma_k$. Consequently, we have:

$$\operatorname{mes}(E \cap [0, x_n]) \leqslant \sum_{n=1}^{k-1} \delta_n = \frac{w^*(v_{k-1})}{v_{k-1}} + \sum_{n=1}^{k-2} \delta_n.$$

Then, if $\sigma_{k-1} \leqslant x_n < \sigma_k$, then for $x_n \to \infty$

$$\frac{\operatorname{mes}(E \cap [0,x_n])}{\varphi(v(x_n))} \leqslant \frac{3w^*(v_{k-1})}{\varphi(v(x_n))v_{k-1}} + \frac{4}{\varphi(v(x_n))} \int\limits_{v_1}^{v_{k-1}} \frac{w^*(t)}{t^2}\, dt \leqslant$$

$$\leqslant \frac{3w^*(v_{k-1})}{\varphi(v(x_{k-1}))v_{k-1}} + \frac{4}{\varphi(v(x_n))} \int\limits_{v(x_1)}^{v(x_n)} \frac{w^*(t)}{t^2}\, dt.$$

The function w^* belongs to the class $W(\varphi)$; therefore, it satisfies the condition $w^*(x) = o(x\varphi(x))$, $x \to \infty$. Hence, as can be seen from the last estimate, everything we need follows.

Theorem 2 is proved. □

Without loss of generality, we can assume that $n_l(t) \leqslant w(t)$, $t > 0$. Otherwise, we can consider the function $w(t) + n_l(t)$, which obviously belongs to $\underline{W}(\varphi)$, since it is clear that $W(\varphi) \subset \underline{W}(\varphi)$. Indeed, for any $w \in W(\varphi)$

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int\limits_{\frac{x}{2}}^{x} \frac{w(t)}{t^2}\, dt = 0.$$

Hence,

$$\lim_{x\to\infty} \frac{w(\frac{x}{2})}{\frac{x}{2}\varphi(x)} = 0.$$

Since Φ is convex, then the function φ is concave and $\varphi(x) \leqslant 2\varphi(\frac{x}{2})$. Therefore,

$$\lim_{x\to\infty} \frac{w(x)}{x\varphi(x)} = 0,$$

and, so, $w \in \underline{W}(\varphi)$.

Lemma. *Let $F \in \underline{D}(\Phi)$, where $\Phi \in L$. Then there exists a sequence of numbers σ_j, $\sigma_j \uparrow \infty$, such that for $\sigma = \sigma_j$ and some $m \in \mathbb{N}$:*

$$\ln \mu(\sigma) \leqslant \Phi(m\sigma), \quad \ln \mu_b^*(\sigma) \leqslant \Phi(m\sigma),$$

where $\mu(\sigma)$, $\mu_b^*(\sigma)$ are the maximal members of the series (5) and (6), accordingly.

Proof of Lemma. Since the condition (7) is satisfied, it follows that there exists c $(0 < c < \infty)$, such that

$$\mu_b^*(\sigma) \leqslant \mu(\sigma + c) \quad (\sigma \geqslant 0). \tag{16}$$

By condition of the theorem, there exists a sequence $\{\sigma_j^{'}\}$, $c < \sigma_j' \uparrow \infty$, the number $m' \geqslant 1$, such that

$$\ln M(\sigma_j') \leqslant \Phi(m'\sigma_j'), \quad j \geqslant 1. \tag{17}$$

Therefore,

$$\ln \mu(\sigma_j' - c) \leqslant \ln \mu(\sigma_j') \leqslant \Phi(m'\sigma_j') \leqslant \Phi(m(\sigma_j' - c)).$$

Then, taking into account (16), (17), we get:

$$\frac{\ln \mu_b^*(\sigma_j' - c)}{\Phi(m(\sigma_j' - c))} \leqslant \frac{\ln \mu(\sigma_j')}{\Phi(m'\sigma_j')} \leqslant 1.$$

It can be seen that the required estimates hold for the sequence $\sigma_j = \sigma_j' - c > 0$.

The lemma is proved.

Let us prove the Theorem 1.

Proof of Theorem 1. 1^0. **Sufficiency.** Let (10) hold, with $w \in W(\varphi)$. Then there exists a function $w^* \in W(\varphi)$, such that $w^*(x) = \beta(x)w(x)$, where $(0 < \beta(x) \uparrow \infty$ as $x \to \infty)$.

Let $v = v(\sigma)$, $p = p(\sigma)$ be solutions to the equations

$$w_1(v) = 3 \ln \mu(\sigma), \quad w_1(p) = 3 \ln \mu_b^*(\sigma), \tag{18}$$

where $w_1(x) = \sqrt{\beta(x)} w(x)$. Set

$$R_v = \sum_{\lambda_j > v} |a_j| \, e^{\lambda_j \sigma}, \, h = \frac{w_1(v)}{v}, \, v = v(\sigma).$$

Next, $\ln n = \ln n(\lambda_n) \leqslant n_l(\lambda_n)$. Since a function $n_l(t)$ is concave, we have

$$n_l(\lambda_n) \leqslant \frac{w(v)}{v} \lambda_n$$

for $\lambda_n \geqslant v$. Consequently,

$$R_v \leqslant \mu(\sigma+h)\sum_{\lambda_n>v} e^{-\lambda_n h} \leqslant \mu(\sigma+h)c_0\exp(\max_{t\geqslant v}\psi(t)),$$

where $\psi(t) = 2n_l(t) - ht$, $c_0 = \sum\limits_{n=1}^{\infty}\frac{1}{n^2}$. Take into account the previous estimate for $n_l(t)$, when

$$\max_{t\geqslant v}(\psi(t)) \leqslant 2\frac{w(v)}{v}t - h(t) \leqslant -v(1+o(1))h.$$

Thus,

$$R_v \leqslant c_0\mu(\sigma+h)\exp[-v(1+o(1)h] = c_0\mu(\sigma+h)\exp[-(1+o(1))w_1(v)]. \quad (19)$$

Set $u(\sigma) = \ln 3 + \ln\ln\mu(\sigma)$, $u^*(\sigma) = \ln 3 + \ln\ln\mu_b^*(\sigma)$. Since $F \in \underline{D}(\Phi)$, then, according to the lemma, there exists the sequence $\{\tau_j\}$ $(0 < \tau_j \uparrow \infty)$, such that for some $m \in \mathbb{N}$

$$u(\sigma) \leqslant \ln\Phi(m\sigma), \quad u^*(\sigma) \leqslant \ln\Phi(m\sigma), \quad \sigma = \tau_j.$$

Therefore, taking into account (18) for $\sigma = \tau_j (j \geqslant 1)$, we have

$$\ln w_1(v(\sigma)) = u(\sigma) \leqslant \ln\Phi(m\sigma),$$

$$\ln w_1(v(\sigma)) = u^*(\sigma) \leqslant \ln\Phi(m\sigma) \quad (m \geqslant 1).$$

Hence, we learn that for $\sigma = \tau_j$ $(j \geqslant 1)$

$$\varphi(w_1(v(\sigma))) \leqslant m\sigma, \quad \varphi(w_1(v(p(\sigma)))) \leqslant m\sigma, (m \geqslant 1).$$

Thus,

$$\frac{1}{\sigma} \leqslant \frac{m}{\varphi(w_1(v(\sigma)))}, \quad \frac{1}{\sigma} \leqslant \frac{m}{\varphi(w_1(p(\sigma)))}, \sigma = \tau_j, m \geqslant 1. \quad (20)$$

Taking into account (8) and the fact that that $\sqrt{x} \leqslant w_1(x)$, we have

$$\varphi(x) \leqslant c_1\varphi(w_1(x)), x \geqslant x_0 \ (0 < c_1 < \infty). \quad (21)$$

Thus, from (20) and (21) we obtain the estimates

$$\frac{1}{\sigma} \leqslant \frac{c_2}{\varphi(v(\sigma))}, \quad \frac{1}{\sigma} \leqslant \frac{c_2}{\varphi(p(\sigma))}, \sigma = \tau_j \ (0 < c_2 < \infty). \quad (22)$$

Further, since $w^* \in W(\varphi)$ and $W(\varphi) \subset \underline{W}(\varphi)$, then

$$\lim_{x\to\infty} \frac{w^*(x)}{x\varphi(x)} = 0, \tag{23}$$

$$\lim_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{w^*(t)}{t^2}\, dt = 0. \tag{24}$$

It is obvious that the change of the condition $u(\sigma) \leqslant C\Phi(\sigma)$ $(\sigma = \tau_j)$ with $u(\sigma) \leqslant \Phi(m\sigma)$ $(\sigma = \tau_j)$ $\quad (j \geqslant 1,\, m \geqslant 1)$ keeps the statement of Theorem 2 the same, provided that the other assumptions remain unchanged. Therefore, applying Theorem 2 to the functions u, w_1 and taking into account (22), (23), (24), we have, outside some set $E_1 \subset [0, \infty)$:

$$\operatorname{mes}(E_1 \cap [0,\tau_j]) = o(\varphi(v(\tau_j))) = o(\tau_j),\ \tau_j \to \infty; \tag{25}$$

from (19) we have

$$R_v \leqslant c_0\mu(\sigma)^{1+o(1)} \exp[-w_1(v)(1+o(1))] = \mu(\sigma)^{-2(1+o(1))}.$$

Hence, for $\sigma \geqslant \sigma_1$, but for $\sigma \notin E_1$, we find that $\lambda_{\nu(\sigma)} \leqslant v(\sigma)$, where $\nu = \nu(\sigma)$ is the central index of the series (5). Then, by (10) for $\sigma \to \infty$ outside the set E_1, we have

$$\mu(\sigma) = |a_\nu| e^{\lambda_\nu \sigma} = |a_\nu b_\nu| e^{\lambda_\nu \sigma} |b_\nu|^{-1} \leqslant \mu_b^*(\sigma) e^{w(\lambda_\nu)} \leqslant \mu_b^*(\sigma) e^{w(v)} = \mu_b^*(\sigma)\mu(\sigma)^{o(1)}.$$

This means that when $\sigma \to \infty$:

$$(1 + o(1)) \ln \mu(\sigma) \leqslant \ln \mu_b^*(\sigma). \tag{26}$$

outside the set E_1.

Since $|b_n| \leqslant e^{w(\lambda_n)}$ $\quad (n \geqslant N)$, we have, for $k \geqslant N$:

$$\mu_b^* = |a_k b_k|\, e^{\lambda_k \sigma} \leqslant \mu(\sigma) e^{w(\lambda_k)}, \tag{27}$$

where $k = k(\sigma)$ is the central index of series (6).

Let

$$R_p^* = \sum_{\lambda_n > p} |a_n b_n| e^{\lambda_n \sigma}, \quad p = p(\sigma).$$

But $u^*(\sigma) \leqslant \Phi(m\sigma)$ for $\sigma = \tau_j$ ($\{\tau_j\}$ is a sequence introduced above), where $u^*(\sigma) = \ln 3 + \ln\ln \mu_b^*(\sigma)$. So, applying Theorem 2 and arguing in the same way as in the proof of estimate for R_v, we find:

$$R_p^* \leqslant c_0 \mu_b^*(\sigma)^{-2(1+o(1))}, \; c_0 = \sum_{n=1}^{\infty} \frac{1}{n^2}, \tag{28}$$

for $\sigma \to \infty$ outside some set $E_2 \subset [0, \infty)$,

$$\operatorname{mes}(E_2 \cap [0, \tau_j]) = o(\varphi(p(\tau_j))) = o(\tau_j), \;\; \tau_j \to \infty. \tag{29}$$

Thus, if $\sigma \geqslant \sigma_2$, $\sigma \notin E_2$ we find that $\lambda_{k(\sigma)} \leqslant p(\sigma)$. Consequently, by (27) we have, for $\sigma \to \infty$:

$$\mu_b^*(\sigma) \leqslant \mu(\sigma) e^{w(p(\sigma))} = \mu(\sigma) \mu_b^*(\sigma)^{o(1)},$$

outside the set E_2; so,

$$(1 + o(1)) \ln \mu_b^*(\sigma) \leqslant \ln \mu(\sigma). \tag{30}$$

Let $E = E_1 \cup E_2$. From (20), (25), (29) for $\tau_j \to \infty$ we get:

$$\frac{\operatorname{mes}(E \cap [0, \tau_j])}{\tau_j} \leqslant c_2 \left[\frac{\operatorname{mes}(E_1 \cap [0, \tau_j])}{\varphi(v(\tau_j))} + \frac{\operatorname{mes}(E_2 \cap [0, \tau_j])}{\varphi(p(\tau_j))} \right] = o(1).$$

Thus, from (26), (30), taking into account (25), (29), we finally get, for $\sigma \to \infty$:

$$\ln \mu(\sigma) = (1 + o(1)) \ln \mu_b^*(\sigma),$$

outside the set $E = E_1 \cup E_2$, such that $dE = 0$.

Sufficiency is proved.

2^0. **Necessity.** We will proof by contradiction. Assume that condition 10) is not valid. Consider two expressions:

$$A = \varlimsup_{x \to \infty} \frac{1}{\varphi(x)} \int_{\lambda_N}^{x} \frac{\alpha(t)}{t^2} \, dt,$$

$$B = \varlimsup_{x \to \infty} \frac{1}{\varphi(x)} \int_{\lambda_N}^{x} \frac{\alpha^*(t)}{t^2} \, dt,$$

where $\alpha(t) = \max\limits_{\lambda_N \leqslant t}\{\ln|b(\lambda_n)|\colon n \geqslant N\}$, $\alpha^*(t) = \max\limits_{\lambda_N \leqslant t}\{-\ln|b(\lambda_n)|\colon n \geqslant N\}$, $b(\lambda_n) = b_n$. It is clear that $\alpha(t) > 0$, $\alpha^*(t) > 0$ for $t \geqslant \lambda_N$, moreover $\alpha(t)$, $\alpha^*(t)$ are nondecreasing step-like functions, continuous from the right. Note that $\alpha(t)$ is the least nondecreasing majorant of the sequence $\{\ln|b_n|\}_{n=N}^{\infty}$, and $\alpha^*(t)$ is the least nondecreasing majorant of the sequence $\{-\ln|b_n|\}_{n=N}^{\infty}$.

We show that A and B cannot be equal to zero simultaneously. Indeed, if A and B both equal zero, then there is a majorant $w_\alpha \in W(\varphi)$, such that $\alpha(t) \leqslant w_\alpha(t)$, $\alpha^*(t) \leqslant w_\alpha(t)$. Consequently,

$$|b_n| + |b_n|^{-1} \leqslant e^{\theta(\lambda_n)} \ (n \geqslant N),\ \theta(\lambda_n) = w_\alpha(\lambda_n) + \ln 2,\ \theta \in W(\varphi).$$

We got a contradiction. So, A and B are not equal to zero at the same time. Let for certainty $A \neq 0$, that is:

$$\varlimsup_{x\to\infty} \frac{1}{\varphi(x)} \int\limits_{\lambda_N}^{x} \frac{\alpha(t)}{t^2}\, dt > 0. \tag{31}$$

Let us show that in this case

$$\varlimsup_{n\to\infty} \frac{1}{\varphi(t_n)} \int\limits_{\lambda_N}^{t_n} \frac{\alpha(t)}{t^2}\, dt > 0, \tag{32}$$

where $\{t_n\}$ is a sequence of all points of discontinuity of the function $\alpha(t)$, $t_0 = \lambda_N$. Let us assume that this is not the case, so

$$\lim_{n\to\infty} \frac{1}{\varphi(t_n)} \int\limits_{\lambda_N}^{t_n} \frac{\alpha(t)}{t^2}\, dt = 0, \tag{33}$$

Let $\alpha(t) = \alpha_n$ for $t_n \leqslant t < t_{n+1} \quad (n > 0)$. If $x \in \ [t_{n-1}, t_n)$, then

$$\varlimsup_{x\to\infty} \frac{1}{\varphi(x)} \int\limits_{\lambda_N}^{x} \frac{\alpha(t)}{t^2}\, dt \leqslant$$

$$\leqslant \varlimsup_{x\to\infty} \frac{1}{\varphi(x)} \int\limits_{\lambda_N}^{t_{n-1}} \frac{\alpha(t)}{t^2}\, dt + \varlimsup_{x\to\infty} \frac{1}{\varphi(x)} \int\limits_{t_{n-1}}^{t_n} \frac{\alpha(t)}{t^2}\, dt \leqslant$$

$$\leqslant \overline{\lim_{n\to\infty}} \frac{1}{\varphi(t_{n-1})} \int\limits_{\lambda_N}^{t_{n-1}} \frac{\alpha(t)}{t^2}\, dt + \overline{\lim_{n\to\infty}} \frac{\alpha(t_{n-1})}{\varphi(t_{n-1})t_{n-1}} = 0.$$

Assuming the condition(33), we get a contradiction with the condition (31). So, indeed the condition (32) takes place. Let

$$\overline{\lim_{n\to\infty}} \frac{1}{\varphi(t_n)} \int\limits_{\lambda_N}^{t_n} \frac{\alpha(t)}{t^2}\, dt = \beta > 0 \quad (n \geqslant 1).$$

It follows that there is a sequence $\{\tau_k\} = \{t_{n_k}\}$, such that

$$\frac{1}{\varphi(\tau_k)} \int\limits_{\lambda_N}^{\tau_k} \frac{\alpha(t)}{t^2}\, dt \geqslant \beta_1 > 0 \quad (k \in \mathbb{N}). \tag{34}$$

We introduce a sequence $\{x_n\}_{n=1}^{\infty}$, such that

$$x_n = \frac{G_{n+1} - G_n}{t_{n+1} - t_n}, \quad G_n = t_n I(t_n),$$

where

$$I(t_n) = \int\limits_{\lambda_N}^{t_n} \frac{g(t)}{t^2}\, dt, \quad g(t) = q\alpha(t), \quad 0 < q < 1 \quad (n \geqslant 1).$$

As in [3], we construct a convex Newton polygon L with vertices $P_n = (t_n; G_n)$ for the Dirichlet series

$$F(s) = \sum_{k=1}^{\infty} a_k e^{\lambda_k s} \quad (s = \sigma + it), \tag{35}$$

where

$$a_k = \begin{cases} e^{-G_n}, & \lambda_n = \tau_k, \\ 0, & \lambda_n \neq \tau_k. \end{cases}$$

Note that $\{\tau_k\}_{k=1}^{\infty}$ is a sequence of central exponents of the absolutely convergent series (35) in the whole plane. Taking this fact into account, we estimate from above the maximal term of the series (35).

For $x_{n-1} \leqslant \sigma < x_n$, we have [3]:

$$\ln \mu(\sigma) = \tau_n(-I(\tau_n) + \sigma) < \frac{\tau_n \tau_{n+1}}{\tau_{n+1} - \tau_n} \int\limits_{\tau n}^{\tau_{n+1}} \frac{g(t)}{t^2}\, dt = q\alpha_n \ \ (n \geqslant 1).$$

On the other hand,

$$\mu_b^*(\sigma) \geqslant |a_{j_n} b_{j_n}|\, e^{\lambda_{j_n}\sigma} \quad (\lambda_{j_n} = \tau_n,\, n \geqslant 1).$$

Since $b_{j_n} = e^{\alpha(t_n)} = e^{\alpha_n}$, for $x_{n-1} \leqslant \sigma < x_n$ $(n \geqslant 1)$ we have

$$\ln \mu_b^*(\sigma) \geqslant \alpha_n - \tau_n I(\tau_n) + \tau_n \sigma = \alpha_n + \ln \mu(\sigma) > \alpha_n \ \ (n \geqslant 1).$$

Thus, for $x_{n-1} \leqslant \sigma < x_n$ $(n \geqslant 1)$ we have

$$\frac{\ln \mu(\sigma)}{\ln \mu_b^*(\sigma)} < q < 1. \tag{36}$$

It remains to prove that $F \in \underline{D}(\Phi)$, where Φ is the inverse of φ. Indeed, from (34) it follows that

$$I(\tau_n) \geqslant \beta_1 \varphi(\tau_n) \quad (n \geqslant 1),$$

i. e.,

$$M(\sigma) = \sup_{|t|<\infty} |F(\sigma + it)| \leqslant \sum_{n=1}^{\infty} e^{-\beta_1 \tau_n \varphi(\tau_n) + \tau_n \sigma}.$$

Since $\tau_n = \lambda_{j_n}$, from (4) it follows that $\ln n = o(\lambda_n \varphi(\lambda_n))$ as $n \to \infty$, and, so, $2 \ln n \leqslant \frac{\beta_1}{2} \tau_n \varphi(\tau_n)$, $n \geqslant n_0$. Given this, we get

$$M(\sigma) \leqslant c_0 \exp\left\{ \max_{t \geqslant 0} \left[-\frac{\beta_1}{2} t\varphi(t) + (\sigma + m)t \right] \right\}, \quad 0 < c_0 < \infty.$$

Note that the maximum is attained at a point $t_* \leqslant \Phi(\frac{2(\sigma+m)}{\beta_1})$. Consequently,

$$M(\sigma) \leqslant c_0 e^{(\sigma+m)t_*} \leqslant c_0 e^{A\sigma\Phi(\frac{A\sigma}{\beta_1})}, \quad 0 < A < \infty.$$

However, Φ is a convex function from the class M. Hence, $\ln M(\sigma) \leqslant B\sigma\Phi(\frac{A\sigma}{\beta_1}) < \Phi(KL\sigma)$, where $L = \max(B, \frac{A}{\beta_1})$, $\sigma \geqslant \sigma_0$. Taking into account this, we finally get $\ln M(\sigma) \leqslant \Phi(m\sigma)$ ($\sigma > 0$ is any number, m is a natural number). This means that $F \in \underline{D}(\Phi)$.

So, if (10) fails, there is a function $F \in \underline{D}(\Phi)$ satisfying estimate (36). In the case when $B > 0$, the example is constructed similarly.

The necessity is established, and thus the theorem is fully proved.

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