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# STATISTICAL BOUNDED SEQUENCES OF BI-COMPLEX NUMBERS

**Abstract.** In this paper, we extend statistical bounded sequences of real or complex numbers to the setting of sequences of bi-complex numbers. We define the statistical bounded sequence space of bi-complex numbers $b^*_\infty$ and also define the statistical bounded sequence spaces of ideals $\mathbb{I}^1_\infty$ and $\mathbb{I}^2_\infty$. We prove some inclusion relations and provide examples. We establish that $b^*_\infty$ is the direct sum of $\mathbb{I}^1_\infty$ and $\mathbb{I}^2_\infty$. Also, we prove the decomposition theorem for statistical bounded sequences of bi-complex numbers. Finally, summability properties in the light of J.A. Fridy's work are studied.

**Key words:** *natural density, bi-complex, statistical bounded, norm.*

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**1. Introduction.** In 1892, Segre [12] introduced the notion of bi-complex numbers that form an algebra isomorphic to the tessarines. Thereafter, Srivastava and Srivastava [13], Wagh [17], Sager and Sağır [10], Rochon and Shapiro [9] investigated on sequences of bi-complex numbers. The notion of convergence is one of the main tools of analysis. There are a lot of convergences, e.g., Cesáro, Nörlund and Riesz, etc. Out of these, statistical convergence is one of the most important notions, which brought a back through development in sequence spaces. Many researchers (e.g., Buck [3], Salat [11], Fridy [4], Tripathy [16], Altinok et.al [1], Tripathy and Nath [14], and Tripathy and Sen [15]) studied the statistical convergence and statistical bounded sequences of real or complex numbers. Research work on statistical convergence in sequence spaces has been done by Albayrak et al. [2], Kuzhaev [5], Nath et al. [6].

Throughout the paper, $C_0, C_1$ and $C_2$ denote the set of real, complex, and bi-complex numbers, respectively.

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## 2. Definition and preliminaries.

**2.1 Bi-complex numbers.** Segre [12] defined a bi-complex number as:

$$\xi = z_1 + i_2 z_2 = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,$$

where $z_1, z_2 \in C_1$ and $x_1, x_2, x_3, x_4 \in C_0$ and the independent units $i_1, i_2$ are such, that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. Denote the set of bi-complex numbers $C_2$; it is defined as:

$$C_2 = \{\xi \colon \xi = z_1 + i_2 z_2; z_1, z_2 \in C_1(i_1)\},$$

where $C_1(i_1) = \{x_1 + i_1 x_2 : x_1, x_2 \in C_0\}$. $C_2$ is a vector space over $C_1(i_1)$. There are four idempotent elements in $C_2$: they are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$, out of which $e_1$ and $e_2$ are nontrivial, such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.

A bi-complex number $\xi = z_1 + i_2 z_2$ is said to be singular if and only if $|z_1^2 + z_2^2| = 0$.

Every bi-complex number $\xi = z_1 + i_2 z_2$ can be uniquely expressed as the combination of $e_1$ and $e_2$; namely,

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 = \mu_1 e_1 + \mu_2 e_2,$$

where $\mu_1 = (z_1 - i_1 z_2)$ and $\mu_2 = (z_1 + i_1 z_2)$.

(i) The $i_1$-conjugation of a bi-complex number $\xi = z_1 + i_2 z_2$ is denoted by $\xi^*$ and is defined by $\xi^* = \bar{z_1} + i_2 \bar{z_2}$.

(ii) The $i_2$-conjugation of a bi-complex number $\xi = z_1 + i_2 z_2$ is denoted by $\bar{\xi}$ and is defined by $\bar{\xi} = z_1 - i_2 z_2$.

(iii) The $i_1 i_2$-conjugation of a bi-complex number $\xi = z_1 + i_2 z_2$ is denoted by $\xi'$ and is defined by $\xi' = \bar{z_1} + i_2 \bar{z_2}$, for all $z_1, z_2 \in C_1(i_1)$ and $\bar{z_1}, \bar{z_2}$ are the complex conjugates of $z_1, z_2$, respectively.

Each of the three conjugations' moduli are given by

(i) $|\xi|_{i_1} = \sqrt{\xi.\bar{\xi}}$ (ii) $|\xi|_{i_2} = \sqrt{\xi.\xi^*}$ (iii) $|\xi|_{i_1 i_2} = \sqrt{\xi.\xi'}$.

The bi-complex number $\xi$ is invertible if $|\xi|_{i_1} \neq 0$. The Euclidean norm $\|.\|$ on $C_2$ is defined by

$$\|\xi\|_{C_2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|\mu_1|^2 + |\mu_2|^2}{2}},$$

where $\xi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 = z_1 + i_2 z_2 = \mu_1 e_1 + \mu_2 e_2$ and $\mu_1 = z_1 - i_1 z_2$, $\mu_2 = z_1 + i_1 z_2$; with this, norm $C_2$ is a Banach space, also $C_2$ is a commutative algebra.

**Remark 1**. *[7] $C_2$ becomes a modified Banach algebra with respect to this norm in the sense that*

$$\|\xi.\eta\|_{C_2} \leqslant \sqrt{2}\|\xi\|_{C_2}.\|\eta\|_{C_2}.$$

Using the representation of a bi-complex number, the set $C_2$ can be expressed as

$$C_2 = X_1 e_1 + X_2 e_2,$$

where $X_1 = \{z_1 - i_1 z_2 \colon z_1, z_2 \in C_1(i_1)\}$ and $X_2 = \{z_1 + i_1 z_2 \colon z_1, z_2 \in C_1(i_1)\}$.

Suppose that $X_1$ and $X_2$ are normed spaces with the norm $\|\cdot\|_1, \|\cdot\|_2$, respectively. The hyperbolic norm on $C_2$ is given by

$$\|\xi\|_{i_1 i_2} = \|\mu_1\|_1 e_1 + \|\mu_2\|_2 e_2.$$

Throughout this article, we consider

$$0_1 = 0 + 0i_1;$$
$$0_2 = 0 + 0i_1 + 0i_2 + 0i_1 i_2 = 0_1 e_1 + 0_1 e_2;$$
$$0_h = 0 + 0i_1 i_2 = 0e_1 + 0e_2;$$
$$\theta_2 = (0_2, 0_2, \ldots).$$

### 2.2. Statistical boundedness.

The concept of statistical convergence depends on the notion of natural density of a set of natural numbers.

A subset $E$ of $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),$$

where $\chi_E$ is the characteristic function on $E$.

Let $(\xi_n)$ and $(\eta_n)$ be two sequences, such that $\xi_k = \eta_k$ for almost all k (in short a.a.k.) if $\delta(\{k \in \mathbb{N} \colon \xi_k \neq \eta_k\}) = 0$.

A sequence of bi-complex numbers $\xi = (\xi_k)$ is said to be statistically convergent to $\xi^* \in C_2$ with respect to the Euclidean norm on $C_2$ if, for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : \|\xi_k - \xi^*\|_{C_2} \geqslant \varepsilon\}) = 0,$$

It is denoted as $stat\text{-}\lim \xi_k = \xi^*$.

If $\xi^* = 0_2$, then the sequence $(\xi_k)$ of bi-complex numbers is said to be statistical null.

A sequence of bi-complex number $\xi = (\xi_k)$ is said to be statistically Cauchy with respect to the Euclidean norm on $C_2$ if, for every $\varepsilon > 0$, there exists $x_{k_0} \in \mathbb{N}$, such that

$$\delta(\{k \in N\colon \|\xi_k - \xi_{k_0}\|_{C_2} \geqslant \varepsilon\}) = 0.$$

A sequence $\xi = (\xi_k)$ of bi-complex numbers is said to be statistically bounded if there exists $0 < M \in C_0$, such that

$$\delta(\{k \in \mathbb{N}\colon \|\xi_k\|_{C_2} \geqslant M\}) = 0.$$

Throughout the paper, $w^*$ and $b^\infty$ denote the sets of all and bounded sequences of bi-complex numbers, respectively.
We list the following classes of sequences, which will be used in this article:

$b^* := \{\xi = (\xi_k) \in w^*\colon$ there exists a bi-complex number $\eta$ such that $stat\text{-}\lim\limits_{k\to\infty} \xi_k = \eta\}$.

$b_0^* := \{\xi = (\xi_k) \in w^*\colon stat\text{-}\lim_{k\to\infty} \xi_k = 0_2\}$.

${}^{C}b^* := \{\xi = (\xi_k) \in w^*\colon \xi$ is statistically Cauchy$\}$.

$b_\infty^* := \{\xi = (\xi_k) \in w^*\colon$ there exists $0 < M \in C_0\colon \delta(\{n\colon \|\xi_k\| \geqslant M\}) = 0\}$.

$\mathbb{I}_\infty^1 := \{(\mu_{1k}e_1), \mu_{1k} \in X_1\colon (\mu_{1k})$ is statistically bounded$\}$.

$\mathbb{I}_\infty^2 := \{(\mu_{2k}e_2), \mu_{2k} \in X_2\colon (\mu_{2k})$ is statistically bounded$\}$.

$\mathbb{J}_\infty^1 := \{\xi = (\xi_k) \in w^*, \xi_k = \mu_{1k}e_1 + \mu_{2k}e_2\colon (\mu_{1k})$ is statistically bounded$\}$.

$\mathbb{J}_\infty^2 := \{\xi = (\xi_k) \in w^*, \xi_k = \mu_{1k}e_1 + \mu_{2k}e_2\colon (\mu_{2k})$ is statistically bounded$\}$.

## 3. Main Result.

**Theorem 1**. *If a sequence $(\xi_k)$ of bi-complex numbers $\xi_k = z_{1k} + i_2z_{2k}$, $\forall k \in \mathbb{N}$ is statistically bounded, then the sequences $(z_{1n})$ and $(z_{2n})$ are also statistically bounded.*

**Proof.** Let $(\xi_k)$ be statistically bounded; then there exists an $M$, such that $\delta(\{k : \|\xi_k\|_{C_2} \geqslant M\}) = 0$, which implies $\delta(\{k : \|z_{1k} + i_2z_{2k}\|_{C_2} \geqslant M\}) = 0$ and $\delta(\{k : |z_{jk}| \geqslant M\} \leqslant \delta(\{k : \|z_{1k} + i_2z_{2k}\|_{C_2} \geqslant M\}) = 0$ for $j = 1, 2$. Hence, $(z_{1k})$ and $(z_{2k})$ are statistically bounded.

Conversely, let $(z_{1k})$ and $(z_{2k})$ be statistically bounded. Then, without loss of generality, we can find $M > 0$, such that

$$\delta(\{k\colon |z_{1k}| \geqslant M\}) = 0$$

and

$$\delta(\{k\colon |z_{2k}| \geqslant M\}) = 0.$$

Then we have the result from the following inequality:

$$\delta(\{k : \|z_{1k} + i_2 z_{2k}\|_{C_2} \geqslant M\}) \leqslant \delta(\{k : |z_{1k}| \geqslant M\}) + \delta(\{k : |z_{2k}| \geqslant M\}) = 0$$

(by sub-additivity property). Hence, $(\xi_k)$ is statistically bounded. □

In view of the above theorem, we formulate the following corollaries:

**Corollary 1**. *If a sequence* $(\xi_k)$, *where* $\xi_k = x_{1k} + i_1 x_{2k} + i_2 x_{3k} + i_1 i_2 x_{4k}$ *of bi-complex numbers, is statistically bounded, then the sequences* $(x_{pn})$, $p = 1, 2, 3, 4$. *of real numbers are also statistically bounded.*

**Corollary 2**. *If a sequence* $(\xi_k)$, *where* $\xi_k = \mu_{1k}e_1 + \mu_{2k}e_2$ *of bi-complex numbers, is statistically bounded, then the sequences* $(\mu_{1k})$ *and* $(\mu_{2k})$ *are statistically bounded.*

**Result 1**. The inclusion relations

$$(i)\ b^* \subset b^*_\infty$$
$$(ii)\ {}^{C}b^* \subset b^*_\infty$$

are strict; this follows from the following example:

**Example 1**. Consider a sequences $(\xi_k)$ and $(\eta_k)$ of bi-complex numbers defined by

$$\xi_k = \begin{cases} k^3 i_1 + k^2 i_2 + k i_1 i_2, & \text{if } k = n^3, n \in \mathbb{N}; \\ i_1 - i_2, & \text{if } k = n^2 + 1; \\ 0, & \text{otherwise.} \end{cases}$$

From the above example, it can be observed that $(\xi_k) \notin b^*$, but $(\xi_k) \in b^*_\infty$.

**Result 2**. $b^\infty \subset b^*_\infty$.

The converse parts are not true. Let us consider a sequence $(\xi_k)$ of bi-complex numbers, defined by

$$\xi_k = \begin{cases} k^2 i_1 + k^2 i_2, & \text{if } k = n^2, n \in \mathbb{N}; \\ e_1 - e_2, & \text{if } k = n^2 + 1; \\ e_1 + e_2, & \text{if } k = n^2 + 2; \\ e_1 e_2, & \text{otherwise.} \end{cases}$$

We observe that $(\xi_k) \in b^*_\infty$, but $(\xi_k) \notin b^\infty$.

**Result 3**.

(1) $\mathbb{I}_\infty^1 \subset b_\infty^*$

(2) $\mathbb{I}_\infty^2 \subset b_\infty^*$

(3) $\mathbb{J}_\infty^1 \supset b_\infty^*$

(4) $\mathbb{J}_\infty^2 \supset b_\infty^*$.

The inclusions are strict; this follows from the following examples:

**Example 2**. Let us consider a sequence $(\xi_k)$ of bi-complex numbers, defined by

$$\xi_k = \mu_{1k}e_1 + \mu_{2k}e_2, \forall k \in \mathbb{N}$$

where

$$\mu_{1k} = \begin{cases} ki_1, & \text{if } k = n^3, n \in \mathbb{N}; \\ i_1, & \text{if } k = n^3 + 1; \\ e_1 + e_2, & \text{if } k = n^3 + 2; \\ e_1e_2, & \text{otherwise.} \end{cases}$$

and

$$\mu_{2k} = \begin{cases} \sqrt{k}i_1, & \text{if } k = n^3, n \in \mathbb{N}; \\ k^2i_1, & \text{if } k = n^3 + 1; \\ -(e_1 + e_2)k^2, & \text{if } k = n^3 + 2; \\ e_1e_2, & \text{otherwise.} \end{cases}$$

In the above example, it can be observed that $(\xi_k)$ is in $\mathbb{J}_\infty^2$ but not in $b_\infty^*$.

**Theorem 2**. *The space $b_\infty^*$ is a linear space over $C_1(i_1)$.*

**Proof.** Let $(\xi_k), (\eta_k) \in b_\infty^*$. Therefore, there exists $M > 0$, such that

$$\delta(\{k \in \mathbb{N} \colon \|\xi_k\|_{C_2} \geqslant M\}) = 0,$$
$$\delta(\{k \in \mathbb{N} \colon \|\eta_k\|_{C_2} \geqslant M\}) = 0.$$

Then $(\xi_k + \eta_k) \in b_\infty^*$ follows from the following inclusion relation:

$$\{k \in \mathbb{N} \colon \|\xi_k + \eta_k\|_{C_2} \geqslant 2M\} \subseteq \{k \in \mathbb{N} \colon \|\xi_k\|_{C_2} \geqslant M\} \cup \{k \in \mathbb{N} \colon \|\eta_k\|_{C_2} \geqslant M\}.$$

For $(\xi_k) \in b_\infty^*$ and $\alpha \in C_1(i_1)$, similarly, it can be shown that $(\alpha\xi_k) \in b_\infty^*$. Therefore, the space $b_\infty^*$ is a linear space over $C_1(i_1)$. □

**Lemma 1**. *The spaces $\mathbb{I}_\infty^1, \mathbb{I}_\infty^2, \mathbb{J}_\infty^1$ and $\mathbb{J}_\infty^2$ are linear spaces over $C_1(i_1)$.*

**Lemma 2**. *The space $b^*_\infty$ is a commutative algebra with the identity $1 = 1 + 0i_1 + 0i_2 + 0i_1i_2$ under coordinate-wise addition, real scalar multiplication, and term by term multiplication.*

**Proof.** We know that $C_2$ is a commutative algebra (linear space that is a commutative ring) with the identity $1 = 1+0i_1+0i_2+0i_1i_2$ and $b^*_\infty \subset C_2$. Since $b^*_\infty$ is a linear space over $C_1(i_1)$ and a commutative ring with the product defined on $b^*_\infty$, such that

$$(\alpha\xi_k \cdot \eta_k) = (\xi_k \cdot \alpha\eta_k), \forall(\xi_k), (\eta_k) \in b^*_\infty \text{ and } \forall\alpha \in C_1(i_1).$$

Hence, we see that $b^*_\infty$ is a commutative algebra. □

In view of Remark 1, we have the following lemma:

**Lemma 3**. *The space $b^*_\infty$ is a modified Banach algebra with respect to the norm $\|\xi\| = \inf \|\xi_k\|_{C_2}, \xi = (\xi_k) \in b^*_\infty$.*

**Proof.** We have the following inequality:

$$\|\xi \cdot \eta\| \leqslant \sqrt{2}\|\xi\|\|\eta\|, \text{ for all } \xi, \eta \in b^*_\infty. \tag{1}$$

From the definition of Banach algebra and using the eq.(1), we can easily prove that $b^*_\infty$ is a modified Banach algebra with respect to the norm $\|\cdot\|$. □

**Theorem 3**. *The spaces $\mathbb{I}^1_\infty$ and $\mathbb{I}^2_\infty$ are commutative Banach algebras.*

**Proof.** Let $\mu'_p \in \mathbb{I}^1_\infty$ be an arbitrary Cauchy sequence in $\mathbb{I}^1_\infty$. Then $\mu'_p$ is Cauchy sequence in $b^*_\infty$. Since $b^*_\infty$ is complete, there exists $\eta \in b^*_\infty$, such that

$$\begin{aligned} &\mu'_p \to \eta \\ \implies &\|\mu'_p - \eta\|_{C_2} = 0, \text{ as } p \to \infty \\ \implies &\inf \|\mu'_p - \eta\|_{C_2} = 0, \text{ as } p \to \infty \\ \implies &\inf \|\mu'_{1p}e_1 + \mu'_{2p}e_2 - \mu_1e_1 - \mu_2e_2\|_{C_2} = 0, \text{ as } p \to \infty \\ \implies &\inf \|\mu'_{1p} - \mu_1\|_1 \to 0, \inf \|\mu'_{2p} - \mu_2\|_2 \to 0, \text{ as } p \to \infty. \end{aligned}$$

Since $\mu'_p \in \mathbb{I}^1_\infty$, so $\mu'_{2p} = 0_1$ and, hence, $\mu_2 = 0_1$. So that $\eta \in \mathbb{I}^1_\infty$. Thus, $\mathbb{I}^1_\infty$ is a commutative Banach algebra and the identity element of $\mathbb{I}^1_\infty$ is $(e_1)$. Similarly, we can prove that $\mathbb{I}^2_\infty$ is a commutative Banach algebra with the identity element of $\mathbb{I}^2_\infty$ is $(e_2)$. □

**Corollary 3**. *The spaces $\mathbb{I}^1_\infty$ and $\mathbb{I}^2_\infty$ are Gelfand algebras.*

**Theorem 4**. *If $a = (a_k) \in \mathbb{I}^1_\infty$ and $b = (b_k) \in \mathbb{I}^2_\infty$, then*

*(1)* $e_1 \cdot a \in \mathbb{I}^1_\infty$.

*(2)* $e_2 \cdot a = \theta_2$.

*(3)* $e_1 \cdot b = \theta_2$.

*(4)* $e_2 \cdot b \in \mathbb{I}^2_\infty$.

**Proof.** Let $a = (a_k) = (\mu_{1k}e_1) \in \mathbb{I}^1_\infty$ and $b = (b_k) = (\mu_{2k}e_2) \in \mathbb{I}^2_\infty$.
(1) $a = (a_1, a_2, a_3, \ldots)$

i.e., $e_1 \cdot a = (a_1e_1, a_2e_1, a_3e_1, \ldots) = (a_1, a_2, a_3, \ldots) = a \in \mathbb{I}^1_\infty$.

(2) $e_2 \cdot a = (a_1e_2, a_2e_2, a_3e_2, \ldots) = (0_2, 0_2, 0_2, \ldots) = \theta_2$ .

(3) Similar to (2).

(4) $b = (b_1, b_2, b_3, \ldots)$
i.e., $e_2 \cdot b = (e_2b_1, e_2b_2, e_2b_3, \ldots) = (b_1, b_2, b_3, \ldots) = b \in \mathbb{I}^2_\infty$. □

**Result 4**.

(1) $\mathbb{I}^1_\infty \cup \mathbb{I}^2_\infty = b^*_\infty$.

(2) $\mathbb{J}^1_\infty \cup \mathbb{J}^2_\infty = b^*_\infty$.

(3) $\mathbb{I}^1_\infty \cap \mathbb{I}^2_\infty = \theta_2$.

(4) $\mathbb{J}^1_\infty \cap \mathbb{J}^2_\infty \neq \phi$.

**Result 5**. If $\xi = (\xi_k) \in b^*_\infty$ and $\mu^{'} = (e_1\mu_{1k}) \in \mathbb{I}^1_\infty, \mu^{''} = (e_2\mu_{2k}) \in \mathbb{I}^2_\infty$, then

$$\xi = \mu^{'} + \mu^{''}.$$

**Result 6**. $b^*_\infty = \mathbb{I}^1_\infty \bigoplus \mathbb{I}^2_\infty$.

**Corollary 4**. *$b^*_\infty/\mathbb{I}^1_\infty$ is isomorphic to $\mathbb{I}^2_\infty$.*

We formulate the following theorem without demo.

**Theorem 5**. *If $\xi = (\xi_k) \in \mathbb{J}^1_\infty \cap \mathbb{J}^2_\infty$, where $\xi = e_1\mu_1 + e_2\mu_2$, then $a \in \mathbb{I}^1_\infty$ and $b \in \mathbb{I}^2_\infty$, $a = e_1\mu_1, b = e_2\mu_2$.*

**Definition 1**. *Let us define a relation $\sim$ on $b^*_\infty$ as follows:*
*For $\xi = (\xi_k)$,$\eta = (\eta_k) \in b^*_\infty$,*

$$\xi \sim \eta \Leftrightarrow \|\xi - \eta\|_{i_1i_2} = 0_h.$$

*It can be easily verified that it is equivalence relation on $b^*_\infty$.*

*Now,*

$$\begin{aligned}&\|\xi-\eta\|_{i_1i_2}=0_h\\ &\implies e_1\|\mu_{1k}-\mu'_{1k}\|_1+e_2\|\mu_{2k}-\mu'_{2k}\|_2=0_2=e_10+e_20\\ &\implies e_1\|\mu_{1k}-\mu'_{1k}\|_1=e_10=0 \text{ and } e_2\|\mu_{2k}-\mu'_{2k}\|_2=e_20=0.\end{aligned}$$

*Since,* $\|e_1\|_{i_1i_2}=e_1$ *and* $\|e_2\|_{i_1i_2}=e_2$*. So we can write* $\mu_1\sim\mu'_1$ *and* $\mu_2\sim\mu'_2$*, where* $\mu'_1,\mu_1\in\mathbb{I}^1_\infty$ *and* $\mu'_2,\mu_2\in\mathbb{I}^2_\infty$*. The equivalence class* $[\xi]$ *on* $b^*_\infty$ *is*

$$[\xi]=\{\zeta:\xi\sim\zeta\}$$

$$\implies [\xi]=[\mu_1]+[\mu_2].$$

**Theorem 6**. *Let* $\xi=(\xi_k)$ *and* $\eta=(\eta_k)\in b^*_\infty$ *and let* $B=\{k\colon \xi_k\neq\eta_k\}$*. Then* $\delta(B)=0$ *if* $\eta\in[\xi]$*.*

**Proof.** Since $\eta\in[\xi]$,

$$\|\xi-\eta\|_{i_1i_2}=0_h$$

$$\begin{aligned}&\implies \|(\mu_{1k}e_1+\mu_{2k}e_2)-(\mu'_{1k}e_1+\mu''_{2k}e_2\|_{i_1i_2}=0_h\\ &\implies \|\mu_{1k}-\mu'_{1k}\|_1e_1+\|\mu_{2k}-\mu''_{2k}\|_2e_2=0e_1+0e_2\\ &\implies \|\mu_{1k}-\mu'_{1k}\|_1=0 \text{ and } \|\mu_{2k}-\mu''_{2k}\|_2=0.\end{aligned}$$

Now,

$$\delta(\{k\colon \|\xi_k-\eta_k\|_{C_2}\geqslant\varepsilon\})=\delta\left(\left\{k\colon \sqrt{\frac{\|\mu_{1k}-\mu'_{1k}\|_1^2+\|\mu_{2k}-\mu''_{2k}\|_2^2}{2}}\geqslant\varepsilon\right\}\right)=0.$$

Therefore,

$$\delta\left(\{k\colon \|\xi_k-\eta_k\|_{C_2}\geqslant\varepsilon\}\right)=0.$$

□

**Lemma 4**. *Let* $\xi=(\xi_k)\in b^*_\infty$ *and if* $\xi\in\mathbb{I}^1_\infty\cup\mathbb{I}^2_\infty$*, then* $\xi$ *is singular statistically bounded.*

**Proof.** Here $\xi$ is statistically bounded. So, we only need to prove that for all $k\in\mathbb{N}$, $\xi_k$ is singular.
Let $\xi\in\mathbb{I}^1_\infty\cup\mathbb{I}^2_\infty$; then either $\xi=(\mu_{1k}e_1)$, $\mu_{1k}\in X_1$, or $\xi=(\mu_{2k}e_2)$, $\mu_{2k}\in X_2$. Since $e_i$ are singular and $\mu_{1k}\in X_i$, so, for all $k\in\mathbb{N}$, $\mu_{ik}e_i$ are also singular, where $i=1,2$. □

**Definition 1.** A sequence $\xi = (\xi_k) \in b^*_\infty$ is convergent to $\xi^*$ in $\| \cdot \|_{i_1 i_2}$ if

$$\|\xi_k - \xi^*\|_{i_1 i_2} = 0_h.$$

**Definition 2.** A sequence $\xi = (\xi_k) \in b^*_\infty$ is called Cauchy sequence in $\| \cdot \|_{i_1 i_2}$ if

$$\|\xi_k - \xi_{k_0}\|_{i_1 i_2} = 0_h,$$

or,

$$\xi_k \sim \xi_{k_0}.$$

**Theorem 7**. *If a bounded sequence $\xi = (\xi_k), \xi_k = e_1\mu_{1k} + e_2\mu_{2k}$ is statistically Cauchy, then $\xi$ is a Cauchy sequence in $|| \cdot \|_{i_1 i_2}$.*

**Proof.** Let $\xi = (\xi_k)$ be statistically Cauchy; then, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that

$$\delta(\{k\colon \|\xi_k - \xi_{n_0}\|_{C_2} \geqslant \varepsilon\}) = 0.$$

$$\implies \delta(\{k\colon \|\mu_{1k} - \mu_{1k_0}\|_1 \geqslant \varepsilon^1\}) = 0$$

and

$$\implies \delta(\{k\colon \|\mu_{2k} - \mu_{2k_0}\|_2 \geqslant \varepsilon^2\}) = 0.$$

Which implies that $\varepsilon^j$ are statistical upper bounds of the sequences $(\|\mu_{jk} - \mu_{jk_0}\|_j$ and, hence, the statistical least upper bounds of $(\|\mu_{jk} - \mu_{jk_0}\|_j$ are $\varepsilon^j$. Since $\varepsilon^j$ are arbitrary, so, the statistical least upper bounds of $(\|\mu_{jk} - \mu_{jk_0}\|_j$ are zero.
Hence, $\|\xi_k - \xi_{k_0}\|_{i_1 i_2} = e_1\|\mu_{1k} - \mu_{1k_0}\|_1 + e_2\|\mu_{2k} - \mu_{2k_0}\|_2 = 0_h, j = 1,2.$ □

**Corollary 5**. *If a sequence $\xi = (\xi_k), \xi_k = e_1\mu_{1k} + e_2\mu_{2k}$ is statistically convergent, then $\xi$ is a Cauchy sequence in $\| \cdot \|_{i_1 i_2}$.*

**Theorem 8**. *Let $\xi = (\xi_k)$ be statistically convergent to $\xi^*$. If $\zeta = (\zeta_k) \in [\xi]$, then $\zeta$ is statistically convergent to $\xi^*$ in $\| \cdot \|_{i_1 i_2}$.*

**Proof.** Since $\xi$ is statistically convergent to $\xi^*$, so

$$\|\xi - \xi^*\|_{i_1 i_2} = 0_h.$$

$\zeta \in [\xi] \implies \|\xi - \zeta\|_{i_1 i_2} = 0.$
Now,

$$\|\zeta - \xi^*\|_{i_1 i_2} \leqslant \|\xi - \xi^*\|_{i_1 i_2} + \|\zeta - \xi\|_{i_1 i_2} = 0_h.$$

Hence, $\zeta$ is statistically convergent to $\xi^*$ in $\| \cdot \|_{i_1 i_2}$. □

Tripathy [16] proved the decomposition theorem for statistically bounded sequences of real numbers.

The following theorem is the decomposition theorem for sequences of bi-complex numbers.

**Theorem 9.** *If a sequence $\xi = (\xi_k)$ of bi-complex numbers is statistically bounded, then there exists a bounded sequence $\eta = (\eta_k)$ of bi-complex numbers and a statistically null sequence $\zeta = (\zeta_k)$ of bi-complex numbers, such that $\xi = \eta + \zeta$.*

**Proof.** Let $\xi = (\xi_k)$, where $\xi_k = \mu_{1k}e_1 + \mu_{2k}e_2$, be a statistically bounded sequence. Then $\delta(B) = 0$, where $B = \{k\colon \|\xi_k\|_{C_2} \geqslant M\}$.
Define the sequences $\eta = (\eta_k)$ and $\zeta = (\zeta_k)$ as follows:

$$\eta_k = \begin{cases} \xi_k, & \text{if } k \in B^c; \\ e_1e_2, & \text{otherwise.} \end{cases}$$

$$\zeta_k = \begin{cases} e_1e_2, & \text{if } k \in B^c; \\ \xi_k, & \text{otherwise.} \end{cases}$$

From the above construction of $\eta$ and $\zeta$, we have

$$\xi = \eta + \zeta,$$

where $\eta \in b^{\infty}$ and $\zeta \in b_0^*$. □

Following Lemma 1.1 of Salat [11], we state the following result without proof:

**Proposition 1.** *A sequence $(\xi_k)$ of bi-complex numbers is statistically bounded if and only if there exists a set $K = \{k_1 < k_2 < \ldots\} \subset \mathbb{N}$, such that $\delta(K) = 1$ and $(\xi_{k_n})$ is bounded.*

**4. Summability properties.** We are going to use the idea by Fridy [4].

**Lemma 5.** *Let us consider a sequence $\xi = (\xi_k)$ of bi-complex numbers, such that $|\xi_k|_{i_1} \neq 0_1$ for infinitely many $k$; then there exists a sequence $\eta = (\eta) \in b^*_\infty$, such that*

$$\sum_{k=1}^{\infty} \xi_k\eta_k = \infty.$$

**Proof.** Consider an increasing sequence $(n_k)$ of natural numbers, such that

$$n_k \geqslant k^2 \text{ and } |\xi_{n_k}|_{i_1} \neq 0_1.$$

Let us consider a sequence $\eta = (\eta_k)$ defined by

$$\eta_k = \begin{cases} \frac{1}{\xi_{n_k}}, & \text{if } k = n_j, j \in \mathbb{N}; \\ e_1 - e_2, & \text{if } k = n_j + 1, j \in \mathbb{N}; \\ e_1 + e_2, & \text{otherwise.} \end{cases}$$

Now, $\{k \colon \|\eta_k\| \geqslant 2\} \subset \{n \colon n = k^2,\ k \in \mathbb{N}\}$.
Thus, $\delta(k \colon \|\eta_k\| \geqslant 2\}) \subset \delta(\{n \colon n = k^2,\ k \in \mathbb{N}\}) = 0$ and

$$\sum_{k=1}^{\infty} \xi_k \eta_k = \infty.$$

□

Let $T = (t_{n,k})$ be any summability matrix. Let $\xi = (\xi_k) \in w^*$; then $\xi$ is called a $T$ bounded sequence if

$$T(\xi) = \left( \sum_{k=1}^{\infty} t_{n,k} \xi_k \right) \in b^{\infty}.$$

The set of all $T$ bounded sequences is denoted by

$$b_{\infty}^{T} = \{\xi = (\xi_k) \in w^* : T(\xi) \in b^{\infty}\}.$$

**Theorem 10**. *There is no row finite matrix $T = (t_{n,k})$, such that $b_{\infty}^{T}$ contains $b_{\infty}^{*}$.*

**Proof.** Let $T = (t_{n,k})$ be any row finite summability matrix. Choose $|t_{n_1,k_1'}|_{i_1} \neq 0_1$. Choose $k_1^{''} \geqslant k^{'}$, such that

$$|t_{n_1,k_1^{''}}|_{i_1} \neq 0_1 \text{ and } |t_{n_1,k}|_{i_1} = 0_1 \text{ for all } k \geqslant k_1^{''}.$$

We can select an increasing sequence of rows and columns, such that for each $r$

$$|t_{n_r,k_r}|_{i_1} \neq 0, k_r \geqslant r^2$$

and

$$t_{n_r,k} = 0, \text{ for all } k > k_r.$$

Define the sequence $\xi = (\xi_k)$ as

$$\xi_k = \begin{cases} \dfrac{1}{t_{n_r,k_r}} \left[ r - \sum_{i=0}^{m-1} t_{n_r,k_i} \xi_{k_i} \right], & \text{if } k = k_r; \\ k^2, & \text{if } k = k_{r-1}; \\ (-1)^k, & \text{otherwise.} \end{cases}$$

Then $\xi$ is not a $T$ bounded sequence. But for any sufficiently large $M > 0$, we have

$$\{k\colon \|\xi_k\|_{C_2} \geqslant M\} \subset \{k_r, k_{r-1}, r \in \mathbb{N}\} \subset \{r^2\colon r \in \mathbb{N}\} \cup \{r^2 - 1\colon r \in \mathbb{N}\}.$$

Hence, $\xi \in b_\infty^*$. $\square$

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