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A CLASS OF HARMONIC (p, q)-STARLIKE FUNCTIONS INVOLVING A GENERALIZED (p, q)-BERNARDI INTEGRAL OPERATOR

Abstract. With the aid of q-calculus, this paper introduces a new generalized (p, q)-Bernardi integral operator $\mathcal{B}^p_{n,q} f(z)$. Then, we define a new subclass of harmonic (p, q)-starlike functions of complex order associated with the operator $\mathcal{B}^p_{n,q} f(z)$. For this new subclass, a necessary and sufficient condition, compact and convex combination theorems, a distortion theorem, and extreme points are investigated. Finally, we discuss the weight mean theorem for functions belonging to this class. This research highlights the significant connections between the results presented in this study and previous works.

Key words: *harmonic functions, q-calculus, (p, q)-Bernardi integral operator, distortion bounds, extreme points, convex combination*

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1. Main concepts of quantum calculus. The principle of quantum calculus (or q-calculus) has greatly influenced the study of the theory of geometric functions, and its significant applications in other areas, such as mathematical science and quantum physics. This principle is similar to traditional calculus but with no need for limits. The idea of the q-calculus, including the q-derivative and the q-integral, has been initially provided by Jackson [8]. With the expansion of the q-calculus study, many relevant facts have also been investigated, including the q-Gamma and q-Beta functions, the q-Laplace transform, the q-Taylor expansion, and the q-Mittag-Leffler function (for more review, see [5] and [6]). In the Geometric Function Theory, the q-calculus has been effectively applied in the studies of functions classes, which include the classes of univalent $\mathcal{S}$ and p-valent $\mathcal{S}(p)$ functions. Ismail et al. [7] have introduced q-calculus in the field and scope of geometric function theory; consequently, several Ma and Minda classes of analytic functions on the open unit disc have been developed, and these classes are closely related to the subordination concept. Furthermore, q-calculus operators, such as the fractional of q-integral and q-derivative operators, have been employed to establish various analytic functions. Additionally, numerous works have examined certain classes of functions that are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ using q-calculus (for example, see [14], [15], and [17]).

This work begins with the basic concepts and, consequently, an in-depth analysis of our proposed applications of the q-calculus. Throughout this paper, assume that $0 < q < 1$. The following definitions provide an introduction to the q-calculus operators for a complex-valued function f.

Let $\mathcal{S}(p)$ be the class of analytic and p-valent functions f in $\mathbb{U}$ with the normalized form:

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j, \quad (p \in \mathbb{N}). \tag{1}$$

Definition 1. *For $0 < q < 1$, the q-number $[\kappa]_q$ is given by*

$$[\kappa]_q = \begin{cases} \dfrac{1-q^\kappa}{1-q}, & (\kappa \in \mathbb{C}), \\ \sum\limits_{\kappa=0}^{n-1} q^\kappa, & (\kappa = n \in \mathbb{N}). \end{cases}$$

Definition 2. *The q-derivative operator $\mathfrak{D}_q$ is given by*

$$\mathfrak{D}_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \quad (z \neq 0).$$

The q-derivative of the function f in (1) is given by

$$\mathfrak{D}_q f(z) = [p]_q z^{p-1} + \sum_{j=p+1}^{\infty} [j]_q a_j z^{j-1}.$$

The q-factorial indicated by $[j]_q!$ is defined by

$$[j]_q! = \begin{cases} 1, & j = 0, \\ [j]_q [j-1]_q \ldots [2]_q [1]_q, & j = 1, 2, 3, \ldots, \end{cases}$$

so that

$$f'(z) := \lim_{q\to 1-} \mathfrak{D}_q \Big\{ [p]_q z^{p-1} + \sum_{j=p+1}^{\infty} [j]_q a_j z^{j-1} \Big\} =$$

$$= pz^{p-1} + \sum_{j=p+1}^{\infty} j a_j z^{j-1}.$$

Jackson [9] defined the q-integral of any function $f(z)$ as follows:

$$\int_0^y f(u) d_q u = y(1-q) \sum_{j=0}^{\infty} q^j f\left(yq^j\right).$$

Let $\varrho \in \mathcal{R}$ and $j \in \mathbb{N}$ be positive integers.
The q-generalized Pochhammer is given by

$$[\varrho; j]_q = [\varrho]_q [\varrho+1]_q [\varrho+2]_q \dots [\varrho+j-1]_q.$$

2. Harmonic functions, definitions and motivation. In the complex domain $\mathcal{D} \subset \mathbb{U}$, if the values u and v are real harmonic, then the continuous function $f = u + iv$ is called the harmonic function in $\mathcal{D}$. In any simply connected domain $\mathcal{D}$, the function f can be stated by

$$f = \mathcal{F} + \bar{\mathcal{G}}, \tag{2}$$

where both $\mathcal{F}$ and $\mathcal{G}$ are analytic functions in $\mathcal{D}$. The function $\mathcal{F}$ is called analytic of f, and $\mathcal{G}$ the conjugate-analytic (or co-analytic) of f. Clunie and Sheil-Small [2] discovered that $|\mathcal{F}'(z)| > |\mathcal{G}'(z)|$ is a necessary and sufficient condition for the harmonic functions (2) to be locally multivalent and sense-preserving in $\mathcal{D}$ (also, see [13]).

Let $\mathcal{H}(p,j)$ be the family of harmonic multivalent functions $f = \mathcal{F} + \bar{\mathcal{G}}$ that are orientation-keeping the open unit disc $\mathbb{U}$. The analytic functions $\mathcal{F}$ and $\mathcal{G}$ are defined by

$$\mathcal{F} = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad \text{and} \quad \mathcal{G} = \sum_{j=p}^{\infty} d_j z^j$$

and

$$f = \mathcal{F} + \bar{\mathcal{G}} = z^p + \sum_{j=p+1}^{\infty} a_j z^j + \sum_{j=p}^{\infty} \overline{d_j z^j}, \tag{3}$$

where $p \geqslant 1$ and $|d_p| < 1$.

The family $\mathcal{H}(1, j) = \mathcal{H}(j)$ of harmonic univalent functions has been presented by Jahangiri et al. [10] (also see [12]).

Also, we consider the subclass $\widetilde{\mathcal{H}}(p, j)$ of the family $\mathcal{H}(p, j)$ that consists of functions $f = \mathcal{F} + \bar{\mathcal{G}}$, where the functions $\mathcal{F}$ and $\mathcal{G}$ are defined as:

$$\mathcal{F}(z) = z^p - \sum_{j=p+1}^{\infty} |a_j| z^j \quad \text{and} \quad \mathcal{G}(z) = -\sum_{j=p}^{\infty} |d_j| z^j, \quad (|d_p| < 1). \quad (4)$$

Recently, many articles have concentrated on the study of the concept of multivalent harmonic functions and their applications (for example, see [1] and [3]).

If the analytic functions $f, h \in \mathcal{H}(p, j)$, then the function f is subordinate to the function h, denoted by $(f \prec h)$, if there exists a Schwarz function Φ with

$$\Phi(0) = 0, \ |\Phi(z)| < 1 \ (z \in \mathbb{U}),$$

such that

$$f(z) = h(\Phi(z)).$$

In addition, we get the following equivalence if the function h is univalent in $\mathbb{U}$:

$$f(z) \prec h(z) \Leftrightarrow f(0) = h(0) \text{ and } f(\mathbb{U}) \subset h(\mathbb{U}).$$

Many diverse subclasses of analytic functions, q-starlike functions, and symmetric q-starlike functions have been studied and analyzed by using q-analogous values of integral and derivative operators. The purpose of this paper is to present the generalized q-Bernardi integral operator for harmonic p-valent functions. Additionally, the paper defines a new subclass of the (p, q)-starlike functions in the open unit disc $\mathbb{U}$ by utilizing this operator. Moreover, the paper discusses the advantages and applications of various new geometric subclasses of (p, q)-starlike harmonic functions, including coefficient estimates, compactness, convex combination, extreme points, and distortion bounds are investigated. The weight mean theorem for functions belonging to this class is also studied.

More recently, Srivastava et al. [19] studied the generalized (p, q)-Bernardi integral operator for p-valent functions as follows:

Definition 3. *For $f(z) \in \mathcal{S}(p)$, the generalized (p,q)-Bernardi integral operator for p-valent functions $\mathcal{B}^p_{n,q}f(z)\colon \mathcal{S}(p) \to \mathcal{S}(p)$ is defined by*

$$\mathcal{B}^p_{n,q}f(z) := \begin{cases} \mathcal{B}^p_{1,q}\left(\mathcal{B}^p_{n-1,q}f(z)\right), & (n \in \mathbb{N}) \\ f(z), & (n = 0), \end{cases} \tag{5}$$

where $\mathcal{B}^p_{1,q}f(z)$ is given by

$$\begin{aligned} \mathcal{B}^p_{1,q}f(z) &= \frac{[p+\omega]_q}{z^\omega}\int_0^z t^{\omega-1}f(t)d_qt = \\ &= z^p + \sum_{j=p+1}^{\infty}\frac{[p+\omega]_q}{[j+\omega]_q}a_jz^j,\ (\omega > -p, z \in \mathbb{U}). \end{aligned} \tag{6}$$

From $\mathcal{B}^p_{1,q}f(z)$, we deduce that

$$\mathcal{B}^p_{2,q}f(z) = \mathcal{B}^p_{1,q}(\mathcal{B}^p_{1,q}f(z)) = z^p + \sum_{j=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^2 a_jz^j, \quad (\omega > -p) \tag{7}$$

and

$$\mathcal{B}^p_{n,q}f(z) = z^p + \sum_{j=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n a_jz^j, \quad (n \in \mathbb{N},\ \omega > -p). \tag{8}$$

Remark. *We illustrate the following particular cases:*

1) *If $n = 1$, we obtain the q-Bernardi integral operator for p-valent functions (see [14]).*
2) *When $q \to 1-$, we obtain the Bernardi integral operator (see [16]). Also we get the Srivastava-Attiya operator for p-valent functions (see [20]).*
3) *Taking $p = 1$, we have the q-Srivastava-Attiya operator (see [17]).*
4) *Taking $n = 1$ and $p = 1$, we get the q-Bernardi integral operator (see [15]).*
5) *Taking $\omega = 1$ and $q \to 1-$, we have the Jung-Kim-Srivastava operator for p-valent functions (see [4]).*

For $f(z) \in \mathcal{H}(p,j)$, we define the operator $\mathcal{B}^p_{n,q}f(z)$ as follows:

$$\mathcal{B}^p_{n,q}f(z) = \mathcal{B}^p_{n,q}\mathcal{F}(z) + \overline{\mathcal{B}^p_{n,q}\mathcal{G}(z)}, \tag{9}$$

where

$$\mathcal{B}^p_{n,q}\mathcal{F}(z) = z^p + \sum_{j=p+1}^{\infty} \left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n a_j z^j$$

and

$$\mathcal{B}^p_{n,q}\mathcal{G}(z) = \sum_{j=p}^{\infty} \left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n d_j z^j.$$

For the (p,q)-Bernardi integral operator $\mathcal{B}^p_{n,q}f(z)$ in (9), we introduce a new class $\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$ as follows:

Definition 4. *A multivalent function $f = \mathcal{F} + \bar{\mathcal{G}} \in \mathcal{H}(p,j)$ is said to be in the class $\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$ if*

$$\Upsilon = \mathcal{R}\left[p + \frac{1}{\eta_1}\left\{(1-\eta_2)\frac{\mathcal{B}^p_{n,q}f(z)}{z^p} + \eta_2 \frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q}f(z))}{[p]_q z^{p-1}} - 1\right\}\right] \geqslant \eta_3,\ (n \in \mathbb{N}, p \geqslant 1), \tag{10}$$

where $\eta_1 \in \mathbb{C}\backslash\{0\}, 0 \leqslant \eta_2 \leqslant 1, 0 \leqslant \eta_3 < 1$.

We also define

$$\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3) = \mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3) \bigcap \widetilde{\mathcal{H}}(p,j). \tag{11}$$

Remark. *Several special subclasses are being listed here:*

1) *If $p = 1$, we have a subclass $\mathcal{HF}_{1,q}(\eta_1,\eta_2,\eta_3)$ introduced by Shah et al. [18].*
2) *When $n = 0$, the class $\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$ becomes*

$$\mathcal{R}\left[p + \frac{1}{\eta_1}\left\{(1-\eta_2)\frac{f(z)}{z^p} + \eta_2 \frac{\mathfrak{D}_q(f(z))}{[p]_q z^{p-1}} - 1\right\}\right] \geqslant \eta_3.$$

3) *If $q \to 1-$, the class $\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$ reduces to the subclass*

$$\mathcal{R}\left[p + \frac{1}{\eta_1}\left\{(1-\eta_2)\frac{\mathcal{B}^p_{n,1}f(z)}{z^p} + \eta_2 \frac{(\mathcal{B}^p_{n,1}f(z))'}{p z^{p-1}} - 1\right\}\right] \geqslant \eta_3.$$

3. The Main Results. Firstly, we have to prove the necessary and sufficient condition of the class $\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$.

Theorem 1. *Let $f = \mathcal{F} + \overline{\mathcal{G}} \in \mathcal{H}(p,j)$ as in (3). Let the following inequality hold:*

$$\sum_{j=p+1}^{\infty}\left(1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2\right)\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n|a_j| + \\ +\sum_{j=p}^{\infty}\left(1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2\right)\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n|d_j| \leqslant (p-\eta_3)\eta_1, \quad (12)$$

where $n\in\mathbb{N}, \eta_1\in\mathbb{C}\backslash\{0\}, 0\leqslant\eta_2\leqslant 1$, and $0\leqslant\eta_3<1$. Then the harmonic function $f(z)$ is orientation preserving in $\mathbb{U}$ and $f(z)\in\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$.

Proof. To prove that $f(z)$ is orientation preserving, it suffices to show that $|\mathcal{F}'(z)| > |\mathcal{G}'(z)|$.

$$\mathfrak{D}_q(\mathcal{F}(z))| \geqslant [p]_q|z|^{p-1} - \sum_{j=p+1}^{\infty}[j]_q|a_j||z|^{j-1} > \\ > [p]_q - \sum_{j=p+1}^{\infty}[j]_q|a_j| \geqslant [p]_q - \sum_{j=p+1}^{\infty}\frac{\left(1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2\right)\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n}{(p-\eta_3)\eta_1}|a_j| \geqslant \\ \geqslant \sum_{j=p}^{\infty}\frac{\left(1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2\right)\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n}{(p-\eta_3)\eta_1}|d_j| > \sum_{j=p}^{\infty}[j]_q|d_j||z|^{j-1} = |\mathfrak{D}_q(\mathcal{G}(z))|.$$

Then

$$\lim_{q\to 1^-}|\mathfrak{D}_q(\mathcal{F}(z))| > |\mathfrak{D}_q(\mathcal{G}(z))| = |\mathcal{F}'(z)| > |\mathcal{G}'(z)|.$$

Let us demonstrate that $f(z)\in\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$. From the fact $\Upsilon\geqslant\eta_3$ if and only if $|p-\eta_3+\Upsilon|\geqslant|p+\eta_3-\Upsilon|$, we just need to prove that

$$\left|p-\eta_3+[p+\frac{1}{\eta_1}\left\{(1-\eta_2)\frac{\mathcal{B}^p_{n,q}f(z)}{z^p}+\eta_2\frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q}f(z))}{[p]_qz^{p-1}}-1\right\}\right]\right| \geqslant \\ \geqslant \left|p+\eta_3-\left[p+\frac{1}{\eta_1}\left\{(1-\eta_2)\frac{\mathcal{B}^p_{n,q}f(z)}{z^p}+\eta_2\frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q}f(z))}{[p]_qz^{p-1}}-1\right\}\right]\right|$$

and, equivalently,

$$\left|(2p-\eta_3)\eta_1+(1-\eta_2)\frac{\mathcal{B}^p_{n,q}f(z)}{z^p}+\eta_2\frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q}f(z))}{[p]_qz^{p-1}}-1\right|-$$
$$-\left|\eta_3\eta_1-(1-\eta_2)\frac{\mathcal{B}^p_{n,q}f(z)}{z^p}-\eta_2\frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q}f(z))}{[p]_qz^{p-1}}+1\right|\geqslant 0.$$

Now, using the function $\mathcal{B}^p_{n,q}f(z)$ in (9), we obtain

$$|(2p-\eta_3)\eta_1+(1-\eta_2)\frac{\mathcal{B}^p_{n,q}f(z)}{z^p}+\eta_2\frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q}f(z))}{[p]_qz^{p-1}}-1|-$$
$$-\left|\eta_1\eta_3-(1-\eta_2)\frac{\mathcal{B}^p_{n,q}f(z)}{z^p}-\eta_2\frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q}f(z))}{[p]_qz^{p-1}}+1\right|=$$
$$=\Bigg|(2p-\eta_3)\eta_1+(1-\eta_2)\Bigg\{1+\sum_{j=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n a_jz^{j-p}+\sum_{j=p}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n\overline{d_jz^{j-p}}\Bigg\}+$$
$$+\eta_2\Bigg\{1+\sum_{j=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n\frac{[j]_q}{[p]_q}a_jz^{j-p}+\sum_{j=p}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n\frac{[j]_q}{[p]_q}\overline{d_jz^{j-p}}\Bigg\}-1\Bigg|-$$
$$-\Bigg|\eta_1\eta_3+(1-\eta_2)\Bigg\{1+\sum_{j=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n a_jz^{j-p}+\sum_{j=p}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n\overline{d_jz^{j-p}}\Bigg\}+$$
$$+\eta_2\Bigg\{1+\sum_{j=p+1}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n\frac{[j]_q}{[p]_q}a_jz^{j-p}+\sum_{j=p}^{\infty}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n\frac{[j]_q}{[p]_q}\overline{d_jz^{j-p}}\Bigg\}-1\Bigg|;$$

this implies

$$\Bigg|(2p-\eta_3)\eta_1+\Bigg\{\sum_{j=p+1}^{\infty}\left[(1-\eta_2)+\eta_2\frac{[j]_q}{[p]_q}\right]\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n a_jz^{j-p}+$$
$$+\sum_{j=p}^{\infty}\left[(1-\eta_2)+\eta_2\frac{[j]_q}{[p]_q}\right]\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n\overline{d_jz^{j-p}}\Bigg\}\Bigg|-$$
$$-\Bigg|\eta_3\eta_1+\Bigg\{\sum_{j=p+1}^{\infty}\left[(1-\eta_2)+\eta_2\frac{[j]_q}{[p]_q}\right]\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n a_jz^{j-p}+$$
$$+\sum_{j=p}^{\infty}\left[(1-\eta_2)+\eta_2\frac{[j]_q}{[p]_q}\right]\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n\overline{d_jz^{j-p}}\Bigg\}\Bigg|\geqslant$$
$$\geqslant(2p-\eta_3)\eta_1-\sum_{j=p+1}^{\infty}\left[(1-\eta_2)+\eta_2\frac{[j]_q}{[p]_q}\right]\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n|a_j||z|^{j-p}-$$

$$-\sum_{j=p}^{\infty}\Big[(1-\eta_2)+\eta_2\frac{[j]_q}{[p]_q}\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n|d_j||z|^{j-p}-$$
$$-\eta_3\eta_1-\sum_{j=p+1}^{\infty}\Big[(1-\eta_2)+\eta_2\frac{[j]_q}{[p]_q}\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n|a_j||z|^{j-p}-$$
$$-\sum_{j=p}^{\infty}\Big[(1-\eta_2)+\eta_2\frac{[j]_q}{[p]_q}\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n|d_j||z|^{j-p}=$$
$$=2(p-\eta_3)\eta_1\Big\{1-\sum_{j=p+1}^{\infty}\Big[1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n|a_j||z|^{j-p}-$$
$$-\sum_{j=p}^{\infty}\Big[1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n|d_j||z|^{j-p}\Big\}.$$

The last expression is non-negative. Then $f(z)\in\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$. For the following harmonic function, the coefficient bound (12) is sharp.

$$f(z)=z^p+\sum_{j=p+1}^{\infty}\frac{(p-\eta_3)\eta_1}{[1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2]}\Big(\frac{[j+\omega]_q}{[p+\omega]_q}\Big)^n\varkappa_{1,j}z^j+$$
$$+\sum_{j=p}^{\infty}\frac{(p-\eta_3)\eta_1}{[1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2]}\Big(\frac{[j+\omega]_q}{[p+\omega]_q}\Big)^n\bar{\varkappa}_{2,j}\bar{z}^j,$$

with $\sum\limits_{j=p+1}^{\infty}|\varkappa_{1,j}|+\sum\limits_{j=p}^{\infty}|\varkappa_{2,j}|=1.$ □

If $n=0$, the following corollary is deduced:

Corollary. *Let $f=\mathcal{F}+\bar{\mathcal{G}}\in\mathcal{H}(p,j)$ as in (3). If the following inequality holds:*

$$\sum_{j=p+1}^{\infty}\Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)|a_j|+$$
$$+\sum_{j=p}^{\infty}\Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)|d_j|\leqslant(p-\eta_3)\eta_1,\quad(13)$$

where $\eta_1\in\mathbb{C}\backslash\{0\}, 0\leqslant\eta_2\leqslant1,$ and $0\leqslant\eta_3<1$, then the harmonic function $f(z)\in\mathcal{HF}_{p,q}(\eta_1,\eta_2,\eta_3)$.

When $q\to1-$, Theorem 1 becomes as below:

Corollary. *Let* $f = \mathcal{F} + \bar{\mathcal{G}} \in \mathcal{H}(p, j)$ *as in* (3)*. If the following inequality holds:*

$$\sum_{j=p+1}^{\infty} \left(1 + \left(\frac{j}{p} - 1\right)\eta_2\right)\left(\frac{p+\omega}{j+\omega}\right)^n |a_j| + \\ + \sum_{j=p}^{\infty} \left(1 + \left(\frac{j}{p} - 1\right)\eta_2\right)\left(\frac{p+\omega}{j+\omega}\right)^n |d_j| \leqslant (p - \eta_3)\eta_1, \quad (14)$$

where $n \in \mathbb{N}, \eta_1 \in \mathbb{C}\backslash\{0\}, 0 \leqslant \eta_2 \leqslant 1$, *and* $0 \leqslant \eta_3 < 1$, *then the harmonic function* $f(z) \in \mathcal{HF}_{p,q}(\eta_1, \eta_2, \eta_3)$.

Next, we prove that the inequality condition is necessary for p-valent functions $f = \mathcal{F} + \bar{\mathcal{G}} \in \widetilde{\mathcal{H}}(p, j)$ in the subclass $\widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3)$.

Theorem 2. *Let* $f = \mathcal{F} + \bar{\mathcal{G}} \in \widetilde{\mathcal{H}}(p, j)$ *as in* (4)*. Then the harmonic functions* $f(z) \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3)$ *if and only if*

$$\sum_{j=p+1}^{\infty} \left(1 + \left(\frac{[j]_q}{[p]_q} - 1\right)\eta_2\right)\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n |a_j| + \\ + \sum_{j=p}^{\infty} \left(1 + \left(\frac{[j]_q}{[p]_q} - 1\right)\eta_2\right)\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n |d_j| \leqslant (p - \eta_3)\eta_1, \quad (15)$$

where $n \in \mathbb{N}, \eta_1 \in \mathbb{C}\backslash\{0\}, 0 \leqslant \eta_2 \leqslant 1$, *and* $0 \leqslant \eta_3 < 1$.

Proof. Since $\widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3) \subset \mathcal{HF}_{p,q}(\eta_1, \eta_2, \eta_3)$, the sufficient condition holds due to Theorem 1. Now, we have to prove the necessity condition. Suppose that $f \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3)$; then, due to (11),

$$\mathcal{R}\left[p + \frac{1}{\eta_1}\left\{(1-\eta_2)\frac{\mathcal{B}^p_{n,q} f(z)}{z^p} + \eta_2 \frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q} f(z))}{[p]_q z^{p-1}} - 1\right\} - \eta_3\right] \geqslant 0. \quad (16)$$

yields. Also, equivalently,

$$\mathcal{R}\left[(p-\eta_3)\eta_1 + \left\{(1-\eta_2)\frac{\mathcal{B}^p_{n,q} f(z)}{z^p} + \eta_2 \frac{\mathfrak{D}_q(\mathcal{B}^p_{n,q} f(z))}{[p]_q z^{p-1}} - 1\right\}\right] \geqslant 0. \quad (17)$$

Using the function $\mathcal{B}^p_{n,q} f(z)$ in (9), we conclude that

$$\mathcal{R}[(p-\eta_3)\eta_1 + (1-\eta_2)\Big\{1 - \sum_{j=p+1}^{\infty} \left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n |a_j| z^{j-p} -$$

$$-\sum_{j=p}^{\infty}\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n|d_j|\bar{z}^{j-p}\Big\}+\eta_2\Big\{1-\sum_{j=p+1}^{\infty}\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n\frac{[j]_q}{[p]_q}|a_j|z^{j-p}- \\ -\sum_{j=p}^{\infty}\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n\frac{[j]_q}{[p]_q}|d_j|\bar{z}^{j-p}\Big\}-1\Big]\geqslant 0.$$

This yields

$$\mathcal{R}[(p-\eta_3)\eta_1-\sum_{j=p+1}^{\infty}[1+(\frac{[j]_q}{[p]_q}-1)\eta_2](\frac{[p+\omega]_q}{[j+\omega]_q})^n|a_j|z^{j-p}- \\ -\sum_{j=p}^{\infty}[1+(\frac{[j]_q}{[p]_q}-1)\eta_2](\frac{[p+\omega]_q}{[j+\omega]_q})^n|d_j|\bar{z}^{j-p}]\geqslant 0. \quad (18)$$

Note that the desired condition (18) must hold for all values z in $\mathbb{U}$. Selecting the values z on the positive real axis, where $0\leqslant z=r<1$, we get

$$(p-\eta_3)\eta_1-\sum_{j=p+1}^{\infty}\Big[1+(\frac{[j]_q}{[p]_q}-1)\eta_2\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n|a_j|r^{j-p}- \\ -\sum_{j=p}^{\infty}\Big[1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n|d_j|r^{j-p}\geqslant 0. \quad (19)$$

When $r\to 1$ and if the condition (12) is not satisfied, the inequality (19) is also not satisfied. In the range $(0,1)$, we may thus identify at least one $z_0=r_0$ for which the quotient (19) is greater than 1. This conflicts with the prerequisite for $f\in\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$, so we obtain the condition (12). □

Lemma 1. *[11] The class $\mathcal{E}\subset\mathcal{H}(p,j)$ is compact if and only if $\mathcal{E}$ is closed and locally uniformly bounded.*

Theorem 3. *The subclass $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$ is a convex and compact subclass of the family of functions $f=\mathcal{F}+\overline{\mathcal{G}}\in\widetilde{\mathcal{H}}(p,j)$, where $\mathcal{F}$ and $\mathcal{G}$ have been mentioned in (4).*

Proof. Let $f_i(z)\in\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$ be defined as

$$f_i(z)=z^p-\sum_{j=p+1}^{\infty}|a_{i,j}|\,z^j-\sum_{j=p}^{\infty}\overline{|d_{i,j}|\,z^j},\quad (i=1,2). \quad (20)$$

Then

$$\mathcal{J}(z) = \daleth f_{1,j}(z) + (1-\daleth) f_{2,j}(z) =$$
$$= z^p - \sum_{j=p+1}^{\infty} (\daleth\,|a_{1,j}| + (1-\daleth)|a_{2,j}|) z^j - \sum_{j=p}^{\infty} (\daleth\,|d_{1,j}| + (1-\daleth)|d_{2,j}|)\bar{z}^j,$$

also belongs to the subclass $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$ for $0 \leqslant \daleth \leqslant 1$.

By the result of Theorem 2, we obtain

$$\daleth[\sum_{j=p+1}^{\infty} [1 + (\frac{[j]_q}{[p]_q} - 1)\eta_2](\frac{[p+\omega]_q}{[j+\omega]_q})^n |a_{1,j}| +$$
$$+ \sum_{j=p}^{\infty} \Big[1 + \Big(\frac{[j]_q}{[p]_q} - 1\Big)\eta_2\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |d_{1,j}|] +$$
$$+ (1-\daleth)\Big[\sum_{j=p+1}^{\infty} [1 + \Big(\frac{[j]_q}{[p]_q} - 1\Big)\eta_2]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |a_{2,j}| +$$
$$+ \sum_{j=p}^{\infty} \Big[1 + \Big(\frac{[j]_q}{[p]_q} - 1\Big)\eta_2\Big]\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |d_{2,j}|\Big] \leqslant$$
$$\leqslant \daleth((p-\eta_3)\eta_1) + (1-\daleth)((p-\eta_3)\eta_1) = (p-\eta_3)\eta_1.$$

As $\mathcal{J}(z) \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$, $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$ is a convex set.

However, if $f_i(z) \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3), (i \in \mathbb{N})$ as in the form (20), we get, using Theorem 2:

$$\sum_{j=p+1}^{\infty} \Big(1 + \Big(\frac{[j]_q}{[p]_q} - 1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |a_{i,j}| +$$
$$+ \sum_{j=p}^{\infty} \Big(1 + \Big(\frac{[j]_q}{[p]_q} - 1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |d_{i,j}| \leqslant (p-\eta_3)\eta_1. \quad (21)$$

Then, for $|z| \leqslant r$ $(0 < r < 1)$,

$$|f_i(z)| \geqslant (1 - |d_{i,p}|)r^p - \sum_{j=p+1}^{\infty} (|a_{i,j}| + |d_{i,j}|)r^j \geqslant$$
$$\geqslant (1 - |d_{i,p}|)r^p - \frac{(p-\eta_3)\eta_1([p+1+\omega]_q)^n}{[1 + (\frac{[p+1]_q}{[p]_q} - 1)\eta_2]([p+\omega]_q)^n} \times$$

$$\times \sum_{j=p+1}^{\infty} \Big\{ \frac{\Big[1+\Big(\frac{[p+1]_q}{[p]_q}-1\Big)\eta_2\Big]([p+\omega]_q)^n}{(p-\eta_3)\eta_1([p+1+\omega]_q)^n} (|a_{i,j}|+|d_{i,j}|) \Big\} r^{p+1} \geqslant$$

$$\geqslant (1-|d_{i,p}|)r^p - \frac{(p-\eta_3)\eta_1([p+1+\omega]_q)^n}{[1+(\frac{[p+1]_q}{[p]_q}-1)\eta_2]([p+\omega]_q)^n} \times$$

$$\times \sum_{j=p+1}^{\infty} \{ \frac{\Big[1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big]([p+\omega]_q)^n}{(p-\eta_3)\eta_1([j+\omega]_q)^n} (|a_{i,j}|+|d_{i,j}|) \} r^{p+1} \geqslant$$

$$\geqslant (1-|d_{i,p}|)r^p - \frac{(p-\eta_3)\eta_1([p+1+\omega]_q)^n}{\Big[1+\Big(\frac{[p+1]_q}{[p]_q}-1\Big)\eta_2\Big]([p+\omega]_q)^n} \times$$

$$\times \Big\{ 1 - \frac{|d_{i,p}|}{(p-\eta_3)\eta_1} \Big\} r^{p+1} \geqslant (1-|d_{i,p}|)r^p - \mathcal{A}_p r^{p+1},$$

where

$$\mathcal{A}_p := \Big(\frac{[p+1+\omega]_q}{[p+\omega]_q}\Big)^n \Big[\frac{(p-\eta_3)\eta_1}{1+\Big(\frac{[p+1]_q}{[p]_q}-1\Big)\eta_2} - \frac{1}{1+\Big(\frac{[p+1]_q}{[p]_q}-1\Big)\eta_2} |d_{i,p}| \Big].$$

Similarly, we have

$$|f_i(z)| \leqslant (1+|d_{i,p}|)r^p + \mathcal{A}_p r^{p+1}. \tag{22}$$

Then the subclass $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$ is locally uniformly bounded. If we let $f_i \to f$, then we deduce that $|a_{i,j}| \to |a_j|$ and $|d_{i,j}| \to |d_j|$ as $i \to \infty$, for $j = 2, 3, \ldots$. We find from (21) that

$$\sum_{j=p+1}^{\infty} \Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |a_j| +$$

$$+ \sum_{j=p}^{\infty} (1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |d_j| \leqslant (p-\eta_3)\eta_1. \tag{23}$$

As a result, $f(z) \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$, and the class $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$ is also closed. This shows the compactness of the class $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$. □

Corollary. *For $|z| = r < 1$, if $f \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$, then*

$$(1-|d_p|)\, r^p - \mathcal{A}_p r^{p+1} \leqslant |f(z)| \leqslant (1+|d_p|)\, r^p + \mathcal{A}_p r^{p+1}.$$

Moreover,

$$\{u\colon |u| < (1-Q_1) - (1-Q_2)\,|d_p|\} \subset f(\mathbb{U}),$$

where $Q_1 := \frac{(p-\eta_3)\eta_1([p+1+\omega]_q)^n}{\{1+(\frac{[p+1]_q}{[p]_q}-1)\eta_2\}([p+\omega]_q)^n}$ *and* $Q_2 := \frac{([p+1+\omega]_q)^n}{\{1+(\frac{[p+1]_q}{[p]_q}-1)\eta_2\}([p+\omega]_q)^n}$.

If $p = 1$, we obtain the following results:

Corollary. [18] *For* $|z| = r < 1$, *if* $f \in \widetilde{\mathcal{HF}}_{1,q}(\eta_1, \eta_2, \eta_3)$, *then*

$$(1-|d_1|)\,r - \mathcal{T}r^2 \leqslant |f(z)| \leqslant (1+|d_1|)\,r + \mathcal{T}r^2,$$

where

$$\mathcal{T} := \Big(\frac{[2+\omega]_q}{[1+\omega]_q}\Big)^n\Big[\frac{(1-\eta_3)\eta_1}{1+([2]_q-1)\eta_2} - \frac{1}{1+([2]_q-1)\eta_2}\,|d_1|\Big].$$

Moreover,

$$\Big\{u\colon |u| < (1-\widetilde{Q}_1) - (1-\widetilde{Q}_2)\,|d_p|\Big\} \subset f(\mathbb{U}),$$

where $\widetilde{Q}_1 := \frac{(1-\eta_3)\eta_1([2+\omega]_q)^n}{\{1+([2]_q-1)\eta_2\}([1+\omega]_q)^n}$ *and* $\widetilde{Q}_2 := \frac{([2+\omega]_q)^n}{\{1+([2]_q-1)\eta_2\}([1+\omega]_q)^n}$.

Theorem 4. *Let* $f_s(z) = z^p - \sum\limits_{j=p+l}^{\infty} |a_{j,s}|z^j - \sum\limits_{j=p+l}^{\infty} |d_{j,s}|\bar{z}^j$ $(s = 1, 2, \ldots)$ *be in the subfamily* $\widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3)$. *Then the function*

$$\mathcal{J}(z) = \sum_{s=1}^{m} \kappa_s f_s(z),\ \ (\kappa_s \geqslant 0, \sum_{s=1}^{\infty} \kappa_s = 1)$$

also belongs to the subclass $\widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3)$. *This means that* $\widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3)$ *is closed under the convex combination.*

Proof. Since $f_s(z) \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3)$, then

$$\sum_{j=p+1}^{\infty} \Big(1 + \Big(\frac{[j]_q}{[p]_q} - 1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |a_{j,s}| + \\ + \sum_{j=p}^{\infty} \Big(1 + \Big(\frac{[j]_q}{[p]_q} - 1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n |d_{j,s}| \leqslant (p-\eta_3)\eta_1.$$

Now,

$$\mathcal{J}(z) = z^p - \sum_{j=p+1}^{\infty} (\sum_{s=1}^{\infty} \kappa_s |a_{j,s}|) z^j - \sum_{j=p}^{\infty} (\sum_{s=1}^{\infty} \kappa_s |d_{j,s}|) \bar{z}^j. \tag{24}$$

By Theorem 2, we conclude that

$$\sum_{j=p+1}^{\infty}(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n\Big(\sum_{s=1}^{\infty}\kappa_s\,|a_{j,s}|\Big)+$$
$$+\sum_{j=p}^{\infty}\Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n\Big(\sum_{s=1}^{\infty}\kappa_s\,|d_{j,s}|\Big)\times$$
$$\times\sum_{s=1}^{\infty}\kappa_s\Big\{\sum_{j=p+1}^{\infty}(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n(|a_{j,s}|)+$$
$$+\sum_{j=p}^{\infty}(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n(|d_{j,s}|)\Big\}\leqslant\sum_{s=1}^{\infty}\kappa_s(p-\eta_3)\eta_1=(p-\eta_3)\eta_1.$$

Hence, $\mathcal{J}\in\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$. □

Next, we state the extreme points of closed convex hulls of the class $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$.

Theorem 5. *The function $f(z)\in\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$ if and only if*

$$f(z)=\sum_{j=p}^{\infty}(\phi_j\mathcal{F}_j+\varphi_j\mathcal{G}_j),\tag{25}$$

where $\mathcal{F}_p=z^p$,

$$\mathcal{F}_j=z^p-\frac{(p-\eta_3)\eta_1}{1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2}\Big(\frac{[j+\omega]_q}{[p+\omega]_q}\Big)^n z^j;\quad (j=p+1,p+2,\dots)$$

and

$$\mathcal{G}_j=z^p-\frac{(p-\eta_3)\eta_1}{1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2}\Big(\frac{[j+\omega]_q}{[p+\omega]_q}\Big)^n \bar{z}^j;\quad (j=p,p+1,\dots),$$

with $\sum_{j=p}^{\infty}(\phi_j+\varphi_j)=1$, $\phi_j\geqslant 0$, and $\varphi_j\geqslant 0$. Particularly, the extreme points of the class $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$ are $\{\mathcal{F}_j\}$ and $\{\mathcal{G}_j\}$.

Proof. Assume that $f(z)$ is given by

$$\begin{aligned} f(z) &= \sum_{j=p}^{\infty}(\phi_j\mathcal{F}_j + \varphi_j\mathcal{G}_j) = \\ &= \sum_{j=p}^{\infty}(\phi_j + \varphi_j)z^p - \sum_{j=p+1}^{\infty}\frac{(p-\eta_3)\eta_1}{1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2}\left(\frac{[j+\omega]_q}{[p+\omega]_q}\right)^n\phi_j z^j - \\ &\quad - \sum_{j=p}^{\infty}\frac{(p-\eta_3)\eta_1}{1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2}\left(\frac{[j+\omega]_q}{[p+\omega]_q}\right)^n\varphi_j \bar{z}^j. \quad (26)\end{aligned}$$

From the above function, we find that

$$|a_j| = \frac{(p-\eta_3)\eta_1}{1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2}\left(\frac{[j+\omega]_q}{[p+\omega]_q}\right)^n\phi_j$$

and

$$|d_j| = \frac{(p-\eta_3)\eta_1}{1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2}\left(\frac{[j+\omega]_q}{[p+\omega]_q}\right)^n\varphi_j.$$

Now

$$\begin{aligned} &\sum_{j=p+1}^{\infty}\frac{1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2}{(p-\eta_3)\eta_1}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n|a_j| + \\ &+\sum_{j=p}^{\infty}\frac{1+(\frac{[j]_q}{[p]_q}-1)\eta_2}{(p-\eta_3)\eta_1}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n|d_j| = \sum_{j=p}^{\infty}(\phi_j+\varphi_j)-\phi_p = 1-\phi_p \leqslant 1. \quad (27)\end{aligned}$$

This leads to the result $f \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$.

Conversely: Let $f \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$; assume that

$$\phi_j = \frac{1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2}{(p-\eta_3)\eta_1}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n|a_j|; \quad (j = p+1,\, p+2, \ldots)$$

and

$$\varphi_j = \frac{1+\left(\frac{[j]_q}{[p]_q}-1\right)\eta_2}{(p-\eta_3)\eta_1}\left(\frac{[p+\omega]_q}{[j+\omega]_q}\right)^n|d_j|; \quad (j = p,\ p+1, \ldots),$$

with $\sum\limits_{j=p}^{\infty}(\phi_j+\varphi_j)=1$.

We get $f(z)=\sum\limits_{j=p}^{\infty}(\phi_j\mathcal{F}_j+\varphi_j\mathcal{G}_j)$ by substituting the values of $|a_j|$ and $|d_j|$ from the previous relations to (4). □

Theorem 6. *For $i=1{,}2$, let $f_i\in\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$. Then, for any real number k, the weighted mean $u_k(z)$ defined by*

$$u_k(z)=\left\{\frac{(1-k)f_1(z)+(1+k)f_2(z)}{2}\right\}, \tag{28}$$

also belongs to $\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$.

Proof. From the value of $u_k(z)$, we can rewrite

$$u_k(z)=z^p-\sum_{j=p+1}^{\infty}\frac{(1-k)a_{j,1}+(1+k)a_{j,2}}{2}z^j- \\ -\sum_{j=p}^{\infty}\frac{(1-k)\overline{d_{j,1}}+(1+k)\overline{d_{j,2}}}{2}\bar{z}^j.$$

In order to demonstrate that $u_k(z)\in\widetilde{\mathcal{HF}}_{p,q}(\eta_1,\eta_2,\eta_3)$, we consider

$$\sum_{j=p+1}^{\infty}\Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n\Big|\frac{(1-k)a_{j,1}+(1+k)a_{j,2}}{2}\Big|+ \\ +\sum_{j=p}^{\infty}\Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n\Big|\frac{(1-k)\overline{d_{j,1}}+(1+k)\overline{d_{j,2}}}{2}\Big|= \\ =\sum_{j=p+1}^{\infty}\Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n\Big(\frac{(1-k)}{2}|a_{j,1}|+\frac{(1-k)}{2}|d_{j,1}|\Big)+ \\ \sum_{j=p}^{\infty}\Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n\Big(\frac{(1+k)}{2}|a_{j,2}|+\frac{(1+k)}{2}|d_{j,2}|\Big)= \\ =\frac{(1-k)}{2}\sum_{j=p+1}^{\infty}\Big(1+\Big(\frac{[j]_q}{[p]_q}-1\Big)\eta_2\Big)\Big(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n(|a_{j,1}|+|d_{j,1}|)+ \\ +\frac{(1+k)}{2}\sum_{j=p}^{\infty}(1+(\frac{[j]_q}{[p]_q}-1)\eta_2)(\frac{[p+\omega]_q}{[j+\omega]_q}\Big)^n(|a_{j,2}|+|d_{j,2}|)\leqslant$$

$$\leqslant \frac{(1-k)}{2}(p-\eta_3)\eta_1 + \frac{(1+k)}{2}(p-\eta_3)\eta_1 = (p-\eta_3)\eta_1.$$

Therefore, from Theorem 2, we get $u_k(z) \in \widetilde{\mathcal{HF}}_{p,q}(\eta_1, \eta_2, \eta_3)$. □

Concluding Remarks. In this article, we have introduced a new subclass of (p, q)-starlike functions of complex order using the generalized (p, q)-Bernardi integral operator for harmonic p-valent functions. Then we verified that the class is harmonic p-valent and sense-preserving in the open unit disc. For this class, which is formed with the aid of q-starlike functions, we have discussed the necessary and sufficient conditions. Additionally, we have examined the distortion bounds, topological properties, extreme points, and some important properties. This study will guide further papers and illuminate new concepts in the field of geometric function theory.

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