Probl. Anal. Issues Anal. Vol. 12 (30), No 2, 2023, pp. 37–50
DOI: 10.15393/j3.art.2023.12730

UDC 517.986.9, 512.5

V. Keerthika, G. Muhiuddin, M. E. Elnair, B. Elavarasan

HYBRID NORM PRODUCT AND RELATION STRUCTURES IN HEMIRINGS

Abstract. In fuzzy logic, the triangular norm (t - norm) is an operator that represents conjunctions. The concept of t - norm turned out to be a basic tool for probabilistic metric spaces, but also in several areas of mathematics, including fuzzy set theory, fuzzy decision making, probability and statistics, etc. In the study of hybrid structures, we noticed that hybrid ideals play an important role. By using $\mathfrak{T}_\Upsilon$ - hybrid ideals in hemirings, the concepts of hybrid relations and the strongest $\mathfrak{T}_\Upsilon$ - hybrid relations are investigated in this paper. The notion of hybrid $\mathfrak{T}_\Upsilon$ - product and their relevant results are also discussed, and we prove that the direct $\mathfrak{T}_\Upsilon$ - product of two $\mathfrak{T}_\Upsilon$ - hybrid left h-ideals in hemiring is also a $\mathfrak{T}_\Upsilon$ - hybrid left h - ideal.

Key words: *hemiring, hybrid structure, t – norm, $\mathfrak{T}_\Upsilon$ – hybrid ideals, $\mathfrak{T}_\Upsilon$ – hybrid relations*

2020 Mathematical Subject Classification: *16Y60, 08A99*

1. Introduction. Vandiver [15] proposed the semiring concept in 1934. Semirings provide a modelling framework that allows for further investigation into the central ideas in these problems using an algebraic framework, which has been shown to be beneficial in a variety of information sciences and applied mathematics areas. Semirings can be identified in a variety of mathematical fields, including functional analysis, topology, graph theory, probability theory, and commutative and non-commutative ring theory. Ideal theory is well known to play a significant role in the development of hemirings. This has resulted in the development of some more constrained ideal concepts, such as k-ideals [5] and h-ideals [7], in the study of semiring theory. However, if the semiring is a hemiring, the ideals of a semiring correlate with the usual ideals of a ring in a sequentially commutative semiring. D. R. La Torre [6] investigated the characteristics of

k-ideals and h-ideals of hemirings in depth, using k-ideals and h-ideals, respectively. For hemirings, he thought up several corresponding ring theorems.

The proposal of fuzzy sets was originally invented by Zadeh [16]. Numerous scholars have developed fuzzy set theory in a variety of directions, and it has stimulated the interest of mathematicians working in a variety of fields. Fuzzy set theory has a broad range of applications, from robot design, water resource planning, and computer simulation to engineering.

In 1999, Molodtsov [10] introduced soft sets and defined the new theory's fundamental results. It is a simple mathematical way of dealing with objects with a variety of characteristics. Soft-set theory is being increasingly used in other fields, also to solve real-world problems. Molodtsov was a pioneer in applying soft-set theory to a wide range of fields, including Riemann integration, measurement theory, smoothness of functions, game theory, operations research, Perron integration, probability theory, smoothness of functions, and so on. Maji et al. [8] were the first to use soft sets in a decision-making problem. It is based on the knowledge reduction notion of rough set theory. He defined and researched several fundamental concepts of soft-set theory in 2003.

Jun et al. [3] conceived the idea of hybrid structure through the initial universe set by having the ability to combine fuzzy sets and soft sets. Using this concept, they initiated the hybrid field, a hybrid linear space, and a hybrid subalgebra. This approach was developed for all of us. Numerous algebraic systems have been exposed to hybrid structure, with a variety of outcomes (see [2], [9], [11], [12], [13]).

Triangular norms were introduced by Schweizer and Sklar [14] and are significant in fuzzy set theory. They can be used to model the intersection of fuzzy sets, as well as the conjunction of certain propositions in fuzzy logic.

The triangular norm ($\mathfrak{T}$-norm) was developed by enlarging the triangular inequality using the $\mathfrak{T}$-norm theory in the study of probabilistic metric spaces [14]. Belatedly, Alsina et al. [1] proposed the $\mathfrak{T}$-norm to fuzzy set theory after the fact and suggested that they be used for the union and intersection of fuzzy sets. In 2005, Zhan [17] explored the notion of $\mathfrak{T}$-fuzzy left h-ideals in hemirings.

In this paper, using hybrid $\mathfrak{T}_\Upsilon$-norm, we explore the idea of $\mathfrak{T}_\Upsilon$-hybrid relations and strongest $\mathfrak{T}_\Upsilon$-hybrid relations in hemirings; also, we show that the direct $\mathfrak{T}_\Upsilon$-product of two $\mathfrak{T}_\Upsilon$-hybrid left h-ideals is also a $\mathfrak{T}_\Upsilon$-

hybrid left h-ideal.

2. Preliminaries. In this section, we compile a few definitions and observations to help us with our main results.

Definition 1. *A set $\mathbb{D}(\neq \varnothing)$ is a semiring with two binary operations, addition and multiplication, that satisfy the axioms listed below:*

(i) $(\mathbb{D}, +)$ *is a commutative semigroup,*

(ii) $(\mathbb{D}, \cdot)$ *is a semigroup,*

(iii) $(r_0 + c_0) \cdot w_0 = r_0 \cdot w_0 + c_0 \cdot w_0$ *and* $r_0 \cdot (c_0 + w_0) = r_0 \cdot c_0 + r_0 \cdot w_0$ $\forall\ r_0, c_0, w_0 \in \mathbb{D}$.

An element $0 \in \mathbb{D}$ *is referred to as a zero element if* $0 \cdot w = w \cdot 0 = 0$ $\forall\ w \in \mathbb{D}$.

A semiring with zero element is defined as hemiring [4].

Throughout this paper, $\mathbb{D}$ denotes a hemiring, and for a set Q, its power set is $\mathbb{P}(Q)$.

Definition 2. *Let $K \in \mathbb{P}(\mathbb{D})$. K is termed as a left (resp., right) ideal of $\mathbb{D}$ if $(K, +)$ is closed and $\mathbb{D}K \subseteq K$ (resp., $K\mathbb{D} \subseteq K$). K of $\mathbb{D}$ is referred to as an ideal if it is both a left and a right ideal of $\mathbb{D}$.*

Definition 3. [6] A left (resp., right) ideal K of $\mathbb{D}$ is referred to be a left (resp., right) h-ideal of $\mathbb{D}$, such that

$$(\forall\ m, z \in \mathbb{D})(\forall\ c, l \in K)(m + c + z = l + z \rightarrow m \in K).$$

Definition 4. [3] For an universal set $\mathbb{B}$, a hybrid structure in $\mathbb{D}$ over $\mathbb{B}$ is $\tilde{w}_\varkappa := (\tilde{w}, \varkappa)\colon \mathbb{D} \rightarrow \mathbb{P}(\mathbb{B}) \times [0,1]$, $d \mapsto (\tilde{w}(d), \varkappa(d))$, where $\tilde{w}\colon \mathbb{D} \rightarrow \mathbb{P}(\mathbb{B})$ and $\varkappa\colon \mathbb{D} \rightarrow [0,1]$ are mappings.

A relation $\ll$ defined on the gathering of all hybrid structures, denoted by $H(\mathbb{D})$, in $\mathbb{D}$ over $\mathbb{B}$ as follows:

$$(\forall\ \tilde{w}_\varkappa, \tilde{n}_\varsigma \in H(\mathbb{D}))\,(\tilde{w}_\varkappa \ll \tilde{n}_\varsigma \Leftrightarrow \tilde{w} \,\tilde{\subseteq}\, \tilde{n}, \varkappa \succeq \varsigma)$$

where $\tilde{w} \,\tilde{\subseteq}\, \tilde{n}$ means: $\tilde{w}(q) \subseteq \tilde{n}(q)$ and $\varkappa \succeq \varsigma$ means: $\varkappa(q) \geqslant \varsigma(q)$ $\forall\ q \in \mathbb{D}$. Then the set $(H(\mathbb{D}), \ll)$ is partially ordered.

Definition 5. *$\tilde{u}_\varkappa \in H(\mathbb{D})$ is described as a hybrid left (resp., right) ideal in $\mathbb{D}$ if $\tilde{u}_\varkappa$ fulfils the criteria:*

$$\text{(i)}\ \ (\forall\ c_0, g_0 \in \mathbb{D}) \begin{pmatrix} \tilde{u}(c_0 + g_0) \supseteq \tilde{u}(c_0) \cap \tilde{u}(g_0) \\ \varkappa(c_0 + g_0) \leqslant \varkappa(c_0) \vee \varkappa(g_0) \end{pmatrix},$$

$$\text{(ii)}\ \ (\forall\ c_0, g_0 \in \mathbb{D}) \begin{pmatrix} \tilde{u}(c_0 g_0) \supseteq \tilde{u}(g_0)(\text{resp., } \tilde{u}(c_0 g_0) \supseteq \tilde{u}(c_0)) \\ \varkappa(c_0 g_0) \leqslant \varkappa(g_0)(\text{resp., } \varkappa(c_0 g_0) \leqslant \varkappa(c_0)) \end{pmatrix}.$$

It is obvious that for a hybrid left ideal $\tilde{u}_{\varkappa}$ in $\mathbb{D}, \tilde{u}(0) \supseteq \tilde{u}(g_0)$ and $\varkappa(0) \leqslant \varkappa(g_0)\ \forall\ g_0 \in \mathbb{D}$.

Definition 6. *A hybrid left (resp., right) ideal $\tilde{j}_{\varkappa}$ is described as a hybrid left (resp., right) h-ideal in $\mathbb{D}$ if $\tilde{j}_{\varkappa}$ fulfils the criteria: For $y_0, k_0, v_0, s_0 \in \mathbb{D}$,*

$$(v_0 + y_0 + s_0 = k_0 + s_0) \implies \begin{pmatrix} \tilde{j}(v_0) \supseteq \tilde{j}(y_0) \cap \tilde{j}(k_0) \\ \varkappa(v_0) \leqslant \varkappa(y_0) \vee \varkappa(k_0) \end{pmatrix}.$$

Definition 7. *A hybrid t-norm $\mathfrak{T}_{\Upsilon}$ is a structure $\mathfrak{T}_{\Upsilon} := (\mathfrak{T}, \Upsilon)$, where $\mathfrak{T} \colon \mathbb{P}(\mathbb{B}) \times \mathbb{P}(\mathbb{B}) \to \mathbb{P}(\mathbb{B})$ and $\Upsilon \colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ are mappings, fulfils the criteria:*

(i) $\begin{pmatrix} \forall\ H \in \mathbb{P}(\mathbb{B}) \\ \forall\ h \in [0,1] \end{pmatrix} \begin{pmatrix} \mathfrak{T}(H, \mathbb{B}) = H \\ \Upsilon(h, 0) = h \end{pmatrix}$,

(ii) $\begin{pmatrix} \forall\ H, R, L \in \mathbb{P}(\mathbb{B}) \\ \forall\ h, r, l \in [0,1] \end{pmatrix} \begin{pmatrix} \mathfrak{T}(H, R) \subseteq \mathfrak{T}(H, L) & \text{if } R \subseteq L \\ \Upsilon(h, r) \geqslant \Upsilon(h, l) & \text{if } r \geqslant l \end{pmatrix}$,

(iii) $\begin{pmatrix} \forall\ H, R \in \mathbb{P}(\mathbb{B}) \\ \forall\ h, r \in [0,1] \end{pmatrix} \begin{pmatrix} \mathfrak{T}(H, R) = \mathfrak{T}(R, H) \\ \Upsilon(h, r) = \Upsilon(r, h) \end{pmatrix}$,

(iv) $\begin{pmatrix} \forall\ H, R, L \in \mathbb{P}(\mathbb{B}) \\ \forall\ h, r, l \in [0,1] \end{pmatrix} \begin{pmatrix} \mathfrak{T}(H, \mathfrak{T}(R, L)) = \mathfrak{T}(\mathfrak{T}(H, R), L) \\ \Upsilon(h, \Upsilon(r, l)) = \Upsilon(\Upsilon(h, r), l) \end{pmatrix}$.

For a hybrid t-norm $\mathfrak{T}_{\Upsilon}$ on $\mathbb{P}(\mathbb{B})$, it is denoted by $\Theta_{\mathfrak{T}} = \{\Lambda \in \mathbb{P}(\mathbb{B}) \colon \mathfrak{T}(\Lambda, \Lambda) = \Lambda\}$ and Υ on $[0,1]$, it is denoted by $\Theta^{\Upsilon} = \{\lambda \in \mathbb{I} \colon \Upsilon(\lambda, \lambda) = \lambda\}$.

Lemma 1. *Each hybrid t-norm $\mathfrak{T}_{\Upsilon}$ has the following attributes: $\mathfrak{T}(\Xi, \Lambda) \subseteq \Xi \cap \Lambda$ and $\Upsilon(\xi, \lambda) \geqslant \xi \vee \lambda$, $\forall\ \Xi, \Lambda \in \mathbb{P}(\mathbb{B})$ and $\forall\ \xi, \lambda \in [0,1]$.*

Proof. Straight forward. □

Definition 8. *Let $\mathfrak{T}_{\Upsilon}$ be a hybrid t-norm. Then the hybrid structure $\tilde{e}_{\varrho}$ in $\mathbb{D}$ satisfies the imaginable property if $Im(\tilde{e}) \subseteq \Theta_{\mathfrak{T}}$ and $Im(\varrho) \subseteq \Theta^{\Upsilon}$.*

Definition 9. *Let $\mathfrak{T}_{\Upsilon}$ be a hybrid t-norm. Then the hybrid structure $\tilde{e}_{\varrho}$ in $\mathbb{D}$ is said to be imaginable if it satisfies the imaginable property.*

Definition 10. *Let $\mathfrak{T}_{\Upsilon}$ be a hybrid t-norm. A hybrid structure $\tilde{l}_{\varrho}$ is described as a $\mathfrak{T}_{\Upsilon}$-hybrid left (resp., right) ideal of $\mathbb{D}$ if it satisfy the following:*

(i) $(\forall\ s, a \in \mathbb{D}) \begin{pmatrix} \tilde{l}(s + a) \supseteq \mathfrak{T}(\tilde{l}(s), \tilde{l}(a)) \\ \varrho(s + a) \leqslant \Upsilon(\varrho(s), \varrho(a)) \end{pmatrix}$,

(ii) $(\forall\ s, a \in \mathbb{D}) \begin{pmatrix} \tilde{l}(sa) \supseteq \tilde{l}(a) (\textit{resp., } \tilde{l}(sa) \supseteq \tilde{l}(s)) \\ \\ \varrho(sa) \leqslant \varrho(a) (\textit{resp., } \varrho(sa) \leqslant \varrho(s)) \end{pmatrix}$.

Definition 11. *Let $\mathfrak{T}_\Upsilon$ be a hybrid t-norm. A $\mathfrak{T}_\Upsilon$-hybrid left (resp., right) ideal $\tilde{l}_\varpi$ of $\mathbb{D}$ is referred to be a $\mathfrak{T}_\Upsilon$-hybrid left (resp., right) h-ideal of $\mathbb{D}$ if it satisfies the following: For $w, g, z, s \in \mathbb{D}$,*

$$(z + w + s = g + s) \Rightarrow \left(\begin{array}{c} \tilde{l}(z) \supseteq \mathfrak{T}(\tilde{l}(w), \tilde{l}(g)) \\ \varpi(z) \leqslant \Upsilon(\varpi(w), \varpi(g)) \end{array} \right).$$

Definition 12. *For $\tilde{w}_\varsigma \in H(\mathbb{D})$ and $(\Gamma, \gamma) \in \mathbb{P}(\mathbb{B}) \times [0,1]$, we define*

$$\mathbb{D}_{\tilde{w}}^\Gamma := \{r \in \mathbb{D} \colon \tilde{w}(r) \supseteq \Gamma\} \text{ and } \mathbb{D}_\varsigma^\gamma := \{r \in \mathbb{D} \colon \varsigma(r) \leqslant \gamma\}.$$

Definition 13. *For $\tilde{w}_\varsigma \in H(\mathbb{D}), Q \in \mathbb{P}(\mathbb{B})$ and $\gamma \in [0,1]$, the set*

$$\tilde{w}_\varsigma[Q, \gamma] := \{a_1 \in \mathbb{D} \mid \tilde{w}(a_1) \supseteq Q \quad \text{and} \quad \varsigma(a_1) \leqslant \gamma\}$$

is described as $[Q, \gamma]$-hybrid cut of $\tilde{w}_\varsigma$. Note that $\mathbb{D}_{\tilde{w}}^\Gamma \cap \mathbb{D}_\varsigma^\gamma = \tilde{w}_\varsigma[\Gamma, \gamma]$.

Let $\mathbb{D}$ be a hemiring. Then we use the following notations:

(i) $\mathfrak{T}_\Upsilon$ is the hybrid t-norm.

(ii) $H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$ is the collection of all $\mathfrak{T}_\Upsilon$-hybrid left h-ideals of $\mathbb{D}$.

(iii) $H\mathfrak{T}_{\Upsilon_L}(\mathbb{D} \times \mathbb{D})$ is the collection of all $\mathfrak{T}_\Upsilon$-hybrid left h-ideals of $\mathbb{D} \times \mathbb{D}$.

3. $\mathfrak{T}$-product of $H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$. In this section, we define some definitions and their results on $\mathfrak{T}_\Upsilon$-hybrid relations, direct $\mathfrak{T}_\Upsilon$-product and the strongest $\mathfrak{T}_\Upsilon$-hybrid relations in hemirings.

Definition 14. *A $\mathfrak{T}_\Upsilon$-hybrid relation on $\mathbb{D}$ is a hybrid structure*

$$\tilde{v}_\varkappa := (\tilde{v}, \varkappa) \colon \mathbb{D} \times \mathbb{D} \to \mathbb{P}(\mathbb{B}) \times \mathbb{I},$$

where $\tilde{v} \colon \mathbb{D} \times \mathbb{D} \to \mathbb{P}(\mathbb{B})$ and $\varkappa \colon \mathbb{D} \times \mathbb{D} \to \mathbb{I}$ are mappings.

Definition 15. *Let $\tilde{v}_\varkappa$ be a hybrid relation on $\mathbb{D}$ and $\tilde{n}_\iota \in H(\mathbb{D})$. Then $\tilde{v}_\varkappa$ is a $\mathfrak{T}_\Upsilon$-hybrid relation on $\tilde{n}_\iota$ if*

$$(\forall\ s, a \in \mathbb{D}) \left(\begin{array}{c} \tilde{v}(s, a) \subseteq \mathfrak{T}(\tilde{n}(s), \tilde{n}(a)) \\ \varkappa(s, a) \geqslant \Upsilon(\iota(s), \iota(a)) \end{array} \right).$$

Definition 16. *Let $\tilde{b}_\varsigma$ be a hybrid relation on $\mathbb{D}$ and $\tilde{v}_\varpi \in H(\mathbb{D})$. Then $\tilde{b}_\varsigma$ is described as a strongest $\mathfrak{T}_\Upsilon$-hybrid relation on $\tilde{v}_\varpi$ if*

$$(\forall\ t_0, f_0 \in \mathbb{D}) \left(\begin{array}{c} \tilde{b}(t_0, f_0) = \mathfrak{T}(\tilde{v}(t_0), \tilde{v}(f_0)) \\ \varsigma(t_0, f_0) = \Upsilon(\varpi(t_0), \varpi(f_0)) \end{array} \right).$$

Definition 17. *Let $\tilde{v}_\varkappa$, $\tilde{b}_\iota \in H(\mathbb{D})$ and the hybrid relation $\tilde{v}_\varkappa \otimes \tilde{b}_\iota := (\tilde{v} \times \tilde{b}, \varkappa \times \iota)$ on $\mathbb{D} \times \mathbb{D}$. Then the direct $\mathfrak{T}_\Upsilon$-product of $\tilde{v}_\varkappa$ and $\tilde{b}_\iota$ is defined by*

$$(\forall\ s, w \in \mathbb{D}) \begin{pmatrix} (\tilde{v} \times \tilde{b})(s, w) = \mathfrak{T}(\tilde{v}(s), \tilde{b}(w)) \\ (\varkappa \times \iota)(s, w) = \Upsilon(\varkappa(s), \iota(w)) \end{pmatrix}.$$

Lemma 2. *Let $\tilde{v}_\varkappa, \tilde{n}_\iota \in H(\mathbb{D})$. Then for any $\epsilon \in \mathbb{P}(\mathbb{B})$ and $c \in \mathbb{I}$, $\mathbb{D}^\epsilon_{\tilde{v}\times\tilde{n}} \subseteq \mathbb{D}^\epsilon_{\tilde{v}} \times \mathbb{D}^\epsilon_{\tilde{n}}$ and $\mathbb{D}^c_{\varkappa\times\iota} \subseteq \mathbb{D}^c_\varkappa \times \mathbb{D}^c_\iota$.*

Proof. Let $(s, a) \in \mathbb{D}^\epsilon_{\tilde{v}\times\tilde{n}}$. Then $(\tilde{v} \times \tilde{n})(s, a) \supseteq \epsilon$ implies $\mathfrak{T}(\tilde{v}(s), \tilde{n}(a)) \supseteq \epsilon$. By Lemma 1, we have $\tilde{v}(s) \supseteq \epsilon$ and $\tilde{n}(a) \supseteq \epsilon$; thus, $s \in \mathbb{D}^\epsilon_{\tilde{v}}$ and $a \in \mathbb{D}^\epsilon_{\tilde{n}}$, and, hence, $(s, a) \in \mathbb{D}^\epsilon_{\tilde{v}} \times \mathbb{D}^\epsilon_{\tilde{n}}$. So $\mathbb{D}^\epsilon_{\tilde{v}\times\tilde{n}} \subseteq \mathbb{D}^\epsilon_{\tilde{v}} \times \mathbb{D}^\epsilon_{\tilde{n}}$.

Let $(s, a) \in \mathbb{D}^c_{\varkappa\times\iota}$. Then $(\varkappa \times \iota)(s, a) \leqslant c$ implies $\Upsilon(\varkappa(s), \iota(a)) \leqslant c$. By Lemma 1, we have $\varkappa(s) \leqslant c$ and $\iota(a) \leqslant c$; thus, $s \in \mathbb{D}^c_\varkappa$ and $a \in \mathbb{D}^c_\iota$, and, hence, $(s, a) \in \mathbb{D}^c_\varkappa \times \mathbb{D}^c_\iota$. So $\mathbb{D}^c_{\varkappa\times\iota} \subseteq \mathbb{D}^c_\varkappa \times \mathbb{D}^c_\iota$. □

Lemma 3. *For $\tilde{v}_\varkappa, \tilde{h}_\iota \in H(\mathbb{D})$, if $\tilde{v}_\varkappa$ is a strongest $\mathfrak{T}_\Upsilon$-hybrid relation on $\tilde{h}_\iota$, then, for $\epsilon \in \mathbb{P}(\mathbb{B})$ and $c_0 \in [0, 1]$, we have:*

(i) $(\mathbb{D} \times \mathbb{D})^\epsilon_{\tilde{v}} \subseteq \mathbb{D}^\epsilon_{\tilde{h}} \times \mathbb{D}^\epsilon_{\tilde{h}}$.

(ii) $(\mathbb{D} \times \mathbb{D})^{c_0}_\varkappa \subseteq \mathbb{D}^{c_0}_\iota \times \mathbb{D}^{c_0}_\iota$.

Proof.

(i) Let $(s, a) \in (\mathbb{D} \times \mathbb{D})^\epsilon_{\tilde{v}}$. Then $\tilde{v}(s, a) \supseteq \epsilon$ implies $\mathfrak{T}(\tilde{h}(s), \tilde{h}(a)) \supseteq \epsilon$. By Lemma 1, we have $\tilde{h}(s) \supseteq \epsilon$ and $\tilde{h}(a) \supseteq \epsilon$, and, hence, $(s, a) \in \mathbb{D}^\epsilon_{\tilde{h}} \times \mathbb{D}^\epsilon_{\tilde{h}}$. So, $(\mathbb{D} \times \mathbb{D})^\epsilon_{\tilde{v}} \subseteq \mathbb{D}^\epsilon_{\tilde{h}} \times \mathbb{D}^\epsilon_{\tilde{h}}$.

(ii) Let $(s, a) \in (\mathbb{D} \times \mathbb{D})^{c_0}_\varkappa$. Then $\varkappa(s, a) \leqslant c_0$ implies $\Upsilon(\iota(s),\iota(a)) \leqslant c_0$. By Lemma 1, we have $\iota(s) \leqslant c_0$ and $\iota(a) \leqslant c_0$. Hence, $(s, a) \in \mathbb{D}^{c_0}_\iota \times \mathbb{D}^{c_0}_\iota$. So, $(\mathbb{D} \times \mathbb{D})^{c_0}_\varkappa \subseteq \mathbb{D}^{c_0}_\iota \times \mathbb{D}^{c_0}_\iota$.

□

Proposition 1. *Let $\tilde{v}_\vartheta \in H(\mathbb{D})$ and $\tilde{b}_\varsigma \in H(\mathbb{D} \times \mathbb{D})$ be the strongest $\mathfrak{T}_\Upsilon$-hybrid relation on $\tilde{v}_\vartheta$. If $\tilde{b}_\varsigma$ in $\mathbb{D} \times \mathbb{D}$ is an imaginable $\mathfrak{T}_\Upsilon$-hybrid left h-ideal, then $\tilde{v}(s_0) \subseteq \tilde{v}(0)$ and $\vartheta(s_0) \geqslant \vartheta(0)$ $\forall$ $s_0 \in \mathbb{D}$.*

Proof. If $\tilde{b}_\varsigma \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D} \times \mathbb{D})$, then $\forall$ $s_0 \in \mathbb{D}$, we have $\tilde{b}(s_0, s_0) \subseteq \tilde{b}(0, 0)$ and $\varsigma(s_0, s_0) \geqslant \varsigma(0, 0)$, which imply that

$$\tilde{v}(s_0) = \mathfrak{T}(\tilde{v}(s_0),\ \tilde{v}(s_0)) \subseteq \mathfrak{T}(\tilde{v}(0),\ \tilde{v}(0)) = \tilde{v}(0)$$

and $\vartheta(s_0) = \Upsilon(\vartheta(s_0), \vartheta(s_0)) \geqslant \Upsilon(\vartheta(0), \vartheta(0)) = \vartheta(0)$. $\square$

Theorem 1. *Let $\tilde{l}_\varsigma, \tilde{d}_\iota \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$. Then $\tilde{l}_\varsigma \otimes \tilde{d}_\iota \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D} \times \mathbb{D})$.*

Proof. Let $w = (w_0, w_1)$ and $p = (p_0, p_1)$ be any elements of $\mathbb{D} \times \mathbb{D}$. Then,

$$\begin{aligned}(\tilde{l} \times \tilde{d})(w + p) &= (\tilde{l} \times \tilde{d})((w_0, w_1) + (p_0, p_1)) = (\tilde{l} \times \tilde{d})(w_0 + p_0, w_1 + p_1) = \\ &= \mathfrak{T}(\tilde{l}(w_0 + p_0), \tilde{d}(w_1 + p_1)) \supseteq \mathfrak{T}(\mathfrak{T}(\tilde{l}(w_0), \tilde{l}(p_0)), \mathfrak{T}(\tilde{d}(w_1), \tilde{d}(p_1))) = \\ &= \mathfrak{T}(\mathfrak{T}(\tilde{l}(w_0), \tilde{d}(w_1)), \mathfrak{T}(\tilde{l}(p_0), \tilde{d}(p_1))) = \mathfrak{T}((\tilde{l} \times \tilde{d})(w_0, w_1), (\tilde{l} \times \tilde{d})(p_0, p_1)) = \\ &= \mathfrak{T}((\tilde{l} \times \tilde{d})(w), (\tilde{l} \times \tilde{d})(p)),\end{aligned}$$

$$\begin{aligned}(\varsigma \times \iota)(w + p) &= (\varsigma \times \iota)((w_0, w_1) + (p_0, p_1)) = (\varsigma \times \iota)(w_0 + p_0, w_1 + p_1) = \\ &= \Upsilon(\varsigma(w_0 + p_0), \iota(w_1 + p_1)) \leqslant \Upsilon(\Upsilon(\varsigma(w_0), \varsigma(p_0)), \Upsilon(\iota(w_1), \iota(p_1))) = \\ &= \Upsilon(\Upsilon(\varsigma(w_0), \iota(w_1)), \Upsilon(\varsigma(p_0), \iota(p_1))) = \Upsilon((\varsigma \times \iota)(w_0, w_1), (\varsigma \times \iota)(p_0, p_1)) = \\ &= \Upsilon((\varsigma \times \iota)(w), (\varsigma \times \iota)(p)).\end{aligned}$$

Also,

$$\begin{aligned}(\tilde{l} \times \tilde{d})(wp) &= (\tilde{l} \times \tilde{d})(w_0 p_0, w_1 p_1) = \\ &= \mathfrak{T}(\tilde{l}(w_0 p_0), \tilde{d}(w_1 p_1)) \supseteq \mathfrak{T}(\tilde{l}(p_0), \tilde{d}(p_1)) = \\ &= (\tilde{l} \times \tilde{d})(p_0, p_1) = (\tilde{l} \times \tilde{d})(p),\end{aligned}$$

$$\begin{aligned}(\varsigma \times \iota)(wp) &= (\varsigma \times \iota)(w_0 p_0, w_1 p_1) = \Upsilon(\varsigma(w_0 p_0), \iota(w_1 p_1)) \leqslant \\ &\leqslant \Upsilon(\varsigma(p_0), \iota(p_1)) = (\varsigma \times \iota)(p_0, p_1) = (\varsigma \times \iota)(p).\end{aligned}$$

Hence, $\tilde{l}_\varsigma \otimes \tilde{d}_\iota$ is a $\mathfrak{T}_\Upsilon$-hybrid left ideal.

Let $s = (s_1, s_2), z = (z_1, z_2), i = (i_1, i_2)$ and $t = (t_1, t_2)$ be such that $s + i + z = t + z$. Then $(s_1, s_2) + (i_1, i_2) + (z_1, z_2) = (t_1, t_2) + (z_1, z_2)$ and so $s_1 + i_1 + z_1 = t_1 + z_1$ and $s_2 + i_2 + z_2 = t_2 + z_2$. It follows that

$$\begin{aligned}(\tilde{l} \times \tilde{d})(s) &= (\tilde{l} \times \tilde{d})(s_1, s_2) = \mathfrak{T}(\tilde{l}(s_1), \tilde{d}(s_2)) \supseteq \\ &\supseteq \mathfrak{T}(\mathfrak{T}(\tilde{l}(i_1), \tilde{l}(t_1)), \mathfrak{T}(\tilde{d}(i_2), \tilde{d}(t_2))) = \mathfrak{T}(\mathfrak{T}(\tilde{l}(i_1), \tilde{d}(i_2)), \mathfrak{T}(\tilde{l}(t_1), \tilde{d}(t_2))) = \\ &= \mathfrak{T}((\tilde{l} \times \tilde{d})(i_1, i_2), (\tilde{l} \times \tilde{d})(t_1, t_2)) = \mathfrak{T}((\tilde{l} \times \tilde{d})(i), (\tilde{l} \times \tilde{d})(t)),\end{aligned}$$

$$\begin{aligned}(\varsigma \times \iota)(s) &= (\varsigma \times \iota)(s_1, s_2) = \Upsilon(\varsigma(s_1), \iota(s_2)) \leqslant \\ &\leqslant \Upsilon(\Upsilon(\varsigma(i_1), \varsigma(t_1)), \Upsilon(\iota(i_2), \iota(t_2))) = \Upsilon(\Upsilon(\varsigma(i_1), \iota(i_2)), \Upsilon(\varsigma(t_1), \iota(t_2))) = \\ &= \Upsilon((\varsigma \times \iota)(i_1, i_2), (\varsigma \times \iota)(t_1, t_2)) = \Upsilon((\varsigma \times \iota)(i), (\varsigma \times \iota)(t)).\end{aligned}$$

Therefore, $\tilde{l}_\varsigma \otimes \tilde{d}_\iota \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D} \times \mathbb{D})$. □

Corollary 1. *If $\tilde{v}_\varpi, \tilde{c}_\vartheta \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$ are imaginable, then $\tilde{v}_\varpi \otimes \tilde{c}_\vartheta \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D} \times \mathbb{D})$ is also an imaginable.*

Proof. By Theorem 1, we have $\tilde{v}_\varpi \otimes \tilde{c}_\vartheta \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D}\times\mathbb{D})$. Let $l = (l_0, l_1) \in \mathbb{D}\times\mathbb{D}$. Then

$$\begin{aligned}
&\mathfrak{T}((\tilde{v} \times \tilde{c})(l), (\tilde{v} \times \tilde{c})(l)) = \mathfrak{T}((\tilde{v} \times \tilde{c})(l_0, l_1),(\tilde{v} \times \tilde{c})(l_0, l_1)) = \\
&= \mathfrak{T}(\mathfrak{T}(\tilde{v}(l_0), \tilde{c}(l_1)), \mathfrak{T}(\tilde{v}(l_0), \tilde{c}(l_1))) = \mathfrak{T}(\mathfrak{T}(\tilde{v}(l_0), \tilde{v}(l_0)), \mathfrak{T}(\tilde{c}(l_1), \tilde{c}(l_1))) = \\
&\qquad = \mathfrak{T}(\tilde{v}(l_0), \tilde{c}(l_1)) = (\tilde{v} \times \tilde{c})(l_0, l_1) = (\tilde{v} \times \tilde{c})(l).
\end{aligned}$$

Also,

$$\begin{aligned}
&\Upsilon((\varpi \times \vartheta)(l), (\varpi \times \vartheta)(l)) = \Upsilon((\varpi \times \vartheta)(l_0, l_1), (\varpi \times \vartheta)(l_0, l_1)) = \\
&= \Upsilon(\Upsilon(\varpi(l_0), \vartheta(l_1)),\Upsilon(\varpi(l_0), \vartheta(l_1))) = \Upsilon(\Upsilon(\varpi(l_0), \varpi(l_0)),\Upsilon(\vartheta(l_1), \vartheta(l_1))) = \\
&\qquad = \Upsilon(\varpi(l_0), \vartheta(l_1)) = (\varpi \times \vartheta)(l_0, l_1) = (\varpi \times \vartheta)(l).
\end{aligned}$$

Therefore $\tilde{v}_\varpi \otimes \tilde{c}_\vartheta$ is imaginable. □

The converse of the Corollary 1 is not true in general, as shown in the following example:

Example 1. Let $\mathbb{D}$ be a hemiring with $|\mathbb{D}| \geqslant 2$. For $M_0, H_0 \in \mathbb{P}(\mathbb{B})$ and $m_0, h_0 \in [0, 1]$, define the structure $\mathfrak{T}_\Upsilon := (\mathfrak{T}, \Upsilon)$, where $\mathfrak{T}(M_0, H_0) = M_0 \cap H_0$ and $\Upsilon(m_0, h_0) = m_0 \vee h_0$. Then $\mathfrak{T}_\Upsilon$ is a hybrid t-norm.

Let $J_0 \in \mathbb{P}(\mathbb{B})$ and $j_1 \in [0, 1]$. Define the hybrid structures $\tilde{v}_\varrho, \tilde{c}_\varpi \in H(\mathbb{D})$ by $\tilde{v}(a_0) = J_0$; $\varrho(a_0) = j_1$ and

$$\tilde{c}(a_0) := \begin{cases} J_0 & \text{if } a_0 = 0, \\ \mathbb{B} & \text{otherwise} \end{cases} \quad \text{and} \quad \varpi(a_0) := \begin{cases} j_1 & \text{if } a_0 = 0, \\ 0 & \text{otherwise} \end{cases}$$

for any $a_0 \in \mathbb{D}$.

If $a_1 = 0$, then $\tilde{c}(a_1) = J_0$ and $\varpi(a_1) = j_1$; so,

$$(\tilde{v} \times \tilde{c})(a_0, a_1) = \mathfrak{T}(\tilde{v}(a_0), \tilde{c}(a_1)) = \mathfrak{T}(J_0, J_0) = J_0,$$

$$(\varrho \times \varpi)(a_0, a_1) = \Upsilon(\varrho(a_0), \varpi(a_1)) = \Upsilon(j_1, j_1) = j_1.$$

If $a_1 \neq 0$, then $\tilde{c}(a_1) = \mathbb{B}$ and $\varpi(a_1) = 0$; so,

$$(\tilde{v} \times \tilde{c})(a_0, a_1) = \mathfrak{T}(\tilde{v}(a_0), \tilde{c}(a_1)) = \mathfrak{T}(J_0, \mathbb{B}) = J_0,$$

$$(\varrho \times \varpi)(a_0, a_1) = \Upsilon(\varrho(a_0), \varpi(a_1)) = \Upsilon(j_1, 0) = j_1.$$

Thus, $\tilde{v}_\varrho \otimes \tilde{c}_\varpi$ is an imaginable $\mathfrak{T}_\Upsilon$-hybrid left h-ideal of $\mathbb{D} \times \mathbb{D}$. Here $\tilde{v}_\varrho$ is an imaginable $\mathfrak{T}_\Upsilon$-hybrid left h-ideal of $\mathbb{D}$, but $\tilde{c}_\varpi$ is not an imaginable $\mathfrak{T}_\Upsilon$-hybrid left h-ideal of $\mathbb{D}$, as for $a_0 \neq 0$ we have $\tilde{c}(0) = J_0 \subset \mathbb{B} = \tilde{c}(a_0)$ and $\varpi(0) = j_1 < 1 = \varpi(a_0)$.

Definition 18. *Let $\tilde{v}_\varkappa$ and $\tilde{n}_\iota$ be hybrid structures in $\mathbb{D}$. Then the $\mathfrak{T}_\Upsilon$-product of $\tilde{v}_\varkappa$ and $\tilde{n}_\iota$, written as $[\tilde{v}_\varkappa \cdot \tilde{n}_\iota]_{\mathfrak{T}_\Upsilon}$, is defined by*

$$(\forall\ s \in \mathbb{D}) \left(\begin{array}{c} [\tilde{v} \cdot \tilde{n}]_{\mathfrak{T}}(s) = \mathfrak{T}(\tilde{v}(s), \tilde{n}(s)) \\ [\varkappa \cdot \iota]_\Upsilon(s) = \Upsilon(\varkappa(s), \iota(s)) \end{array} \right).$$

Definition 19. *Let $\mathfrak{T}_\Upsilon$ be a hybrid t-norm. Then the hybrid t-norm $\mathfrak{T}^*_\Upsilon$ is said to be dominated if it satisfies:*

$$\left(\begin{array}{c} \forall\ Z, P, G, C \in \mathbb{P}(\mathbb{B}) \\ \forall\ \vartheta, \varrho, \sigma, \varkappa \in [0, 1]) \end{array} \right) \left(\begin{array}{c} \mathfrak{T}^*(\mathfrak{T}(Z, P), \mathfrak{T}(G, C)) \supseteq \mathfrak{T}(\mathfrak{T}^*(Z, G), \mathfrak{T}^*(P, C)) \\ \Upsilon^*(\Upsilon(\vartheta, \varrho), \Upsilon(\sigma, \varkappa)) \leqslant \Upsilon(\Upsilon^*(\vartheta, \sigma), \Upsilon^*(\varrho, \varkappa)) \end{array} \right).$$

Theorem 2. *Let $\tilde{e}_\varpi, \tilde{b}_\iota \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$. Then $[\tilde{e}_\varpi \cdot \tilde{b}_\iota]_{\mathfrak{T}^*_\Upsilon} \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$.*

Proof. Let $v_0, g_0 \in \mathbb{D}$. Then

$$\begin{aligned}
[\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(v_0 + g_0) &= \mathfrak{T}^*(\tilde{e}(v_0 + g_0), \tilde{b}(v_0 + g_0)) \supseteq \\
&\supseteq \mathfrak{T}^*(\mathfrak{T}(\tilde{e}(v_0), \tilde{e}(g_0)), \mathfrak{T}(\tilde{b}(v_0), \tilde{b}(g_0))) \supseteq \\
&\supseteq \mathfrak{T}(\mathfrak{T}^*(\tilde{e}(v_0), \tilde{b}(v_0)), \mathfrak{T}^*(\tilde{e}(g_0), \tilde{b}(g_0))) = \mathfrak{T}([\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(v_0), [\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(g_0)),
\end{aligned}$$

$$\begin{aligned}
[\varpi \cdot \iota]_{\Upsilon^*}(v_0 + g_0) &= \Upsilon^*(\varpi(v_0 + g_0), \iota(v_0 + g_0)) \leqslant \\
&\leqslant \Upsilon^*(\Upsilon(\varpi(v_0), \varpi(g_0)), \Upsilon(\iota(v_0), \iota(g_0))) \leqslant \\
&\leqslant \Upsilon(\Upsilon^*(\varpi(v_0), \iota(v_0)), \Upsilon^*(\varpi(g_0), \iota(g_0))) = \Upsilon([\varpi \cdot \iota]_{\Upsilon^*}(v_0), [\varpi \cdot \iota]_{\Upsilon^*}(g_0)).
\end{aligned}$$

Also,

$$[\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(v_0 g_0) = \mathfrak{T}^*(\tilde{e}(v_0 g_0), \tilde{b}(v_0 g_0)) \supseteq \mathfrak{T}^*(\tilde{e}(g_0), \tilde{b}(g_0)) = [\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(g_0),$$

$$\begin{aligned}
[\varpi \cdot \iota]_{\Upsilon^*}(v_0 g_0) &= \Upsilon^*(\varpi(v_0 g_0), \iota(v_0 g_0)) \leqslant \\
&\leqslant \Upsilon^*(\varpi(g_0), \iota(g_0)) = [\varpi \cdot \iota]_{\Upsilon^*}(g_0).
\end{aligned}$$

Hence, $[\tilde{e}_\varpi \cdot \tilde{b}_\iota]_{\mathfrak{T}^*_\Upsilon}$ is a $\mathfrak{T}_\Upsilon$-hybrid left ideal of $\mathbb{D}$.

Now, if $v_0, m_0, x_0, d_0 \in \mathbb{D}$ are such that $v_0 + x_0 + m_0 = d_0 + m_0$, then

$$[\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(v_0) = \mathfrak{T}^*(\tilde{e}(v_0), \tilde{b}(v_0)) \supseteq \mathfrak{T}^*(\mathfrak{T}(\tilde{e}(x_0), \tilde{e}(d_0)), \mathfrak{T}(\tilde{b}(x_0), \tilde{b}(d_0))) \supseteq$$
$$\supseteq \mathfrak{T}(\mathfrak{T}^*(\tilde{e}(x_0), \tilde{b}(x_0)), \mathfrak{T}^*(\tilde{e}(d_0), \tilde{b}(d_0))) = \mathfrak{T}([\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(x_0), [\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(d_0)),$$

$$[\varpi \cdot \iota]_{\Upsilon^*}(v_0) = \Upsilon^*(\varpi(v_0), \iota(v_0)) \leqslant \Upsilon^*(\Upsilon(\varpi(x_0), \varpi(d_0)), \Upsilon(\iota(x_0), \iota(d_0))) \leqslant$$
$$\leqslant \Upsilon(\Upsilon^*(\varpi(x_0), \iota(x_0)), \Upsilon^*(\varpi(d_0), \iota(d_0))) = \Upsilon([\varpi \cdot \iota]_{\Upsilon^*}(x_0), [\varpi \cdot \iota]_{\Upsilon^*}(d_0)).$$

Therefore, $[\tilde{e}_\varpi \cdot \tilde{b}_\iota]_{\mathfrak{T}^*_\Upsilon} \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$. $\square$

Theorem 3. *Let $z : \mathbb{D} \to \mathbb{D}'$ be an "onto" homomorphism of hemirings. Let $\tilde{e}_\varphi$ and $\tilde{b}_\varsigma$ be $\mathfrak{T}_\Upsilon$-hybrid left h-ideals of $\mathbb{D}'$. If $[\tilde{e}_\varphi \cdot \tilde{b}_\varsigma]_{\mathfrak{T}^*_\Upsilon}$ is a $\mathfrak{T}^*_\Upsilon$-product of $\tilde{e}_\varphi$ and $\tilde{b}_\varsigma$ and $[z^{-1}(\tilde{e}_\varphi) \cdot z^{-1}(\tilde{b}_\varsigma)]_{\mathfrak{T}^*_\Upsilon}$ is the $\mathfrak{T}^*_\Upsilon$-product $z^{-1}(\tilde{e}_\varphi)$ and $z^{-1}(\tilde{b}_\varsigma)$, then*

$$z^{-1}([\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}) = [z^{-1}(\tilde{e}) \cdot z^{-1}(\tilde{b})]_{\mathfrak{T}^*},$$
$$z^{-1}([\varphi \cdot \varsigma]_{\Upsilon^*}) = [z^{-1}(\varphi) \cdot z^{-1}(\varsigma)]_{\Upsilon^*}.$$

Proof. Let $s \in \mathbb{D}$. Then

$$z^{-1}([\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*})(s) = [\tilde{e} \cdot \tilde{b}]_{\mathfrak{T}^*}(z(s)) = \mathfrak{T}^*(\tilde{e}(z(s)), \tilde{b}(z(s))) =$$
$$= \mathfrak{T}^*(z^{-1}(\tilde{e})(s), z^{-1}(\tilde{b})(s)) = [z^{-1}(\tilde{e}) \cdot z^{-1}(\tilde{b})]_{\mathfrak{T}^*}(s),$$

$$z^{-1}([\varphi \cdot \varsigma]_{\Upsilon^*})(s) = [\varphi \cdot \varsigma]_{\Upsilon^*}(z(s)) = \Upsilon^*(\varphi(z(s)), \varsigma(z(s))) =$$
$$= \Upsilon^*(z^{-1}(\varphi)(s), z^{-1}(\varsigma)(s)) = [z^{-1}(\varphi) \cdot z^{-1}(\varsigma)]_{\Upsilon^*}(s).$$

The proof is completed. $\square$

Theorem 4. *Let $\tilde{q}_\varrho$ be an imaginable hybrid structure in a hemiring $\mathbb{D}$ and let $\tilde{v}_\varkappa$ be the strongest $\mathfrak{T}_\Upsilon$-hybrid relation on $\tilde{q}_\varrho$. Then the statements below are equivalent:*

(i) $\tilde{q}_\varrho \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$,

(ii) $\tilde{v}_\varkappa \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D} \times \mathbb{D})$ *is imaginable.*

Proof. $(i) \Rightarrow (ii)$ Let $h = (h_0, h_1), u = (u_0, u_1) \in \mathbb{D} \times \mathbb{D}$. Then

$$\tilde{v}(h + u) = \tilde{v}((h_0, h_1) + (u_0, u_1)) = \tilde{v}(h_0 + u_0, h_1 + u_1) =$$
$$= \mathfrak{T}(\tilde{q}(h_0 + u_0), \tilde{q}(h_1 + u_1)) \supseteq \mathfrak{T}(\mathfrak{T}(\tilde{q}(h_0), \tilde{q}(u_0)), \mathfrak{T}(\tilde{q}(h_1), \tilde{q}(u_1))) =$$
$$= \mathfrak{T}(\mathfrak{T}(\tilde{q}(h_0), \tilde{q}(h_1)), \mathfrak{T}(\tilde{q}(u_0), \tilde{q}(u_1))) =$$
$$= \mathfrak{T}(\tilde{v}(h_0, h_1), \tilde{v}(u_0, u_1)) = \mathfrak{T}(\tilde{v}(h), \tilde{v}(u)),$$

$$\varkappa(h+u) = \varkappa((h_0,h_1)+(u_0,u_1)) = \varkappa(h_0+u_0, h_1+u_1) = \\ = \Upsilon(\varrho(h_0+u_0), \varrho(h_1+u_1)) \leqslant \Upsilon(\Upsilon(\varrho(h_0),\varrho(u_0)), \Upsilon(\varrho(h_1),\varrho(u_1))) = \\ = \Upsilon(\Upsilon(\varrho(h_0),\varrho(h_1)), \Upsilon(\varrho(u_0),\varrho(u_1))) = \Upsilon(\varkappa(h_0,h_1), \varkappa(u_0,u_1)) = \\ = \Upsilon(\varkappa(h), \varkappa(u)).$$

Also,

$$\tilde{v}(hu) = \tilde{v}(h_0u_0, h_1u_1) = \mathfrak{T}(\tilde{q}(h_0u_0), \tilde{q}(h_1,u_1)) \supseteq \\ \supseteq \mathfrak{T}(\tilde{q}(u_0), \tilde{q}(u_1)) = \tilde{v}(u_0,u_1) = \tilde{v}(u),$$

$$\varkappa(hu) = \varkappa(h_0u_0, h_1u_1) = \Upsilon(\varrho(h_0u_0), \varrho(h_1u_1)) \leqslant \\ \leqslant \Upsilon(\varrho(u_0), \varrho(u_1)) = \varkappa(u_0,u_1) = \varkappa(u).$$

Hence, $\tilde{v}_\varkappa$ in $\mathbb{D}\times\mathbb{D}$ is a $\mathfrak{T}_\Upsilon$-hybrid left ideal.

Now, let $m=(m_0,m_1)$, $t=(t_0,t_1)$, $h=(h_0,h_1)$, $l=(l_0,l_1)\in\mathbb{D}\times\mathbb{D}$ be such that $h+m+l=t+l$. Then $(h_0,h_1)+(m_0,m_1)+(l_0,l_1) = = (t_0,t_1)+(l_0,l_1)$. Thus, $h_0+m_0+l_0=t_0+l_0$ and $h_1+m_1+l_1=t_1+l_1$. Now,

$$\tilde{v}(h) = \tilde{v}(h_0,h_1) = \mathfrak{T}(\tilde{q}(h_0),\tilde{q}(h_1)) \supseteq \mathfrak{T}(\mathfrak{T}(\tilde{q}(m_0),\tilde{q}(t_0)), \mathfrak{T}(\tilde{q}(m_1),\tilde{q}(t_1))) = \\ = \mathfrak{T}(\mathfrak{T}(\tilde{q}(m_0),\tilde{q}(m_1)), \mathfrak{T}(\tilde{q}(t_0),\tilde{q}(t_1))) = \mathfrak{T}(\tilde{v}(m_0,m_1), \tilde{v}(t_0,t_1)) = \\ = \mathfrak{T}(\tilde{v}(m), \tilde{v}(t)),$$

$$\varkappa(h) = \varkappa(h_0,h_1) = \Upsilon(\varrho(h_0),\varrho(h_1)) \leqslant \Upsilon(\Upsilon(\varrho(m_0),\varrho(t_0)), \Upsilon(\varrho(m_1),\varrho(t_1))) = \\ = \Upsilon(\Upsilon(\varrho(m_0),\varrho(m_1)), \Upsilon(\varrho(t_0),\varrho(t_1))) = \Upsilon(\varkappa(m_0,m_1), \varkappa(t_0,t_1)) = \\ = \Upsilon(\varkappa(m), \varkappa(t)).$$

Therefore, $\tilde{v}_\varkappa \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D}\times\mathbb{D})$.

For $e=(e_0,e_1)\in\mathbb{D}\times\mathbb{D}$, we have

$$\mathfrak{T}(\tilde{v}(e),\tilde{v}(e)) = \mathfrak{T}(\tilde{v}(e_0,e_1),\tilde{v}(e_0,e_1)) = \mathfrak{T}(\mathfrak{T}(\tilde{q}(e_0),\tilde{q}(e_1)), \mathfrak{T}(\tilde{q}(e_0),\tilde{q}(e_1))) = \\ = \mathfrak{T}(\mathfrak{T}(\tilde{q}(e_0),\tilde{q}(e_0)), \mathfrak{T}(\tilde{q}(e_1),\tilde{q}(e_1))) = \mathfrak{T}(\tilde{q}(e_0),\tilde{q}(e_1)) = \tilde{v}(e_0,e_1) = \tilde{v}(e),$$

$$\Upsilon(\varkappa(e),\varkappa(e)) = \Upsilon(\varkappa(e_0,e_1),\varkappa(e_0,e_1)) = \\ = \Upsilon(\Upsilon(\varrho(e_0),\varrho(e_1)), \Upsilon(\varrho(e_0),\varrho(e_1))) = \\ = \Upsilon(\Upsilon(\varrho(e_0),\varrho(e_0)), \Upsilon(\varrho(e_1),\varrho(e_1))) = \Upsilon(\varrho(e_0),\varrho(e_1)) = \varkappa(e_0,e_1) = \varkappa(e).$$

Thus, $\tilde{v}_{\varkappa} \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D} \times \mathbb{D})$ is imaginable.

$(ii) \Rightarrow (i)$ Consider that $\tilde{v}_{\varkappa}$ is an imaginable $\mathfrak{T}_{\Upsilon}$-hybrid left h-ideal in $\mathbb{D} \times \mathbb{D}$. For $g_0, b_0 \in \mathbb{D}$, we get

$$\begin{aligned}\tilde{q}(g_0 + b_0) &= \mathfrak{T}(\tilde{q}(g_0 + b_0), \tilde{q}(g_0 + b_0)) = \tilde{v}(g_0 + b_0, g_0 + b_0) = \\ &= \tilde{v}((g_0, g_0) + (b_0, b_0)) \supseteq \mathfrak{T}(\tilde{v}(g_0, g_0), \tilde{v}(b_0, b_0)) = \\ &= \mathfrak{T}(\mathfrak{T}(\tilde{q}(g_0), \tilde{q}(g_0)), \mathfrak{T}(\tilde{q}(b_0), \tilde{q}(b_0))) = \mathfrak{T}(\tilde{q}(g_0), \tilde{q}(b_0)),\end{aligned}$$

$$\begin{aligned}\varrho(g_0 + b_0) &= \Upsilon(\varrho(g_0 + b_0), \varrho(g_0 + b_0)) = \varkappa(g_0 + b_0, g_0 + b_0) = \\ &= \varkappa((g_0, g_0) + (b_0, b_0)) \leqslant \Upsilon(\varkappa(g_0, g_0), \varkappa(b_0, b_0)) = \\ &= \Upsilon(\Upsilon(\varrho(g_0), \varrho(g_0)), \Upsilon(\varrho(b_0), \varrho(b_0))) = \Upsilon(\varrho(g_0), \varrho(b_0)).\end{aligned}$$

Also,

$$\begin{aligned}\tilde{q}(g_0 b_0) &= \mathfrak{T}(\tilde{q}(g_0 b_0), \tilde{q}(g_0 b_0)) = \tilde{v}((g_0, g_0)(b_0, b_0)) \supseteq \tilde{v}(b_0, b_0) = \\ &= \mathfrak{T}(\tilde{q}(b_0), \tilde{q}(b_0)) = \tilde{q}(b_0),\end{aligned}$$

$$\begin{aligned}\varrho(g_0 b_0) &= \Upsilon(\varrho(g_0 b_0), \varrho(g_0 b_0)) = \varkappa((g_0, g_0)(b_0, b_0)) \leqslant \varkappa(b_0, b_0) = \\ &= \Upsilon(\varrho(b_0), \varrho(b_0)) = \varrho(b_0).\end{aligned}$$

Hence, $\tilde{q}_{\varrho}$ in $\mathbb{D}$ is an imaginable $\mathfrak{T}_{\Upsilon}$-hybrid left ideal.

Let $m_0, c_0, w_0, v_0 \in \mathbb{D}$ be such that $w_0 + m_0 + v_0 = c_0 + v_0$. Then $(w_0, w_0) + (m_0, m_0) + (v_0, v_0) = (c_0, c_0) + (v_0, v_0)$.

It follows that

$$\begin{aligned}\tilde{q}(w_0) &= \mathfrak{T}(\tilde{q}(w_0), \tilde{q}(w_0)) = \tilde{v}(w_0, w_0) \supseteq \mathfrak{T}(\tilde{v}(m_0, m_0), \tilde{v}(c_0, c_0)) = \\ &= \mathfrak{T}(\mathfrak{T}(\tilde{q}(m_0), \tilde{q}(m_0)), \mathfrak{T}(\tilde{q}(c_0), \tilde{q}(c_0))) = \mathfrak{T}(\tilde{q}(m_0), \tilde{q}(c_0))\end{aligned}$$

and

$$\begin{aligned}\varrho(w_0) &= \Upsilon(\varrho(w_0), \varrho(w_0)) \leqslant \Upsilon(\varkappa(m_0, m_0), \varkappa(c_0, c_0)) = \\ &= \Upsilon(\Upsilon(\varrho(m_0), \varrho(m_0)), \Upsilon(\varrho(c_0), \varrho(c_0))) = \Upsilon(\varrho(m_0), \varrho(c_0)).\end{aligned}$$

Consequently, $\tilde{q}_{\varrho} \in H\mathfrak{T}_{\Upsilon_L}(\mathbb{D})$ is imaginable. $\square$

4. Conclusion. In this article, we explored the notion of $\mathfrak{T}_{\Upsilon}$-hybrid relations, strongest $\mathfrak{T}_{\Upsilon}$-hybrid relations, and direct $\mathfrak{T}_{\Upsilon}$-product in a hemiring by using the idea of hybrid $\mathfrak{T}_{\Upsilon}$-norm, and investigated some of their significant properties. In the future research, we could extend the $\mathfrak{T}_{\Upsilon}$-hybrid ideals of hemirings to $\mathfrak{T}_{\Upsilon}$-hybrid bi-ideals of hemirings.

Acknowledgment. The authors are grateful to the referees for their valuable comments and suggestions for improving the paper.

References

[1] Alsina C., Trillas E., Valverde L. *On some logical connectives for fuzzy set theory.* J. Math. Anal. Appl., 1983, vol. 93, no. 1, pp. 15 – 26. DOI: https://doi.org/10.1016/0022-247X(83)90216-0

[2] Elavarasan B., Muhiuddin G., Porselvi K., Jun Y. B. *Hybrid structures applied to ideals in near-rings.* Complex Intell. Syst., 2021, vol. 7, pp. 1489 – 1498. DOI: https://doi.org/10.1007/s40747-021-00271-7

[3] Jun Y. B., Song S. Z., Muhiuddin G. *Hybrid structures and applications.* Annals of Communications in Mathematics, 2018, vol. 1, no. 1, pp. 11 – 25.

[4] Jun Y. B., Oztürk M. A., Song S. Z. *On fuzzy h-ideals in hemirings.* Inform. Sci., 2004, vol. 162, pp. 211 – 226. DOI: https://doi.org/10.1016/j.ins.2003.09.007

[5] Henriksen M. *Ideals in semirings with commutative addition.* Amer. Math. Soc. Notices, 1958, vol. 6, pp. 321.

[6] La Torre D. R. *On h-ideals and k-ideals in hemirings.* Publ. Math. Debrecen, 1965, vol. 12, pp. 219 – 226.

[7] Lizuka K. *On the Jacobson radical of a semiring.* Tohoku Math. J., 1959, vol. 11, no. 3, pp. 409 – 421. DOI: https://doi.org/10.2748/tmj/1178244538

[8] Maji K., Roy A. R., Biswas R. *An application of soft sets in a decision making problem.* Comput. Math. with Appl., 2002, vol. 44, pp. 1077 – 1083. DOI: https://doi.org/10.1016/S0898-1221(02)00216-X

[9] Meenakshi S., Muhiuddin G., Elavarasan B., Al-Kadi D. *Hybrid ideals in near-subtraction semigroups.* AIMS Mathematics, 2022, vol. 7, no. 7, pp. 13493 – 13507. DOI: https://doi.org/10.3934/math.2022746

[10] Molodtsov D. *Soft set theory—first results.* Comput. Math. with Appl., 1999, vol. 37, no. 4-5, pp. 19 – 31. DOI: https://doi.org/10.1016/S0898-1221(99)00056-5

[11] Muhiuddin G., John J. C. G., Elavarasan B., Porselvi K., Al-kadi D. *Properties of k-hybrid ideals in ternary semiring.* J. Intell. Fuzzy Syst., 2022, vol. 42, no. 6, pp. 5799 – 5807. DOI: https://doi.org/10.3233/JIFS-212311

[12] Porselvi K., Elavarasan B. *On hybrid interior ideals in semigroups.* Probl. Anal. Issues Anal., 2019, vol. 8(26), no. 3, pp. 137 – 146.
DOI: https://doi.org/10.15393/j3.art.2019.6150

[13] Porselvi K., Muhiuddin G., Elavarasan B., Assiry A. *Hybrid nil radical of a ring.* Symmetry, 2022, vol. 14, no. 7, pp. 1367.
DOI: https://doi.org/10.3390/sym14071367

[14] Schweizer B., Sklar A. *Probabilistic metric spaces.* North-Holland Amsterdam, 1983.

[15] Vandiver H. S. *Note on a simple type of algebra in which cancellation law of addition does not hold.* Bull. Amer. Math. Soc., 1934, vol. 40, pp. 914 – 920.

[16] Zadeh L. A. *Fuzzy sets.* Inform Control, 1965, vol. 8, no. 3, pp. 338 – 353.
DOI: https://doi.org/10.1016/S0019-9958(65)90241-X

[17] Zhan J. *On properties of fuzzy left h-ideals in hemirings with t-norms.* Int. J. Math. Sci., 2005, vol. 19, pp. 3127 – 3144.
DOI: https://doi.org/10.1155/IJMMS.2005.3127

Received November 22, 2022.
In revised form, January 20, 2023.
Accepted February 24, 2023.
Published online April 12, 2023.

V. Keerthika
Department of Mathematics, Karunya Institute of Technology and Sciences
Coimbatore - 641114, Tamilnadu, India
Email: keerthika0613@gmail.com

G. Muhiuddin
Department of Mathematics, University of Tabuk
P.O. Box-741, Tabuk-71491, Saudi Arabia
Email: chishtygm@gmail.com

Mohamed E. Elnair
Department of Mathematics, Faculty of Science, University of Tabuk
P.O. Box 741, Tabuk 71491, Saudi Arabia
Email: abomunzir124@gmail.com

B. Elavarasan
Department of Mathematics, Karunya Institute of Technology and Sciences
Coimbatore - 641114, Tamilnadu, India
Email: belavarasan@gmail.com