Probl. Anal. Issues Anal. Vol. 12 (30), No 2, 2023, pp. 51–67

DOI: 10.15393/j3.art.2023.12630

UDC 517.98, 515.124

Ö. Kişi, M. Gürdal

ON $\mathcal{I}_2$ AND $\mathcal{I}_2^*$-CONVERGENCE IN ALMOST SURELY OF COMPLEX UNCERTAIN DOUBLE SEQUENCES

Abstract. In this study, we investigate the notions of $\mathcal{I}_2$-convergence almost surely (a.s.) and $\mathcal{I}_2^*$-convergence a.s. of complex uncertain double sequences in an uncertainty space, and obtain some of their features and identify the relationships between them. In addition, we put forward the concepts of $\mathcal{I}_2$ and $\mathcal{I}_2^*$-Cauchy sequence a.s. of complex uncertain double sequences and investigate their relationships.

Key words: *uncertainty theory, complex uncertain variable, $\mathcal{I}_2$-convergence, $\mathcal{I}_2^*$-convergence*

2020 Mathematical Subject Classification: *40A05, 40A35, 40G15, 60B10*

1. Introduction and Background. The concept of statistical convergence was introduced by Fast [7] and then it was further investigated from the sequence-space point of view by several authors (see, for example, [8], [15]). Statistical convergence has become one of the most active areas of research due to its wide applicability in various branches of mathematics, such as number theory, mathematical analysis, probability theory, etc. The study of statistical convergence of double sequence has been initiated by Moricz [16], Mursaleen and Edely [17], Tripathy [23], independently. The notion of the ideal convergence is a common generalization of the classical notions of convergence and statistical convergence. The notion of $\mathcal{I}$-convergent was further investigated from the sequence-space point of view and linked with the summability theory by Kostyrko et al. [12]. Later, this concept has been generalized in many directions. Das et al. [2] presented the notion of $\mathcal{I}$-convergence of double sequences in a metric space and worked out some features of this convergence. More details on statistical convergence, $\mathcal{I}$-convergence, and on applications of this concept can be found in Dündar and Altay [6], Gürdal and Huban [9], Gürdal and Şahiner [10], Nabiev et al. [18], and Savaş and Gürdal [21], [22].

Liu [13] was the first to introduce the uncertainty theory based on an uncertain measure that satisfies normality, duality, subadditivity, and product axioms. Nowadays uncertainty theory has become one of the most active areas of research due to its wide applicability in various domains such as uncertain programming, uncertain optimal control, uncertain risk analysis, uncertain differential equation, etc. For more details, one may refer to [14]. In order to identify complex uncertain sequences, the notion of uncertain variables was defined over the uncertain space. Complex uncertain sequences are measurable functions from an uncertain space to the set of all complex numbers $\mathbb{C}$. Over the last few years, the study of convergence of sequences in the complex uncertain space has drawn attention of the researchers. In 2016, Chen et. al. [1] investigated various types of convergence of sequences, such as convergence almost surely, convergence in measure, convergence in mean, and convergence in distribution, in the complex uncertain space. He mainly studied the interrelationship between the notions.

The notion of statistical convergence was first developed in terms of complex uncertain sequences by Tripathy and Nath [24]. Later on, a lot of work has been carried out in this direction till today (see, for example, [3], [4], [5], [11], [19], [20], [24]).

The aim of this study is to present the notion of $\mathcal{I}_2^*$-convergence almost surely in complex uncertain theory, examine several properties, and identify the relationships between $\mathcal{I}_2$ and $\mathcal{I}_2^*$-convergence almost surely of complex uncertain sequence. In addition, we put forward $\mathcal{I}_2$ and $\mathcal{I}_2^*$-Cauchy sequence almost surely of complex uncertain sequence.

2. Preliminaries In this section, we gather the necessary results and techniques on which we will rely to accomplish our main results.

Utilizing the notion of ideals, Kostyrko et al. [12] determined the notion of $\mathcal{I}$ and $\mathcal{I}^*$-convergence.

Assume $Y \neq \varnothing$; $\mathcal{I} \subset 2^Y$ is called an ideal on Y provided that (a) for all $S,T \in \mathcal{I}$ implies $S \cup T \in \mathcal{I}$; (b) for all $S \in \mathcal{I}$ and $T \subset S$ implies $T \in \mathcal{I}$.

Assume $Y \neq \varnothing$; $\mathcal{F} \subset 2^Y$ is named a filter on Y provided that (a) for all $S,T \in \mathcal{F}$ implies $S \cap T \in \mathcal{F}$; (b) for all $S \in \mathcal{F}$ and $T \supset S$ implies $T \in \mathcal{F}$.

An ideal $\mathcal{I}$ is called non-trivial provided that $Y \notin \mathcal{I}$ and $\mathcal{I} \neq \varnothing$. A non-trivial ideal $\mathcal{I} \subset P(Y)$ is known as an admissible ideal in Y iff $\mathcal{I} \supset \{\{w\} : w \in Y\}$. At that time, the filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{Y - S : S \in \mathcal{I}\}$

is called the filter connected with the ideal.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible (also admissible ideal) when $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

$$\mathcal{I}_2^0 = \{K \subseteq \mathbb{N} \times \mathbb{N} \colon (\exists m(K) \in \mathbb{N}) (\exists i, j \geqslant m(K) \Rightarrow (i,j) \notin K)\}$$

Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and, obviously, an ideal $\mathcal{I}_2$ is strongly admissible iff $\mathcal{I}_2^0 \subseteq \mathcal{I}_2$.

Definition 1. *[2] Assume (Y, ρ) is a metric space; A sequence $s = (s_{mn})$ in X is called $\mathcal{I}_2$-convergent to $s_0 \in Y$, if for any $\mu > 0$ we get*

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \rho(s_{mn}, s_0) \geqslant \mu\} \in \mathcal{I}_2.$$

In that case, we indicate

$$\mathcal{I}_2 - \lim_{m,n \to \infty} s_{mn} = s_0.$$

Das et al. [2] defined the condition $(AP2)$ and obtained some significant relations between $\mathcal{I}_2$ and $\mathcal{I}_2^*$-convergence for sequences in a metric space.

Definition 2. *[13] Let L be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it supplies the following axioms:*
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in L$;
Axiom 3 (Subadditivity): For all countable sequence of $\{\Lambda_j\} \in L$, we get

$$\mathcal{M}\Big\{\bigcup_{j=1}^{\infty} \Lambda_j\Big\} \leqslant \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, L, \mathcal{M})$ is named an uncertainty space and each element Λ in L is called an event. In order to obtain an uncertain measure of a compound event, a product uncertain measure is defined by Liu [13] as:

$$\mathcal{M}\Big\{\prod_{k=1}^{\infty} \Lambda_k\Big\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

Definition 3. *[1] A complex uncertain variable is a measurable function ζ from an uncertainty $(\Gamma, L, \mathcal{M})$ to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set*

$$\{\zeta \in B\} = \{\gamma \in \Gamma \colon \zeta(\gamma) \in B\}$$

is an event.

Definition 4. *[1] A complex uncertain sequence $\{\phi_m\}$ is called to be convergent a.s. to ϕ, provided that for all $\mu > 0$ there is an event Λ with $\mathcal{M}\{\Lambda\} = 1$, such that*

$$\lim_{m\to\infty} \|\phi_m(\gamma) - \phi(\gamma)\| = 0$$

for all $\gamma \in \Lambda$. In that case, we indicate $\phi_m \xrightarrow{A_s} \phi$.

Definition 5. *[24] A complex uncertain sequence $\{\phi_m\}$ is called statistically convergent a.s. to ϕ, provided that for all $\mu > 0$ there is an event Λ with $\mathcal{M}\{\Lambda\} = 1$, such that*

$$\lim_{m\to\infty} \frac{1}{m} \left|\{k \leqslant m : \|\phi_k(\gamma) - \phi(\gamma)\| \geqslant \mu\}\right| = 0,$$

for each $\gamma \in \Lambda$. In that case, we denote $\phi_m \xrightarrow{S^{A_s}} \phi$.

Definition 6. *[5] A complex uncertain double sequence $\{\phi_{mn}\}$ is called convergent a.s. to ϕ, provided that for all $\mu > 0$ there is an event Λ with $\mathcal{M}\{\Lambda\} = 1$, such that*

$$\lim_{m,n\to\infty} \|\phi_{mn}(\gamma) - \phi(\gamma)\| = 0$$

for all $\gamma \in \Lambda$. In that case, we indicate $\phi_{mn} \xrightarrow{A_s} \phi$.

Definition 7. *[4] A complex uncertain sequence $\{\phi_{mn}\}$ is called statistically convergent a.s. to ϕ, provided that for all $\mu > 0$ there is an event Λ with $\mathcal{M}\{\Lambda\} = 1$, such that*

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\{j \leqslant m, k \leqslant n\colon \ \|\phi_{jk}(\gamma) - \phi(\gamma)\| \geqslant \mu\}\right| = 0,$$

for each $\gamma \in \Lambda$. In that case, we denote $\phi_m \xrightarrow{S^{A_s}} \phi$.

3. Main Results.

Definition 8. *A complex uncertain double sequence $\{\phi_{mn}\}$ is said to be $\mathcal{I}_2$-convergent a.s. to ϕ, provided that for all $\mu > 0$ there is an event Λ with $\mathcal{M}\{\Lambda\} = 1$, such that*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N}\colon \ \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant \mu\} \in \mathcal{I}_2$$

for all $\gamma \in \Lambda$. Symbolically we denote $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$.

Theorem 1. *If $\phi_{mn} \xrightarrow{A_s} \phi$, then $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$.*

Proof. It follows directly from the fact that $\mathcal{I}_2 = \mathcal{I}_2^f$ is the ideal of all finite subsets of $\mathbb{N} \times \mathbb{N}$. □

The converse of Theorem 1 is not typically true, as shown in the case below.

Example 1. Contemplate the uncertainty space $(\Gamma, L, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \ldots\}$ with power set and $\mathcal{M}(\Gamma) = 1$ and $\mathcal{M}(\varnothing) = 0$ and

$$\mathcal{M}(\Lambda) = \begin{cases} \sup_{\gamma_{m+n} \in \Lambda} \frac{m+n}{2(m+n)+1}, & \text{if } \sup_{\gamma_{m+n} \in \Lambda} \frac{m+n}{2(m+n)+1} < \frac{1}{2}, \\ 1 - \sup_{\gamma_{m+n} \in \Lambda^c} \frac{m+n}{2(m+n)+1}, & \text{if } \sup_{\gamma_{m+n} \in \Lambda^c} \frac{m+n}{2(m+n)+1} < \frac{1}{2}, \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

for $m, n = 1, 2, \ldots$. Also, the complex uncertain variables determined by

$$\phi_{mn}(\gamma) = \begin{cases} i\beta_{mn}, & \text{if } \gamma \in \{\gamma_1, \gamma_4, \gamma_9, \ldots\}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\beta_{mn} = \begin{cases} mn, & \text{if } m = u^2, n = v^2, \ u, v \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

and $\phi \equiv 0$. Take $\mathcal{I}_2 = \mathcal{I}_2^f$. For any $\mu > 0$ and an event Λ with $\mathcal{M}(\Lambda) = 1$, we get

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant \mu\} =$$
$$= \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\phi_{mn}(\gamma)\| \geqslant \mu\} = \{(1,1), (4,4), (9,9), \ldots\} \in \mathcal{I}_2$$

for each $\gamma \in \Lambda$. Thus, the sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2$-convergent a.s. to $\phi \equiv 0$ ($\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} 0$), however it is not convergent a.s. to $\phi \equiv 0$.

Theorem 2. *If $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$, then ϕ is uniquely determined.*

Proof. If possible, assume $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi_1$ and $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi_2$ for some $\phi_1(\gamma) \neq \phi_2(\gamma)$, for all $\gamma \in \Lambda$. Let $\mu > 0$ be arbitrary. At that time, for any $\mu > 0$, we obtain:

$$U = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\phi_{mn}(\gamma) - \phi_1(\gamma)\| < \frac{\mu}{2}\right\} \in \mathcal{F}(\mathcal{I}_2) \text{ and}$$

$$V=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \left\|\phi_{mn}(\gamma)-\phi_2(\gamma)\right\|<\frac{\mu}{2}\right\}\in\mathcal{F}(\mathcal{I}_2).$$

As $U\cap V\in\mathcal{F}(\mathcal{I}_2)$ and $\varnothing\notin\mathcal{F}(\mathcal{I}_2)$, this means that $U\cap V\neq\varnothing$. Take $(p,r)\in U\cap V$. Then

$$\left\|\phi_{pr}(\gamma)-\phi_1(\gamma)\right\|<\frac{\mu}{2}\ \text{and}\ \left\|\phi_{pr}(\gamma)-\phi_2(\gamma)\right\|<\frac{\mu}{2}.$$

As a result, we get

$$\begin{aligned}\left\|\phi_1(\gamma)-\phi_2(\gamma)\right\| &=\left\|\phi_{pr}(\gamma)-\phi_2(\gamma)+\phi_1(\gamma)-\phi_{pr}(\gamma)\right\|\leqslant\\ &\leqslant\left\|\phi_{pr}(\gamma)-\phi_2(\gamma)\right\|+\left\|\phi_{pr}(\gamma)-\phi_1(\gamma)\right\|<\\ &<\tfrac{\mu}{2}+\tfrac{\mu}{2}<\mu.\end{aligned}$$

Hence, ϕ is uniquely determined. □

Theorem 3. *Let (ϕ_{mn}) and (ϕ^*_{mn}) be complex uncertain double sequences. If $\phi_{mn}\xrightarrow{A_s(\mathcal{I}_2)},\phi$ and $\phi^*_{mn}\xrightarrow{A_s(\mathcal{I}_2)}\phi^*$, then*

i) $\phi_{mn}\mp\phi^*_{mn}\xrightarrow{A_s(\mathcal{I}_2)}\phi\mp\phi^*$,

ii) $\alpha\phi_{mn}\xrightarrow{A_s(\mathcal{I}_2)}\alpha\phi$, where $\alpha\in\mathbb{C}$.

Proof. i) Assume $\mu>0$. Then

$$U=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \left\|\phi_{mn}(\gamma)-\phi(\gamma)\right\|<\frac{\mu}{2}\right\}\in\mathcal{F}(\mathcal{I}_2)\ \text{and}$$
$$V=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \left\|\phi^*_{mn}(\gamma)-\phi^*(\gamma)\right\|<\frac{\mu}{2}\right\}\in\mathcal{F}(\mathcal{I}_2).$$

As $U\cap V\in\mathcal{F}(\mathcal{I}_2)$ and $\varnothing\notin\mathcal{F}(\mathcal{I}_2)$, then $U\cap V\neq\varnothing$. Let $(p,r)\in U\cap V$. Then we obtain

$$\begin{aligned}\left\|(\phi_{mn}(\gamma)+\phi^*_{mn}(\gamma))-(\phi(\gamma)+\phi^*(\gamma))\right\|&\leqslant\left\|\phi_{mn}(\gamma)-\phi(\gamma)\right\|+\\ &+\left\|\phi^*_{mn}(\gamma)-\phi^*(\gamma)\right\|<\frac{\mu}{2}+\frac{\mu}{2}<\mu,\end{aligned}$$

namely,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\ \left\|(\phi_{mn}(\gamma)+\phi^*_{mn}(\gamma))-(\phi(\gamma)+\phi^*(\gamma))\right\|<\mu\right\}\in\mathcal{F}(\mathcal{I}_2).$$

Hence, $\phi_{mn}+\phi^*_{mn}\xrightarrow{A_s(\mathcal{I}_2)}\phi+\phi^*$.

ii) The proof is simple, thus it is omitted. □

Theorem 4. *Let* (ϕ_{mn}) *and* (ϕ^*_{mn}) *be complex uncertain double sequences. If* $\phi_{mn} \overset{A_s(\mathcal{I}_2)}{\longrightarrow} \phi$ *and* $\phi^*_{mn} \overset{A_s(\mathcal{I}_2)}{\longrightarrow} \phi^*$, *and there are positive numbers* u *and* v, *such that* $\|\phi_{mn}\| \leqslant u$ *and* $\|\phi^*\| \leqslant v$ *for any* m, n, *then*

i) $\phi_{mn}\phi^*_{mn} \overset{A_s(\mathcal{I}_2)}{\longrightarrow} \phi\phi^*$;

ii) $\dfrac{\phi_{mn}}{\phi^*_{mn}} \overset{A_s(\mathcal{I}_2)}{\longrightarrow} \dfrac{\phi}{\phi^*}$, *where* $(\phi^*_{mn}) \neq 0$ *and* $\phi^* \neq 0$.

Proof. i) Assume $\mu > 0$ and $u, v > 0$; then

$$U = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \; \|\phi_{mn} - \phi\| < \frac{\mu}{2v}\right\} \in \mathcal{F}(\mathcal{I}_2) \text{ and}$$
$$V = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \; \|\phi^*_{mn} - \phi^*\| < \frac{\mu}{2u}\right\} \in \mathcal{F}(\mathcal{I}_2).$$

Since $U \cap V \in \mathcal{F}(\mathcal{I}_2)$ and $\varnothing \notin \mathcal{F}(\mathcal{I}_2)$, this means $U \cap V \neq \varnothing$. So, for all $(p, r) \in U \cap V$ we obtain

$$\begin{aligned}\|\phi_{mn}\phi^*_{mn} - \phi\phi^*\| &= \|\phi_{mn}\phi^*_{mn} - \phi_{mn}\phi^* + \phi_{mn}\phi^* - \phi\phi^*\| \leqslant \\ &\leqslant \|\phi_{mn}\phi^*_{mn} - \phi_{mn}\phi^*\| + \|\phi_{mn}\phi^* - \phi\phi^*\| \leqslant \\ &\leqslant u\,\|\phi^*_{mn} - \phi^*\| + v\,\|\phi_{mn} - \phi\| < u\frac{\mu}{2u} + v\frac{\mu}{2v} = \mu,\end{aligned}$$

namely,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \; \|\phi_{mn}\phi^*_{mn} - \phi\phi^*\| < \mu\} \in \mathcal{F}(\mathcal{I}_2).$$

Hence, $\phi_{mn}\phi^*_{mn} \overset{A_s(\mathcal{I}_2)}{\longrightarrow} \phi\phi^*$.

ii). It is similar to the proof of i), so is omitted. □

Theorem 5. *If each subsequence of a complex uncertain double sequence* $\{\phi_{mn}\}$ *is* $\mathcal{I}_2$*-convergent a.s. to* ϕ, *then* $\{\phi_{mn}\}$ *is* $\mathcal{I}_2$*-convergent a.s. to* ϕ.

Proof. Assume that each subsequence of a complex uncertain double sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2$-convergent a.s. to ϕ, but $\{\phi_{mn}\}$ is not $\mathcal{I}_2$-convergent a.s. to ϕ. At that time, there exists some $\mu > 0$, such that

$$T = \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \; \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant \mu\} \notin \mathcal{I}_2.$$

So T has to be an infinite set. Assume

$$T = \{m_1 < m_2 < \ldots m_r < \ldots; \; n_1 < n_2 < \ldots < n_s < \ldots\}.$$

Now, determine a sequence (ϕ^*_{rs}) as $\phi^*_{rs} = \phi_{m_r n_s}$ for all $r, s \in \mathbb{N}$. Then (ϕ^*_{rs}) is a subsequence of (ϕ_{mn}), which is not $\mathcal{I}_2$-convergent a.s. to ϕ: a contradiction. □

The converse of Theorem 5 is not typically true, as shown in the case below.

Example 2. According to Example 1, we see that the complex uncertain double sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2$-convergent a.s. to $\phi \equiv 0$. Now, we construct a subsequence (ϕ^*_{rs}) of $\{\phi_{mn}\}$ by $(\phi^*_{rs}) = (\phi_{m_r n_s})$, where $m_r = r^2$, $n_s = s^2$, $r, s \in \mathbb{N}$, which is not $\mathcal{I}_2$-convergent a.s. to $\phi \equiv 0$.

Theorem 6. *Let $\{\phi_{mn}\}$, $\{\phi^*_{mn}\}$ be two complex uncertain double sequences, such that $\{\phi^*_{mn}\}$ converges a.s. to ϕ and*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \phi_{mn} \neq \phi^*_{mn}\} \in \mathcal{I}_2.$$

Then $\{\phi_{mn}\}$ is $\mathcal{I}_2$-convergent a.s. to ϕ.

Proof. Presume the complex uncertain sequence $\{\phi^*_{mn}\}$ -convergent a.s. to ϕ and $\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \phi_{mn} \neq \phi^*_{mn}\} \in \mathcal{I}_2$. Then, for all $\mu > 0$,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \left\| \phi^*_{mn}(\gamma) - \phi(\gamma) \right\| \geqslant \mu\}$$

is a finite set, and, so,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \left\| \phi^*_{mn}(\gamma) - \phi(\gamma) \right\| \geqslant \mu\} \in \mathcal{I}_2.$$

Now,

$$\begin{aligned} &\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \left\| \phi_{mn}(\gamma) - \phi(\gamma) \right\| \geqslant \mu\} \subseteq \\ &\subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \left\| \phi^*_{mn}(\gamma) - \phi(\gamma) \right\| \geqslant \mu\} \cup \\ &\cup \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \phi_{mn} \neq \phi^*_{mn}\} \in \mathcal{I}_2. \end{aligned}$$

As a result, we obtain $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$. □

Definition 9. *A complex uncertain double sequence $\{\phi_{mn}\}$ is named to be $\mathcal{I}_2^*$-convergent a.s. to ϕ if there exists a $L \in \mathcal{F}(\mathcal{I}_2)$ ($Z = (\mathbb{N} \times \mathbb{N}) \backslash L \in \mathcal{I}_2$) and an event Λ with $\mathcal{M}(\Lambda) = 1$, such that*

$$\lim_{\substack{m,n \to \infty \\ (m,n) \in L}} \left\| \phi_{mn}(\gamma) - \phi(\gamma) \right\| = 0$$

for each $\gamma \in \Lambda$. Symbolically we write $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2^*)} \phi$.

Example 3. Take into account the uncertainty space $(\Gamma, L, \mathcal{M})$ to be $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ with $\mathcal{M}(\Lambda) = \sum\limits_{\gamma_m, \gamma_n \in \Lambda} \frac{1}{2^{m+n}}$. Also, the complex uncertain variables defined by

$$\phi_{mn}(\gamma) = \begin{cases} i\beta_{mn}, & \text{if } \gamma = \gamma_{m+n}, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\beta_{mn} = \begin{cases} mn, & \text{if } m = u^2, n = v^2,\ u, v \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

and $\phi \equiv 0$. Take $\mathcal{I}_2 = \mathcal{I}_2^f$. Then there is a set $L = (\mathbb{N} \times \mathbb{N}) \setminus K \in \mathcal{F}(\mathcal{I}_2)$, where $K = \{(1{,}1), (4{,}4), (9{,}9), \ldots\} \in \mathcal{I}_2$, for which

$$\lim_{\substack{m,n \to \infty \\ (m,n) \in L}} \|\phi_{mn}(\gamma) - \phi(\gamma)\| = 0$$

for each $\gamma \in \Lambda$ with $\mathcal{M}(\Lambda) = 1$. Thus, the sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2^*$-convergent a.s. to $\phi \equiv 0$.

Theorem 7. *If $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2^*)} \phi$, then $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$.*

Proof. Let us assume that $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2^*)} \phi$. Then there is a set $L \in \mathcal{F}(\mathcal{I}_2)$ (i.e. $Z = (\mathbb{N} \times \mathbb{N}) \setminus L \in \mathcal{I}_2$) and an event Λ with $\mathcal{M}(\Lambda) = 1$, so that

$$\lim_{\substack{m,n \to \infty \\ (m,n) \in L}} \|\phi_{mn}(\gamma) - \phi(\gamma)\| = 0$$

for all $\gamma \in \Lambda$. This means that there exists $k_0 \in \mathbb{N}$, such that $\|\phi_{mn}(\gamma) - \phi(\gamma)\| < \mu$ for all $(m, n) \in L$ and $m, n \geqslant k_0$. Then we obtain

$$\begin{aligned} T(\mu, \gamma) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \colon \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant \mu\} \subset \\ \subset Z \cup (L \cap ((\{1, 2, \ldots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \ldots, (k_0 - 1)\}))). \end{aligned}$$

Now

$$Z \cup (L \cap ((\{1{,}2, \ldots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \ldots, (k_0 - 1)\}))) \in \mathcal{I}_2.$$

This indicates that $T(\mu, \gamma) \in \mathcal{I}_2$. Therefore, $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$. □

Remark. *But the converse of Theorem 7 is not true in general.*

Example 4. Assume $\mathbb{N} = \bigcup_{u,v=1,1}^{\infty,\infty} D_{uv}$, where

$$D_{uv} = \{(2^{u-1}k, 2^{v-1}t) \colon 2 \text{ does not divide } k \text{ and } t, \quad k, t \in \mathbb{N}, \}$$

be the decomposition of $\mathbb{N} \times \mathbb{N}$, such that all D_{uv} are infinite and $D_{uv} \cap D_{rs} = \varnothing$, for $(u, v) \neq (r, s)$. Presume $\mathcal{I}_2$ be the class of all subsets of $\mathbb{N} \times \mathbb{N}$ that can intersect only finite number of D_{uv}'s. Then $\mathcal{I}_2$ is a nontrivial admissible ideal of $\mathbb{N} \times \mathbb{N}$. Now we consider the uncertainty space $(\Gamma, L, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ with power set and $\mathcal{M}(\Gamma) = 1$ and $\mathcal{M}(\varnothing) = 0$ and

$$\mathcal{M}(\Lambda) = \begin{cases} \sup_{\gamma_{m+n} \in \Lambda} \frac{m+n}{2(m+n)+1}, & \text{if } \sup_{\gamma_{m+n} \in \Lambda} \frac{m+n}{2(m+n)+1} < \frac{1}{2}, \\ 1 - \sup_{\gamma_{m+n} \in \Lambda^c} \frac{m+n}{2(m+n)+1}, & \text{if } \sup_{\gamma_{m+n} \in \Lambda^c} \frac{m+n}{2(m+n)+1} < \frac{1}{2}, \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$$

for $m, n = 1, 2, \ldots$. Also, the complex uncertain variables determined by

$$\phi_{mn}(\gamma) = \begin{cases} i\beta_{mn}, & \text{if } \gamma \in \{\gamma_1, \gamma_2, \gamma_3, \ldots\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\beta_{mn} = \frac{1}{uv}$, if $(m, n) \in D_{uv}$ for $m, n = 1, 2, \ldots$ and $\phi \equiv 0$. It is obvious that the sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2$-convergent a.s. to $\phi \equiv 0$. Hovever, this sequence is not $\mathcal{I}_2^*$-convergent a.s. to $\phi \equiv 0$. Since for any $Z \in \mathcal{I}_2$ there exists $(w, z) \in \mathbb{N} \times \mathbb{N}$, such that $Z \subseteq \bigcup_{u,v=1,1}^{w,z} D_{uv}$, and a result $D_{(w+1)(z+1)} \subseteq (\mathbb{N} \times \mathbb{N}) \backslash Z$. Let $L = (\mathbb{N} \times \mathbb{N}) \backslash Z$, then $L \in F(\mathcal{I}_2)$ for which we can define a subsequence that is not convergent a.s. to $\phi \equiv 0$. As a result, the sequence $\{\phi_{mn}\}$ is not $\mathcal{I}_2^*$-convergent a.s. to $\phi \equiv 0$.

Theorem 8. *Let $\{\phi_{mn}\}$ be a complex uncertain double sequence in an uncertainty space $(\Gamma, L, \mathcal{M})$, such that $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$, then $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2^*)} \phi$, when $\mathcal{I}_2$ satisfies the condition (AP_2).*

Proof. Let us assume that $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$. Then there is an event Λ with $\mathcal{M}(\Lambda) = 1$ and for any $\mu > 0$ the set

$$T(\mu, \gamma) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \colon \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant \mu\} \in \mathcal{I}_2$$

for each $\gamma \in \Lambda$. Now, we establish a countable family of mutually disjoint sets $\{T_k(\gamma)\}_{k\in\mathbb{N}}$ in $\mathcal{I}_2$ by considering

$$T_1(\gamma) := \{(m,n) \in \mathbb{N}\times\mathbb{N} \colon \ \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant 1\}$$

and

$$\begin{aligned} T_k(\gamma) :=& \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} \colon \frac{1}{k} \leqslant \|\phi_{mn}(\gamma) - \phi(\gamma)\| < \frac{1}{k-1}\Big\} = \\ &= T\Big(\gamma, \frac{1}{k}\Big) \backslash T\Big(\gamma, \frac{1}{k-1}\Big). \end{aligned}$$

Since $\mathcal{I}_2$, the condition (AP_2) holds, so, for the above countable collection $\{T_k(\gamma)\}_{k\in\mathbb{N}}$, there exists another countable family of subsets $\{V_k(\gamma)\}_{k\in\mathbb{N}}$ supplying $T_i(\gamma)\Delta V_i(\gamma)$ is finite for all $i \in \mathbb{N}$ and $V(\gamma) = \bigcup\limits_{i=1}^{\infty} V_i(\gamma) \in \mathcal{I}_2$. We shall prove that for $L \in F(\mathcal{I}_2)$ we have $\lim\limits_{\substack{m,n\to\infty \\ (m,n)\in L}} \|\phi_{mn}(\gamma) - \phi(\gamma)\| = 0$. Let $\delta > 0$ be arbitrary. Utilizing the Archimedean property, we can select $k \in \mathbb{N}$, such that $\frac{1}{k+1} < \delta$. Then

$$\begin{aligned} &\{(m,n) \in \mathbb{N}\times\mathbb{N} \colon \ \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant \delta\} \subseteq \\ &\subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant \tfrac{1}{k+1}\right\} = \bigcup_{i=1}^{k+1} T_i(\gamma) \in \mathcal{I}_2. \end{aligned}$$

Since $T_i(\gamma)\Delta V_i(\gamma)$ is finite, $(i = 1, 2, \ldots, k+1)$ there is an $p_0 \in \mathbb{N}$, such that

$$\begin{aligned} &\bigcup_{i=1}^{k+1} V_i(\gamma) \cap \{(m,n) \colon m \geqslant p_0 \wedge n \geqslant p_0\} = \\ &\qquad\qquad = \bigcup_{i=1}^{k+1} T_i(\gamma) \cap \{(m,n) : m \geqslant p_0 \wedge n \geqslant p_0\}. \end{aligned}$$

Select $(m,n) \in (\mathbb{N}\times\mathbb{N}) \backslash V(\gamma) \in F(\mathcal{I}_2)$, such that $m \geqslant p_0 \wedge n \geqslant p_0$. So, we have to get $(m,n) \notin \bigcup\limits_{i=1}^{k+1} V_i(\gamma)$ and, so, $(m,n) \notin \bigcup\limits_{i=1}^{k+1} T_i(\gamma)$. Afterwards, there is an event with $\mathcal{M}(\Lambda) = 1$, such that $\|\phi_{mn}(\gamma) - \phi(\gamma)\| < \frac{1}{k+1} < \mu$ for all $\gamma \in \Lambda$. Hence $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2^*)} \phi$. $\square$

Definition 10. *A complex uncertain double sequence $\{\phi_{mn}\}$ is named to be $\mathcal{I}_2$-Cauchy sequence a.s., provided that for all $\mu > 0$ there exist m_0,$n_0 \in \mathbb{N}$ and an event Λ with $\mathcal{M}(\Lambda) = 1$, such that*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \ \|\phi_{mn}(\gamma) - \phi_{m_0n_0}(\gamma)\| \geqslant \mu\} \in \mathcal{I}_2,$$

for all $\gamma \in \Lambda$.

Theorem 9. *If a complex uncertain double sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2$-convergent a.s. to ϕ, then it is $\mathcal{I}_2$-Cauchy sequence a.s.*

Proof. Let $\phi_{mn} \xrightarrow{A_s(\mathcal{I}_2)} \phi$. Then, for all $\mu > 0$, there is an event Λ with $\mathcal{M}(\Lambda) = 1$, so that

$$P(\mu,\gamma) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \ \|\phi_{mn}(\gamma) - \phi(\gamma)\| \geqslant \mu\} \in \mathcal{I}_2$$

for each $\gamma \in \Lambda$. Obviously, $(\mathbb{N} \times \mathbb{N}) \backslash P(\mu,\gamma) \in F(\mathcal{I}_2)$ and, so, it is non-empty. Select $(m_0,n_0) \in (\mathbb{N} \times \mathbb{N}) \backslash P(\mu,\gamma)$. Then we obtain

$$\|\phi_{m_0n_0}(\gamma) - \phi(\gamma)\| < \mu,$$

for all $\gamma \in \Lambda$. Let

$$R(\mu,\gamma) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \ \|\phi_{mn}(\gamma) - \phi_{m_0n_0}(\gamma)\| \geqslant 2\mu\} \in \mathcal{I}_2$$

for all $\gamma \in \Lambda$. Now, we demonstrate that the following inclusion is true: $R(\mu,\gamma) \subseteq P(\mu,\gamma)$. For if $(r,s) \in R(\mu,\gamma)$, we get

$$\begin{aligned} 2\mu &\leqslant \|\phi_{rs}(\gamma) - \phi_{m_0n_0}(\gamma)\| \leqslant \|\phi_{rs}(\gamma) - \phi(\gamma)\| + \|\phi_{m_0n_0}(\gamma) - \phi(\gamma)\| < \\ &< \|\phi_{rs}(\gamma) - \phi(\gamma)\| + \mu, \end{aligned}$$

which yields $(r,s) \in P(\mu,\gamma)$. As a result, we conlude that $R(\mu,\gamma) \in \mathcal{I}_2$, namely, $\{\phi_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence a.s. □

Remark 1. *The converse of the Theorem 9 is an open problem and we leave it.*

Definition 11. *A complex uncertain double sequence $\{\phi_{mn}\}$ is named to be $\mathcal{I}_2^*$-Cauchy sequence a.s., provided that there is a set $L \in F(\mathcal{I}_2)$ (i.e. $Z = (\mathbb{N} \times \mathbb{N}) \setminus L \in \mathcal{I}_2$) and there is an event Λ with $\mathcal{M}(\Lambda) = 1$, so that for all $\mu > 0$ and for (m,n), $(m_0,n_0) \in L$, $m,n,m_0,n_0 > k_0 = k_0(\mu)$*

$$\|\phi_{mn}(\gamma) - \phi_{m_0n_0}(\gamma)\| < \mu,$$

for each $\gamma \in \Lambda$. Also, we write

$$\lim_{m,n,m_0,n_0 \to \infty} \|\phi_{mn}(\gamma) - \phi_{m_0 n_0}(\gamma)\| = 0,$$

where (m,n), $(m_0, n_0) \in L$.

Theorem 10. *If a complex uncertain sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2^*$-Cauchy sequence a.s., then it is $\mathcal{I}_2$-Cauchy sequence a.s.*

Proof. Let the complex uncertain sequence $\{\phi_{mn}\}$ be $\mathcal{I}_2^*$-Cauchy sequence a.s. Then there exists a set $L \in F(\mathcal{I}_2)$ and there exists an event Λ with $\mathcal{M}(\Lambda) = 1$, so that

$$\|\phi_{mn}(\gamma) - \phi_{m_0 n_0}(\gamma)\| < \mu,$$

for any $\mu > 0$ and for all $m, n, m_0, n_0 > k_0 = k_0(\mu)$.

Assume $Z = (\mathbb{N} \times \mathbb{N}) \setminus L \in \mathcal{I}_2$. Then, for any $\mu > 0$

$$T(\mu,\gamma) = \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \|\phi_{mn}(\gamma) - \phi_{m_0 n_0}(\gamma)\| \geqslant \mu\} \subset$$
$$\subset Z \cup (L \cap ((\{1, 2, \ldots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \ldots, (k_0 - 1)\}))) \in \mathcal{I}_2.$$

Hence, the sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2$ -Cauchy sequence a.s. □

Remark 2. *But the converse of Theorem 10 is not true in general.*

Example 5. Assume $\mathbb{N} = \bigcup_{u,v=1,1}^{\infty,\infty} D_{uv}$, where

$$D_{uv} = \left\{\left(2^{u-1}k, 2^{v-1}t\right) : 2 \text{ does not divide } k \text{ and } t,\ k, t \in \mathbb{N}\right\}$$

be the decomposition of $\mathbb{N} \times \mathbb{N}$, such that all D_{uv} are infinite and $D_{uv} \cap D_{rs} = \varnothing$, for $(u, v) \neq (r, s)$. Presume $\mathcal{I}_2$ be the class of all subsets of $\mathbb{N} \times \mathbb{N}$ that can intersect only finite number of D_{uv}'s. Then $\mathcal{I}_2$ is a nontrivial admissible ideal of $\mathbb{N} \times \mathbb{N}$. Now, consider the uncertainty space $(\Gamma, L, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ with the power set and $\mathcal{M}(\Gamma) = 1$ and $\mathcal{M}(\varnothing) = 0$ and

$$\mathcal{M}(\Lambda) = \sum_{\gamma_m, \gamma_n \in \Lambda} \frac{1}{2^{m+n}} \text{ for } m, n = 1, 2, 3, \ldots.$$

Also, the complex uncertain variables are identified by

$$\phi_{mn}(\gamma) = i\beta_{mn}, \text{ if } \gamma \in \{\gamma_1, \gamma_2, \gamma_3, \ldots\},$$

where $\beta_{mn} = \frac{1}{uv+1}$, if $(m,n) \in D_{uv}$ for $m,n = 1,2,\ldots$ and $\phi \equiv 0$. It is obvious that the sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2$-convergent a.s. to $\phi \equiv 0$. By Theorem 9, the sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence a.s.

Next, we have to demonstrate that the complex uncertain double sequence $\{\phi_{mn}\}$ is not $\mathcal{I}_2^*$-Cauchy sequence a.s. For this, suppose, if possible, that the sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2^*$-Cauchy sequence a.s. Then $\exists$ a set $L \in \mathcal{F}(\mathcal{I}_2)$ and for every $\mu > 0$, for all (m,n), $(m_0,n_0) \in L$, $\exists\, k_0 = k_0(\mu) \in \mathbb{N}$ and there exists an event Λ with $M\{\Lambda\} = 1$, such that

$$\|\phi_{mn}(\gamma) - \phi_{m_0n_0}(\gamma)\| < \mu,\ \forall m,n,m_0,n_0 > k_0 = k_0(\mu) \tag{1}$$

for each $\gamma \in \Lambda$. Since $(\mathbb{N}\times\mathbb{N})\backslash L \in \mathcal{I}_2$, there exists a $(w,z) \in \mathbb{N}\times\mathbb{N}$ such that $(\mathbb{N}\times\mathbb{N})\backslash L \subset D_{11} \cup D_{22} \cup \ldots \cup D_{wz}$. But $D_{ij} \subset L$ $\forall i > w$, $j > z$. In particular, $D_{(w+1)(z+1)}$,$D_{(w+2)(z+2)} \subseteq L$. We see that from the construction of D_{uv}'s, for given any $k_0(\mu) \in \mathbb{N}$ there are $(m,n) \in D_{(w+1)(z+1)}$, $(m_0,n_0) \in D_{(w+2)(z+2)}$ such that $m,n,m_0,n_0 > k_0 = k_0(\mu)$. Therefore

$$\left\|\phi_{mn}(\gamma) - \phi_{m_0n_0}(\gamma)\right\| = \left\|\frac{ij}{(w+1)(z+1)} - \frac{ij}{(w+2)(z+2)}\right\| = \\ = \frac{1}{(w+1)(z+1)} - \frac{1}{(w+2)(z+2)}.$$

If we take $\varepsilon = 1/(3(w+1)(z+1))$, then there is $k_0 = k_0(\mu) \in \mathbb{N}$, whenever (m,n), $(m_0,n_0) \in L$ with $m,n,m_0,n_0 > k_0 = k_0(\mu)$, such that the Equation 1 holds. This is a contradiction, so our assumption was wrong and, so, $\{\phi_{mn}\}$ is not $\mathcal{I}_2^*$-Cauchy sequence a.s.

Theorem 11. *Let $\{\phi_{mn}\}$ be a complex uncertain double sequence in an uncertainty space $(\Gamma, L, \mathcal{M})$, such that $\{\phi_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence a.s. Then $\{\phi_{mn}\}$ is $\mathcal{I}_2^*$-Cauchy sequence a.s. if $\mathcal{I}_2$ supplies the condition (AP2).*

Proof. Let $\{\phi_{mn}\}$ be an $\mathcal{I}_2$-Cauchy sequence a.s. Then, for all $\mu > 0$ there exists m_0,$n_0 \in \mathbb{N}$ and an event Λ with $\mathcal{M}(\Lambda) = 1$, so that

$$T(\mu,\gamma) = \{(m,n) \in \mathbb{N}\times\mathbb{N} \colon\ \|\phi_{mn}(\gamma) - \phi_{m_0n_0}(\gamma)\| \geqslant \mu\} \in \mathcal{I}_2,$$

for all $\gamma \in \Lambda$.

In particular, for $\mu = \frac{1}{s}$, $s \in \mathbb{N}$ we get

$$U_s(\mu) = \left\{ (m,n) \in \mathbb{N}\times\mathbb{N} \colon\ \|\phi_{mn}(\gamma) - \phi_{m_0n_0}(\gamma)\| < \frac{1}{s} \right\}$$

for all $\gamma \in \Lambda$ with $\mathcal{M}(\Lambda) = 1$.

Since $\mathcal{I}_2$ supplies the condition $(AP2)$, then there is a set $L \in F(\mathcal{I}_2)$ and $L \backslash U_s$ is finite for all $s \in \mathbb{N}$. According to the Archimedean property, we select $i_0 \in \mathbb{N}$ such that $\frac{2}{i_0} < \mu$. Then $L \backslash U_{i_0}$ is a finite set, so there exist $k_0, l_0 \in \mathbb{N}$, such that $m, n, m_0, n_0 \in U_{i_0}$ for all $m, n, m_0, n_0 > k_0, l_0$, i.e., $\|\phi_{mn}(\gamma) - \phi_{k_0 l_0}(\gamma)\| < \frac{1}{i_0}$ and $\|\phi_{m_0 n_0}(\gamma) - \phi_{k_0 l_0}(\gamma)\| < \frac{1}{i_0}$ for all $m, n, m_0, n_0 > k_0, l_0$ and for all $\gamma \in \Lambda$ with $\mathcal{M}(\Lambda) = 1$.

Now,

$$\begin{aligned}\|\phi_{mn}(\gamma) - \phi_{m_0 n_0}(\gamma)\| &= \|\phi_{mn}(\gamma) - \phi_{k_0 l_0}(\gamma) - \phi_{m_0 n_0}(\gamma) + \phi_{k_0 l_0}(\gamma)\| \leqslant \\ &\leqslant \|\phi_{mn}(\gamma) - \phi_{k_0 l_0}(\gamma)\| + \|\phi_{m_0 n_0}(\gamma) - \phi_{k_0 l_0}(\gamma)\| < \\ &< \frac{1}{i_0} + \frac{1}{i_0} = \frac{2}{i_0} < \mu, \ \forall m, n, m_0, n_0 > k_0, l_0\end{aligned}$$

and for all $\gamma \in \Lambda$ with $\mathcal{M}(\Lambda) = 1$.
Hence, the complex uncertain sequence $\{\phi_{mn}\}$ is $\mathcal{I}_2^*$-Cauchy sequence a.s. □

4. Conclusion The main aim of this paper is to present the notion of $\mathcal{I}_2^*$-convergent almost surely of complex uncertain double sequence, study some of their properties, and identify the relationships between $\mathcal{I}_2$ and $\mathcal{I}_2^*$-convergent almost surely of complex uncertain double sequences. Also, we investigate $\mathcal{I}_2$ and $\mathcal{I}_2^*$-Cauchy sequence almost surely and study the relationship between them. These ideas and results are expected to be a source for researchers in the area of convergence of complex uncertain sequences. Also, these concepts can be generalized and applied for further studies.

5. Acknowledgment The authors thank the anonymous referees for their constructive suggestions to improve the quality of the paper.

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Received November 12, 2022.
In revised form, February 23, 2023.
Accepted March 23, 2023.
Published online April 17, 2023.

Ömer Kişi
Bartın University
Department of Mathematics, 74100, Bartın, Turkey
E-mail: okisi@bartin.edu.tr

Mehmet Gürdal
Süleyman Demirel University
Department of Mathematics, 32260, Isparta, Turkey
E-mail: gurdalmehmet@sdu.edu.tr