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WEIGHTED INTEGRABILITY RESULTS FOR FIRST HANKEL-CLIFFORD TRANSFORM

Abstract. We obtain sufficient conditions for the weighted integrability of the first Hankel-Clifford transforms of functions from generalized integral Lipschitz classes. These conditions are analogues and generalization of well-known Titchmarsh conditions for the classical Fourier transform.

Key words: *first Hankel-Clifford transform, Hankel-Clifford translation, generalized Lipschitz spaces, weighted integrability*

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1. Introduction. Let $f\colon \mathbb{R} \to \mathbb{C}$ be an integrable function in Lebesgue's sense over $\mathbb{R}$ ($f \in L^1(\mathbb{R})$). Then the Fourier transform of f is defined by

$$\widehat{f}(x) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}} f(t) e^{-itx}\, dt, \quad x \in \mathbb{R}.$$

In the case $1 < p \leqslant 2$, the Fourier transform of a function $f \in L^p(\mathbb{R})$ is defined as the limit of $(2\pi)^{-1/2} \int\limits_a^b f(x) e^{-itx}\, dx$ in the norm of $L^q(\mathbb{R})$, where $1/p + 1/q = 1$ and $a \to -\infty$, $b \to +\infty$.

In particular, $\widehat{f} \in L^q(\mathbb{R})$ and the following Hausdorff-Young inequality:

$$\|\widehat{f}\|_q \leqslant C\|f\|_p := C\Big(\int\limits_{\mathbb{R}} |f(t)|^p\, dt\Big)^{1/p}, \quad f \in L^p(\mathbb{R}), \quad 1 < p \leqslant 2, \tag{1}$$

holds. For $p = 2$, the inequality in (1) is substituted by the Plancherel equality. More about these results can be found in [19, Ch. III and IV] or [3, Ch. 5].

For $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, we consider the modulus of smoothness of order $k \in \mathbb{N}$

$$\omega_k(t, \delta)_p = \sup_{0\leqslant h\leqslant\delta} \|\mathring{\Delta}_h^k f\|_p, \quad \mathring{\Delta}_h^k f(x) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + (k-2j)h/2).$$

The following result of Titchmarsh is well known (see [19, Ch. 4, Theorem 84]):

Theorem 1. *Let $1 < p \leqslant 2$, $0 < \alpha \leqslant 1$, $f \in Lip(\alpha, p)$. Then $\widehat{f}(t) \in L^\beta(\mathbb{R})$ for all β satisfying the inequality*

$$\frac{p}{p + ap - 1} < \beta \leqslant q = \frac{p}{p-1}.$$

We will write that a non-negative measurable function $\lambda(t) \in L^1_{loc}(\mathbb{R}_+)$ belongs to the class A_γ, $\gamma \geqslant 1$, if there exists $C(\gamma) \geqslant 1$, such that

$$\Big(\int_{2^i}^{2^{i+1}} \lambda^\gamma(t)dt\Big)^{1/\gamma} \leqslant C(\gamma) 2^{i(1-\gamma)/\gamma} \int_{2^{i-1}}^{2^i} \lambda(t)dt, \quad i \in \mathbb{Z}. \tag{2}$$

By the Hölder inequality, it is easy to see that $A_{\gamma_1} \subset A_{\gamma_2}$ for $1 \leqslant \gamma_2 < \gamma_1$. It is proved in [8], that this embedding is strict. It is clear that a measurable function $\lambda(t) \geqslant 0$ with the property

$$\sup\{\lambda(t)\colon 2^i \leqslant t < 2^{i+1}\} \leqslant c \inf\{\lambda(t)\colon 2^{i-1} \leqslant t < 2^i\}, \quad i \in \mathbb{Z}$$

is contained in all classes A_γ, $\gamma \geqslant 1$. Further, we assume that $\lambda(t) = \lambda(-t)$ for $t > 0$.

An analogue of (2) for sequences was introduced by Gogoladze and Meskhia [6]. The condition (2) was suggested by Móricz [13], who proved the following result:

Theorem 2. *Let $1 < p \leqslant 2$ and $f \in L^p(\mathbb{R})$. If $1/p + 1/q = 1$, $0 < r < q$, and $\lambda \in A_{p/(p-rp+r)}$, then*

$$\int_{|t|\geqslant 2} \lambda(t) |\widehat{f}(t)|^r dt \leqslant \int_1^\infty \lambda(t) t^{-r/q} \omega^r(f, \pi/t)_p \, dt.$$

A more general result and the proof of its sharpness may be found in [8].

The aim of this paper is to obtain an analogue and generalization of Theorem 2 for the first Hankel-Clifford transform. Also, we estimate the rate of convergence of the corresponding integral. We generalize some results of Lahmadi, El Hamma and Mahfoud from [9] and [5]. Note that in [10] a less general results than in [5] are obtained. Some analogues of Theorem 1 for the Fourier-Bessel (or Hankel) transform were proved by Platonov [16]. Titchmarsh-type conditions for integrability of Fourier-Jacobi transforms and its generalization to Sobolev-Nikol'skii type spaces can be found in [4]. Analogues of Theorem 2 for Fourier-Dunkl and Fourier-Jacobi transforms can be found in [20] and [22].

2. Definitions. Let $1 \leqslant p < \infty$, $\mu \geqslant 0$, and $L^p_\mu(\mathbb{R}_+)$ be the space of all measurable real-valued functions with $\|f\|_{L^p_\mu} = \big(\int\limits_0^\infty |f(x)|^p x^\mu\, dx\big)^{1/p} < \infty$. If χ_E is the indicator of a set $E \subset \mathbb{R}_+$ and $f\chi_E \in L^p_\mu(\mathbb{R}_+)$, then $f \in L^p_\mu(E)$. By $L^\infty_\mu(\mathbb{R}_+)$ we denote the space of bounded functions with the norm $\|f\|_{L^\infty_\mu} = \|f\|_\infty = \sup_{x\in\mathbb{R}_+} |f(x)|$.

The Bessel-Clifford function of the first kind of order $\mu \geqslant 0$ (see, e.g., [7]) is defined by

$$c_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu + k + 1)},$$

where $\Gamma(\alpha)$ is the Euler gamma function, and $c_\mu(x)$ is a solution of the differential equation

$$x\frac{d^2y}{dx^2} + (\mu + 1)\frac{dy}{dx} + y = 0.$$

Let $j_\nu(x)$ be the normalized Bessel function of the first kind and order $\nu > -\frac{1}{2}$, given by

$$j_\nu(x) = \Gamma(\nu + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \nu + 1)} (x/2)^{2n}.$$

Then c_μ and j_μ are connected by

$$c_\mu(x) = \Gamma^{-1}(\mu + 1) j_\mu(2\sqrt{x}), \quad x \geqslant 0. \tag{3}$$

If $\mu \geqslant 0$, then the first Hankel-Clifford transform $h_{1,\mu}(f)$ is defined for appropriate functions f by

$$h_{2,\mu}(f)(y) = y^{\mu} \int_0^{+\infty} c_{\mu}(\lambda x) f(x)\, dx$$

(see [12]). In [12] it is proved that the space $H_{\mu}(\mathbb{R}_+)$ consisting of all infinitely differentiable functions $\varphi(x)$ defined on $\mathbb{R}_+$, such that for all $k, l \in \mathbb{Z}_+ = \{0, 1, \ldots\}$

$$\sup_{x\in\mathbb{R}_+} \left| \frac{x^l d^k (x^{-\mu}\varphi(x))}{dx^k} \right| < \infty$$

is invariant under the operator $h_{1,\mu}$.

Further we use the operator $M_{\mu} f(x) = x^{\mu} f(x)$ and its inverse M_{μ}^{-1}. Since $|c_{\mu}(x)| \leqslant \Gamma^{-1}(\mu + 1)$ for $x \in \mathbb{R}$ (see Lemma 2), the inequality

$$\|(\cdot)^{-\mu} h_{1,\mu}(f)(\cdot)\|_{L^{\infty}_{\mu}} \leqslant \Gamma^{-1}(\mu + 1)\|f\|_{L^1} = \Gamma^{-1}(\mu + 1)\|(\cdot)^{-\mu} f(\cdot)\|_{L^1_{\mu}} \quad (4)$$

holds for $f \in L^1_{\mu}(\mathbb{R}_+)$, i.e., $\|M_{\mu}^{-1} h_{1,\mu}(f)\|_{L^{\infty}_{\mu}} \leqslant \Gamma^{-1}(\mu+1)\|M_{\mu}^{-1} f\|_{L^1_{\nu}}$. In [12] it is noted that in [11] several variants of Parseval-Plancherel equality are discussed. In particular, for $f, g \in L^2_{\mu}(\mathbb{R}_+)$ one has

$$\int_0^{\infty} x^{-\mu} f(x) g(x)\, dx = \int_0^{\infty} y^{-\mu} F_1(y) G_1(y)\, dy,$$

where $F_1(y) = h_{1,\mu}(f)(y)$, $G_1(y) = h_{1,\mu}(g)(y)$. Whence,

$$\|M_{\mu}^{-1} h_{1,\mu}(f)\|_{L^2_{\mu}} = \|(\cdot)^{-\mu} h_{1,\mu}(f)(\cdot)\|_{L^2_{\mu}} = \|(\cdot)^{-\mu} f(\cdot)\|_{L^2_{\mu}} = \|M_{\mu}^{-1} f\|_{L^2_{\mu}}. \quad (5)$$

By the Riesz-Thorin interpolation theorem (see [2, Ch. 1, Theorem 1.1.1]), from (4) and (5) we deduce a Hausdorff-Young type inequality:

$$\|M_{\mu}^{-1} h_{1,\mu}(f)\|_{L^q_{\mu}} \leqslant C(p)\|M_{\mu}^{-1} f\|_{L^p_{\mu}}, \quad 1 \leqslant p \leqslant 2, \quad 1/p + 1/q = 1, \quad (6)$$

for $f \in L^p_{\mu}(\mathbb{R}_+)$. Let $\Delta = \Delta(x, y, z)$ be area of the triangle with sides x, y, z ($\Delta(x, y, z) = (p(p - x)(p - y)(p - z))^{1/2}$, where $p = (x + y + z)/2$). For $\mu \geqslant 0$, we set

$$D_{\mu}(x, y, z) = \frac{\Delta^{2\mu+1}(x, y, z)}{2^{2\mu}(xyz)^{\mu}\Gamma(\mu + \frac{1}{2})\sqrt{\pi}},$$

if the triangle with sides x, y, z exists, and $D_{\mu}(x, y, z) = 0$ otherwise. Then $D_{\mu}(x, y, z)$ is non-negative and symmetric in x, y, z. In [17], Prasad, Singh, and Dixit introduced the generalized Hankel-Clifford translation of $f \in L^1_{\mu}(\mathbb{R}_+)$ by

$$T_x(f)(y) = \int_0^{+\infty} f(z)D_\mu(x,y,z)z^\mu dz, \quad 0 < x, y < \infty.$$

By Lemma 1.3 in [17], we have the following relation for $f \in L^1_\mu(\mathbb{R}_+)$ between the first Hankel-Clifford transform and the generalized Hankel-Clifford translation:

$$h_{1,\mu}(M_\mu T_x(f))(t) = c_\mu(xt)h_{1,\mu}(M_\mu f)(t), \quad t \geqslant 0. \tag{7}$$

The difference of order $m \in \mathbb{N}$ with step $t > 0$ is

$$\Delta^m_{t,\mu,hc} f(x) = (I - \Gamma(\mu+1)T_t)^m f(x).$$

We define the modulus of smoothness of order $m \in \mathbb{N}$ in $L^p_\mu(\mathbb{R}_+)$ by

$$\omega_m(f,\delta)_{p,\mu,hc} = \sup_{0 \leqslant t \leqslant \delta} \|\Delta^m_{t,\mu,hc} f\|_{L^p_\mu}.$$

The complicated form of equality (7) and inequality (6) obstruct to applying of differences of order $m \geqslant 2$ for $h_{1,\mu}$, e.g., there are doubts in the formula of Lemma 2.1 in [9].

Let $\lambda(t)$ be a non-negative measurable function from $L^1_{loc}(\mathbb{R}_+)$ and $\mu \geqslant 0$. If $\gamma \geqslant 1$ and there exists $C(\gamma) \geqslant 1$, such that

$$\|M_\mu^{-1} h_{1,\mu}(f)\|_{L^2_\mu} = \|(\cdot)^{-\mu} h_{1,\mu}(f)(\cdot)\|_{L^2_\mu} \Big(\int_y^{2y} \lambda^\gamma(t) t^\mu \, dt \Big)^{1/\gamma} \leqslant$$

$$\leqslant C(\gamma) y^{(\mu+1)(1-\gamma)/\gamma} \int_{y/2}^{y} \lambda(t) t^\mu \, dt, \quad t > 0, \tag{8}$$

then $\lambda \in A_{\gamma,\mu}$.

Moricz [13] used similar conditions for $y = 2^i$ and $\gamma = 0$ (see (2)), but in the proof of Theorem 1 it is more useful to apply (7).

3. Auxiliary propositions. From Lemma 1, we easily deduce the correctness of definitions of $\omega_1(f,\delta)_{p,\mu,hc}$.

Lemma 1. *Let* $1 \leqslant p < \infty$, $\mu \geqslant 0$, $f \in L^p_\mu(\mathbb{R}_+)$. *Then* $\|\Gamma(\mu+1)T_t f\|_{L^p_\mu} \leqslant \|f\|_{L^p_\mu}$.

The proof of Lemma 1 can be found in [18, Lemma 1.1]

Lemma 2. *Let* $\mu \geqslant 0$. *Then*

(i) $|j_\mu(x)| \leqslant 1$ *for* $x \geqslant 0$ *and* $j_\mu(x) < 1$ *for* $x > 0$;

(ii) $1 - j_\mu(x) \geqslant C > 0$ *for* $x \geqslant 1$;

(iii) the inequality $C_1 x^2 \leqslant 1 - j_\mu(x) \leqslant C_2 x^2$ *holds for some* $C_2 > C_1 > 0$ *and all* $x \in [0, 1]$.

Proof. The assertion of (i) can be found in [15], while the statement of (ii) is proved in [14, Lemma 3.3]. The right-hand inequality of (iii) is well known (see, e.g., [1]), the left-hand one is proved in [21] for $0 \leqslant x \leqslant \eta$. By (i) and the continuity of $j_\mu(x)$, we also have the boundedness from below of $(1 - j_\mu(x))/x^2$ on $[\eta,1]$, and (iii) follows. □

Lemma 3. *Let* $1 \leqslant p \leqslant 2$, $\mu \geqslant 0$, $f \in L^p_\mu(\mathbb{R}_+)$, $y \geqslant 0$. *Then* $h_{1,\mu}(M_\mu T_y f)(x) = c_\mu(yx) h_{1,\mu}(M_\mu f)(x)$ *a.e. on* $\mathbb{R}_+$.

Proof. In the case $p = 1$, see (7). In particular, the formula of Lemma 3 is valid for $f \in S(\mathbb{R}_+)$. By definition, $g(x) \in S(\mathbb{R}_+)$, if the even extension of g to $\mathbb{R}$ belongs to the Schwartz space $S(\mathbb{R})$. It is clear that $S(\mathbb{R}_+)$ is dense in $L^p_\mu(\mathbb{R}_+)$. If $f_n \in S(\mathbb{R}_+)$ and $f_n \to f$ in $L^p_\mu(\mathbb{R}_+)$, then, by Lemma 1, $T_y f_n \to T_y f$ in $L^p_\mu(\mathbb{R}_+)$, and, by (6) for $1 < p \leqslant 2$ and $1/p + 1/q = 1$ or by (4) for $p = 1$ and $q = \infty$, we have

$$\limsup_{n\to\infty} \|M_\mu^{-1}(h_{1,\mu}(M_\mu T_y f_n) - h_{1,\mu}(M_\mu T_y f))\|_{L^q_\mu} \leqslant$$
$$\leqslant C_1 \limsup_{n\to\infty} \|M_\mu^{-1} M_\mu (T_y f - T_y f_n)\|_{L^p_\mu} = 0;$$

therefore,

$$0 = \lim_{n\to\infty} \|M_\mu^{-1}(c_\mu(y\cdot) h_{1,\mu}(M_\mu f_n) - h_{1,\mu}(M_\mu T_y f))\|_{L^q_\mu} =$$
$$= \|M_\mu^{-1}(c_\mu(y\cdot) h_{1,\mu}(M_\mu f) - h_{1,\mu}(M_\mu T_y f))\|_{L^q_\mu},$$

and the equality of Lemma 3 is proved a.e. on $\mathbb{R}_+$. □

4. Main results.

Theorem 3. *Let* $\mu \geqslant 0$, $1 < p \leqslant 2$, $1/p + 1/q = 1$, $f \in L^p_\mu(\mathbb{R}_+)$. *If* $\lambda \in A_{q/(q-r),\mu}$ *for some* $r \in (0,q)$ *and the integral*

$$\int_N^\infty \lambda(y) y^{\mu r/p - r/q} \omega_1^r(f, y^{-1})_{p,\mu,hc} y^\mu \, dy$$

converges for all $N > 0$*, then* $\lambda(t)|h_{1,\mu}(M_\mu f)(t)|^r \in L^1_\mu[N, +\infty)$ *for all* $N > 0$ *and*

$$\int_N^\infty \lambda(t)|h_{1,\mu}(M_\mu f)(t)|^r t^\mu \, dt \leqslant$$
$$\leqslant C \int_{N/2}^\infty \lambda(y) y^{\mu r/p - r/q} \omega_1^r(f, y^{-1})_{p,\mu,hc} y^\mu \, dy. \quad (9)$$

Proof. By Lemma 3 and the Hausdorff-Young-type inequality (6), we have:

$$\int_{\mathbb{R}_+} y^{-q\mu}|h_{1,\mu}(M_\mu f)(t)|^q (1 - j_\mu(2\sqrt{yt}))^q y^\mu \, dy \leqslant$$
$$\leqslant C_1 \Big(\int_0^\infty (x^{-\mu}|M_\mu \Delta^1_{t,\mu,hc} f(x)|)^p x^\mu \, dx \Big)^{q/p} \leqslant C_1 \omega_m^q(f, t)_{p,\mu,hc}.$$

Let $N > 0$ be fixed and $D_i = [2^i N, 2^{i+1} N)$, $i \in \mathbb{Z}_+$. Taking $t_i = 2^{-i} N^{-1}$, by Lemma 2 (ii) we obtain

$$\int_{D_i} |h_{1,\mu}(M_\mu f)(y)|^q y^\mu \, dy \leqslant$$
$$\leqslant C_2 (2^i N)^{q\mu} \int_{D_i} y^{-q\mu}|h_{1,\mu}(M_\mu f)(y)|^q (1 - j_\mu(2\sqrt{y t_i}))^q y^\mu \, dy \leqslant$$
$$\leqslant C_3 (2^i N)^{q\mu} \omega_m^q(f, t_i)_{p,\mu,hc}.$$

By the Hölder inequality and the condition (8), we see that for $0 < r < q$

$$\int_{D_i} \lambda(y)|h_{1,\mu}(M_\mu f)(y)|^r y^\mu \, dy \leqslant$$
$$\leqslant \Big(\int_{D_i} |\lambda(y)|^{q/(q-r)} y^\mu \, dy \Big)^{1-r/q} \Big(\int_{D_i} |h_{1,\mu}(M_\mu f)(y)|^q y^\mu \, dy \Big)^{r/q} \leqslant$$
$$\leqslant C_4 (2^{-i} N^{-1})^{(\mu+1)r/q} (2^i N)^{r\mu} \omega_1^r(f, t_i)_{p,\mu,hc} \int_{D_{i-1}} \lambda(y) y^\mu \, dy \leqslant$$
$$\leqslant C_5 \int_{D_{i-1}} \omega_1^r(f, y^{-1})_{p,\mu,hc} \lambda(y) y^{\mu r/p - r/q} y^\mu \, dy. \quad (10)$$

Summing up (10) over $i \in \mathbb{Z}_+$, we obtain (9). $\square$

From Theorem 3 we deduce an integrability result on the whole $\mathbb{R}_+$.

Theorem 4. *Let $\mu \geqslant 0$, $1 < p \leqslant 2$, $1/p + 1/q = 1$, $f \in L^p_\mu(\mathbb{R}_+)$. If $\lambda \in A_{q/(q-r),\mu}$ for some $r \in (0,q)$, $\lambda \in L^{q/(q-r)}_\mu[0,1)$ and the integral*

$$\int\limits_1^\infty \lambda(y) y^{\mu r/p - r/q} \omega_1^r(f, y^{-1})_{p,\mu,hc} y^\mu \, dy \tag{11}$$

converges, then $\lambda(t)|h_{1,\mu}(M_\mu f)(t)|^r \in L^1_\mu(\mathbb{R}_+)$.

Proof. By (9), we find that

$$\int\limits_1^\infty \lambda(t)|h_{1,\mu}(M_\mu f)(t)|^r t^\mu \, dt \leqslant C \int\limits_{1/2}^\infty \lambda(y) y^{\mu r/p - r/q} \omega_1^r(f, y^{-1})_{p,\mu,hc} y^\mu \, dy. \tag{12}$$

By Lemma 1, we have $\omega_1(f,t)_{p,\mu,hc} \leqslant C_2 \|f\|_{L^p_\mu}$. Using the last inequality, the condition $\lambda \in L^{q/(q-r)}_\mu[0,1)$ and the Hölder inequality, we obtain

$$\int\limits_{1/2}^1 \lambda(y) y^{\mu r/p - r/q} \omega_1^r(f, y^{-1})_{p,\mu,hc} y^\mu \, dy \leqslant C_3 \|f\|^r_{L^p_\mu} \int\limits_{1/2}^1 \lambda(y) y^\mu \, dy \leqslant$$

$$\leqslant C_4 \Big(\int\limits_0^1 |\lambda(y)|^{q/(q-r)} y^\mu \, dy \Big)^{1-r/q} \Big(\int\limits_0^1 y^\mu \, dy \Big)^{r/q} < \infty$$

and both sides of (12) are finite. Finally, by the condition $\lambda \in L^{q/(q-r)}_\mu[0,1)$ we see that

$$\int\limits_0^1 \lambda(y)|h_{1,\mu}(M_\mu f)(y)|^r y^\mu \, dy \leqslant$$

$$\leqslant \Big(\int\limits_0^1 y^{-q\mu} |h_{1,\mu}(M_\mu f)(y)|^q y^\mu dy \Big)^{r/q} \Big(\int\limits_0^1 |\lambda(y)|^{q/(q-r)} y^{qr\mu/(q-r)} y^\mu dy \Big)^{1-r/q} \leqslant$$

$$\leqslant C_5 \|f\|^r_{L^p_\mu} \Big(\int\limits_0^1 |\lambda(y)|^{q/(q-r)} y^\mu \, dy \Big)^{1-r/q} < \infty,$$

since $0 \leqslant y^{qr\mu/(q-r)} \leqslant 1$ for $0 \leqslant y \leqslant 1$. Theorem is proved. □

Corollary 1. *Let f, p, q, μ and r are as in Theorem 4. If $\alpha > (r/q - 1)(\mu + 1)$ and the integral*

$$\int_1^\infty y^{\alpha + r\mu/p - r/q} \omega_1^r(f, y^{-1})_{p,\mu,hc} y^\mu \, dy \tag{13}$$

converges, the $t^\alpha |h_{1,\mu}(M_\mu f)(t)|^r \in L^1_\mu(\mathbb{R}_+)$.

Proof. It is easy to see that $\lambda_\alpha(t) = t^\alpha$ belongs to $A_{q/(q-r),\mu}$ for every $\alpha \in \mathbb{R}$. On the other hand, the condition $\lambda_\alpha \in L_\mu^{q/(q-r)}[0, 1)$ is equivalent to the convergence of integral $\int_0^1 t^{q\alpha/(q-r)+\mu} \, dt$, or to the inequality $\alpha > (r/q - 1)(\mu + 1)$. Using Theorem 4, we obtain the statement of Corollary 1. □

Corollary 2. *Let f, p, q, μ, and r are as in Theorem 4 and $\omega_1(f, t)_{p,\mu,hc} \leqslant Ct^\delta$ for some $\delta > 0$ and all $t \geqslant 0$. If $\alpha > (r/q - 1)(\mu + 1)$, $p\delta + p > \mu + 1$ and*

$$\frac{p(\alpha + \mu + 1)}{\delta p + p - \mu - 1} < r < q, \tag{14}$$

then $t^\alpha |h_{1,\mu}(M_\mu f)(t)|^r \in L^1_\mu(\mathbb{R}_+)$.

Proof. Under conditions of Corollary 2, the integral (13) converges if $\alpha + r\mu/p - r/q - r\delta + \mu < -1$ and $r < q$, i.e., $r < q$ and $r(1/q + \delta - \mu/p) > \alpha + \mu + 1$. If $1/q + \delta - \mu/p = \delta + 1 - (\mu + 1)/p \leqslant 0$, then r does not exist, while for $p\delta + p > \mu + 1$ we obtain (14). □

Remark 1. *The result of Corollary 2 in the case $\alpha = 0$ coincides with Theorem 3 in [5]. In a similar manner, one can obtain the result of Theorem 4 in [5].*

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