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RATIONAL TYPE CYCLIC CONTRACTION IN G-METRIC SPACES

Abstract. Rational type cyclic contraction via $\mathcal{C}$-class function is established in G-metric spaces, which can not be reduced to the contractive condition in standard metric spaces. A common fixed-point result is obtained for the pair of (A, B)-weakly increasing mappings in G-metric spaces.

Key words: *G-metric spaces, Cyclic maps, $\mathcal{C}$-class function, Common fixed point*

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1. Introduction. In 2012, Jleli and Samet [3] observed that some of the fixed-point theorems in G-metric spaces can be deduced from standard metric spaces or quasi-metric spaces (for details see [7], [8]). Shatanawi and Abodayeh [9] introduced a new contractive condition and proved fixed-point and common fixed-point results in G-metric spaces, for which the techniques of Jleli and Samet [3], Samet et al. [6] are inapplicable.
In this paper, we introduce rational type cyclic contraction via $\mathcal{C}$-class function in G-metric space that generalizes the contractive condition of Shatanawi and Abodayeh [9] for larger class of auxiliary functions and deduced common fixed-point result in G-metric spaces. Some examples are also presented to show that our results are effective.

2. Preliminaries.

Definition 1. *An altering distance function is a continuous, non-decreasing mapping $\phi\colon [0,\infty) \to [0,\infty)$, such that $\phi^{-1}(0) = 0$.*

Notation:

(i) Φ is the family of all altering distance functions.

(ii) Ψ is the family of all mappings $\psi\colon [0,\infty) \to [0,\infty)$ with the property: if $\{t_m\}_{m\in\mathbb{N}} \subset [0,\infty)$ and $\psi(t_m) \to 0$, then $t_m \to 0$.

Note that $\Phi \subset \Psi$.

Definition 2. *[5] Let X be a nonempty set. Let $G\colon X \times X \times X \to [0,\infty)$ be a function satisfying the following properties:*

(G_1) $G(x,y,z) = 0$, *if* $x = y = z$,

(G_2) $G(x,x,y) > 0, \forall x,y \in X$ *with* $x \neq y$,

(G_3) $G(x,x,y) \leqslant G(x,y,z), \forall x,y,z \in X$ *with* $z \neq y$,

(G_4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \dots$ *(symmetry in all three variables),*

(G_5) $G(x,y,z) \leqslant G(x,a,a) + G(a,y,z), \forall x,y,z,a \in X$ *(rectangle inequality).*

The function G is called G-metric on X and the pair (X,G) is called a G-metric space.

Definition 3. *[5] A G-metric space (X,G) is said to be symmetric if*

$$G(x,y,y) = G(y,x,x), \forall x,y \in X.$$

Lemma 1. *[5] If (X,G) is a G-metric space, then*

$$G(x,y,y) \leqslant 2G(y,x,x), \forall x,y \in X.$$

Definition 4. *[5] Let (X,G) be a G-metric space, $x \in X$ be a point, and $\{x_n\} \subseteq X$ be a sequence. We say that:*

(1) a sequence $\{x_n\}$ G-converges to x, if $\lim\limits_{n,m\to\infty} G(x_n,x_m,x) = 0$; that is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n,x_m,x) < \varepsilon$, $\forall n,m \geqslant n_0$.

(2) a sequence $\{x_n\}$ is G-Cauchy if $\lim\limits_{n,m,k\to\infty} G(x_n,x_m,x_k) = 0$; that is, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying $G(x_n,x_m,x_k) < \varepsilon$, $\forall n,m,k \geqslant n_0$.

(3) (X,G) is complete if every G-Cauchy sequence in X is G-convergent in X.

Proposition 1. *[5] Let (X,G) be a G-metric space, $\{x_n\} \subseteq X$ be a sequence, and $x \in X$. Then the following are equivalent:*

(a) $\{x_n\}$ *G-converges to* x,

(b) $\lim\limits_{n\to\infty} G(x_n, x_n, x) = 0$,

(c) $\lim\limits_{n\to\infty} G(x_n, x, x) = 0$.

Proposition 2. *[5] A sequence $\{x_n\}$ in a G-metric space (X, G) is G-Cauchy if and only if* $\lim\limits_{n,m\to\infty} G(x_n, x_m, x_m) = 0$.

Definition 5. *[1] A sequence $\{x_n\}$ in a G-metric space (X, G) is asymptotically regular if* $\lim\limits_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

Lemma 2. *[1] Let $\{x_n\}$ be an asymptotically regular sequence in a G-metric space (X, G) and suppose that $\{x_n\}$ is not Cauchy. Then there exist a positive real number $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, such that $\forall k \in \mathbb{N}$:*

$$k \leqslant n_k < m_k < n_{k+1},$$

$$G(x_{n_k}, x_{n_k+1}, x_{m_k-1}) \leqslant \varepsilon < G(x_{n_k}, x_{n_k+1}, x_{m_k})$$

and, also, for all given $p_1, p_2, p_3 \in \mathbb{Z}$:

$$\lim_{n\to\infty} G(x_{n_k+p_1}, x_{m_k+p_2}, x_{m_k+p_3}) = \varepsilon.$$

Definition 6. *[5] Let (X, G) be a G-metric space. We say that a mapping $T\colon X \to X$ is G-continuous at $x \in X$ if $\{Tx_m\} \to Tx$ for all sequences $\{x_m\} \subseteq X$, such that $\{x_m\} \to x$.*

In 2013, Shatanawi and Postolache [10] introduced (A, B)-weakly increasing functions for a pair of mappings:

Definition 7. *Let $(X, \preceq)$ be a partially ordered set and A, B be two closed subsets of X with $X = A \cup B$. Let $f, g\colon X \to X$ be two mappings. Then the pair (f, g) is said to be (A, B)-weakly increasing if $fx \preceq gfx, \forall x \in A$ and $gx \preceq fgx, \forall x \in B$.*

Kirk et al. [4] introduced cyclic mappings and proved fixed point results for cyclic mappings:

Definition 8. *A self-map $f\colon X \to X$ is cyclic if there exist non-empty subsets $A_0, A_1, \ldots, A_{p-1} \subseteq X$, such that*

$$X = \bigcup_{i=1}^{p} A_i \text{ and } f(A_i) \subseteq A_{i+1} \text{ for } 0 \leqslant i \leqslant p-1 \text{ (where } A_p = A_0).$$

Ansari [2] introduced $\mathcal{C}$-class functions as follows:

Definition 9. *A mapping $F\colon [0,\infty)^2 \to \mathbb{R}$ is called a $\mathcal{C}$-class function if it is continuous and satisfies the following conditions:*

(F_1) $F(s,t) \leqslant s$, $\forall s,t \geqslant 0$;

(F_2) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$, $\forall\ s,t \geqslant 0$.

Example 7. Let $s,t \in [0,\infty)$; then we have:

(1) $F(s,t) = s - t$,

(2) $F(s,t) = \dfrac{s-t}{1+t}$,

(3) $F(s,t) = \dfrac{s}{1+t}$,

(2) $F(s,t) = ks, k \in (0,1)$.

3. Main Results. Here we consider functions $\psi \in \Psi$ and generalize the contractivity condition of Shatanawi and Abodayeh ([9], Theorem 2.1) by using $\mathcal{C}$-class function, and prove common fixed point theorems in G-metric spaces.

Theorem 1. *Let $\preceq$ be an ordered relation in a set X. Let (X,G) be a complete G-metric space and $X = A \bigcup B$, where A and B are nonempty closed subsets of X. Let f,g be self mappings on X that satisfy the following conditions:*

(i) The pair (f,g) is (A,B)-weakly increasing.

(ii) $f(A) \subseteq B$ and $g(B) \subseteq A$.

(iii) There exist two functions $\phi \in \Phi, \psi \in \Psi$, such that

$$\phi(G(fx,gfx,gy)) \leqslant F(\phi(M(x,y)),\psi(M(x,y))) \qquad (1)$$

holds for all comparative elements $x,y \in X$ with $x \in A$ and $y \in B$ and

$$\phi(G(gx,fgx,fy)) \leqslant F(\phi(M'(x,y)),\psi(M'(x,y))) \qquad (2)$$

holds for all comparative elements $x,y \in X$ with $x \in B$ and $y \in A$, where F is a $\mathcal{C}$-class function,

$$M(x,y) = \max\Big\{G(x,fx,y), \frac{G(fx,fx,y)[1+G(x,x,gy)]}{1+G(x,fx,y)}, \frac{G(gy,gy,y)[1+G(fx,fx,x)]}{1+G(x,fx,y)}\Big\}$$

and

$$M'(x,y) = \max\Big\{G(x,gx,y), \frac{G(gx,gx,y)[1+G(x,x,fy)]}{1+G(x,gx,y)}, \frac{G(fy,fy,y)[1+G(gx,gx,x)]}{1+G(x,gx,y)}\Big\}.$$

(iv) f or g is continuous.

Then, f and g have a common fixed point in $A\bigcap B$.

Proof. Start with $x_0 \in A$. Since $f(A) \subseteq B$, there exists $x_1 \in B$, such that $fx_0 = x_1$ and, since $g(B) \subseteq A$, there exists $x_2 \in A$, such that $gx_1 = x_2$. Continuing this way, we construct a sequence $\{x_n\}$ in X, such that

$$fx_{2n} = x_{2n+1},\ \text{for } x_{2n} \in A;\ \text{and } gx_{2n+1} = x_{2n+2},\ \text{for } x_{2n+1} \in B, n \geqslant 0.$$

Using condition (i), we have $x_n \preceq x_{n+1}, \forall n \geqslant 0$.
If $x_{2n} = x_{2n+1}$, for some $n \in \mathbb{N}$, then x_{2n} is a fixed point of f in $A\bigcap B$. Since $x_{2n} \preceq x_{2n+1}$, from (1) we have:

$$\phi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) = \phi(G(fx_{2n}, gfx_{2n}, gx_{2n+1})) \leqslant \leqslant F(\phi(M(x_{2n}, x_{2n+1})), \psi(M(x_{2n}, x_{2n+1}))), \quad (3)$$

where

$$M(x_{2n}, x_{2n+1}) = = \max\Big\{G(x_{2n}, fx_{2n}, x_{2n+1}), \frac{G(fx_{2n}, fx_{2n}, x_{2n+1})[1+G(x_{2n}, x_{2n}, gx_{2n+1})]}{1+G(x_{2n}, fx_{2n}, x_{2n+1})}, \frac{G(gx_{2n+1}, gx_{2n+1}, x_{2n+1})[1+G(fx_{2n}, fx_{2n}, x_{2n})]}{1+G(x_{2n}, fx_{2n}, x_{2n+1})}\Big\} =$$

$$= \max\Big\{G(x_{2n}, x_{2n+1}, x_{2n+1}), \frac{G(x_{2n+1}, x_{2n+1}, x_{2n+1})[1+G(x_{2n}, x_{2n}, x_{2n+2})]}{1+G(x_{2n}, x_{2n+1}, x_{2n+1})},$$

$$\left.\frac{G(x_{2n+2},x_{2n+2},x_{2n+1})[1+G(x_{2n+1},x_{2n+1},x_{2n})]}{1+G(x_{2n},x_{2n+1},x_{2n+1})}\right\} =$$
$$= \max\{G(x_{2n},x_{2n+1},x_{2n+1}), G(x_{2n+1},x_{2n+2},x_{2n+2})\} =$$
$$= G(x_{2n+1},x_{2n+2},x_{2n+2}).$$

From (3) and (F_1) we have:

$$\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})) \leqslant$$
$$\leqslant F(\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})), \psi(G(x_{2n+1},x_{2n+2},x_{2n+2}))) \leqslant$$
$$\leqslant \phi(G(x_{2n+1},x_{2n+2},x_{2n+2})),$$

which implies

$$F(\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})), \psi(G(x_{2n+1},x_{2n+2},x_{2n+2}))) =$$
$$= \phi(G(x_{2n+1},x_{2n+2},x_{2n+2})).$$

From (F_2) we have:

$$\phi(G(x_{2n+1},x_{2n+2},x_{2n+2}) = 0 \text{ or } \psi(G(x_{2n+1},x_{2n+2},x_{2n+2}) = 0.$$

Since $\phi \in \Phi$ and $\psi \in \Psi$, we have $G(x_{2n+1},x_{2n+2},x_{2n+2}) = 0$. That is, $x_{2n} = x_{2n+1} = x_{2n+2}$. Hence, x_{2n} is a common fixed point of f and g in $A \bigcap B$. Now, assume that $x_n \neq x_{n+1}, \forall n \geqslant 0$. Since $x_{2n} \preceq x_{2n+1}$, from (1) we have:

$$\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})) = \phi(G(fx_{2n}, gfx_{2n}, gx_{2n+1})) \leqslant$$
$$\leqslant F(\phi(M(x_{2n},x_{2n+1})),\psi(M(x_{2n},x_{2n+1}))), \tag{4}$$

where

$$M(x_{2n},x_{2n+1}) = \max\{G(x_{2n},x_{2n+1},x_{2n+1}), G(x_{2n+1},x_{2n+2},x_{2n+2})\}.$$

If $M(x_{2n},x_{2n+1}) = G(x_{2n+1},x_{2n+2},x_{2n+2}), \forall n \geqslant 0$, then from (4) we have

$$\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})) \leqslant$$
$$\leqslant F(\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})), \psi(G(x_{2n+1},x_{2n+2},x_{2n+2}))).$$

Since F is $\mathcal{C}$-class function, we have:

$$F(\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})),\psi(G(x_{2n+1},x_{2n+2},x_{2n+2}))) = \\ = \phi(G(x_{2n+1},x_{2n+2},x_{2n+2})) \implies \phi(G(x_{2n+1},x_{2n+2},x_{2n+2})) = 0$$

or

$$\psi(G(x_{2n+1},x_{2n+2},x_{2n+2})) = 0, \forall n \geqslant 0.$$

Since $\phi \in \Phi$, we have $G(x_{2n+1},x_{2n+2},x_{2n+2}) = 0, \forall n \geqslant 0$; this implies $x_{2n+1} = x_{2n+2}, \forall n \geqslant 0$: a contradiction. Therefore, $M(x_{2n},x_{2n+1}) = G(x_{2n},x_{2n+1},x_{2n+1}), \forall n \geqslant 0$.
Now, from (4) and (F_1), we get

$$\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})) \leqslant \\ \leqslant F(\phi(G(x_{2n},x_{2n+1},x_{2n+1})),\psi(G(x_{2n},x_{2n+1},x_{2n+1}))) \leqslant \\ \leqslant \phi(G(x_{2n},x_{2n+1},x_{2n+1})), \forall n \geqslant 0. \quad (5)$$

Since $x_{2n+1} \preceq x_{2n+2}$, from (2) we can prove:

$$\phi(G(x_{2n+2},x_{2n+3},x_{2n+3})) \leqslant \\ \leqslant F(\phi(G(x_{2n+1},x_{2n+2},x_{2n+2})),\psi(G(x_{2n+1},x_{2n+2},x_{2n+2}))) \leqslant \\ \leqslant \phi(G(x_{2n+1},x_{2n+2},x_{2n+2})), \forall n \geqslant 0. \quad (6)$$

From (5) and (6), we conclude that

$$\phi(G(x_{n+1},x_{n+2},x_{n+2})) \leqslant F(\phi(G(x_n,x_{n+1},x_{n+1})),\psi(G(x_n,x_{n+1},x_{n+1}))) \leqslant \\ \leqslant \phi(G(x_n,x_{n+1},x_{n+1})), \forall n \geqslant 0. \quad (7)$$

Since $\phi \in \Phi$, we get $G(x_{n+1},x_{n+2},x_{n+2}) \leqslant G(x_n,x_{n+1},x_{n+1}), \forall n \geqslant 0$, which implies that the sequence $\{G(x_n,x_{n+1},x_{n+1})\}$ is a non-negative monotonically decreasing sequence. So, there exists $r \geqslant 0$, such that

$$\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = r. \quad (8)$$

By taking the limit as $n \to \infty$ in (7), we get

$$\phi(r) \leqslant F(\phi(r), \lim_{n\to\infty} \psi(G(x_n,x_{n+1},x_{n+1}))) \leqslant \phi(r),$$

which implies that $F(\phi(r), \lim_{n\to\infty} \psi(G(x_n,x_{n+1},x_{n+1}))) = \phi(r)$.
From (F_2), we get $\phi(r) = 0$ or $\lim_{n\to\infty} \psi(G(x_n,x_{n+1},x_{n+1})) = 0$. Since $\phi \in \Phi$ and $\psi \in \Psi$, we get

$$r = \lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0. \quad (9)$$

From the definition of G-metric space, we have

$$\lim_{n\to\infty} G(x_n, x_n, x_{n+1}) = 0. \tag{10}$$

Now, we prove that $\{x_n\}$ is G-Cauchy. It is sufficient to show that $\{x_{2n}\}$ is a G-Cauchy sequence.
Suppose that $\{x_n\}$ is not Cauchy. Then, by (9), (10), and Lemma 2, there exist $\varepsilon > 0$ and two subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ of $\{x_{2n}\}$, such that $\forall k \in \mathbb{N}$, $k \leqslant n_k < m_k < n_{k+1}$ and for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{n\to\infty} G(x_{2n_k+p_1}, x_{2m_k+p_2}, x_{2m_k+p_3}) = \varepsilon. \tag{11}$$

Since $x_{2m_k} \preceq x_{2n_k+1}$, from (1) we have:

$$\begin{aligned}\phi(G(x_{2m_k+1}, x_{2m_k+2}, x_{2n_k+2})) = \phi(G(fx_{2m_k}, gfx_{2m_k}, gx_{2n_k+1})) \leqslant \\ \leqslant F(\phi(M(x_{2m_k}, x_{2n_k+1})), \psi(M(x_{2m_k}, x_{2n_k+1}))),\end{aligned} \tag{12}$$

where

$$\begin{aligned}M(x_{2m_k}, x_{2n_k+1}) = \max\Big\{&G(x_{2m_k}, x_{2m_k+1}, x_{2n_k+1}),\\ &\frac{G(x_{2m_k+1}, x_{2m_k+1}, x_{2n_k+1})[1 + G(x_{2m_k}, x_{2m_k}, x_{2n_k+2})]}{1 + G(x_{2m_k}, x_{2m_k+1}, x_{2n_k+1})},\\ &\frac{G(x_{2n_k+2}, x_{2n_k+2}, x_{2n_k+1})[1 + G(x_{2m_k+1}, x_{2m_k+1}, x_{2m_k})]}{1 + G(x_{2m_k}, x_{2m_k+1}, x_{2n_k+1})}\Big\}.\end{aligned}$$

Using (9), (10) and (11), we get $\lim_{k\to\infty} M(x_{2m_k}, x_{2n_k+1}) = \varepsilon$. Taking limit as $k \to \infty$ in (12), we get

$$\phi(\varepsilon) \leqslant F(\phi(\varepsilon), \lim_{k\to\infty} \psi(M(x_{2m_k}, x_{2n_k+1}))).$$

Since F is a $\mathcal{C}$-class function, we get

$$\phi(\varepsilon) \leqslant F(\phi(\varepsilon), \lim_{k\to\infty} \psi(M(x_{2m_k}, x_{2n_k+1}))) \leqslant \phi(\varepsilon);$$

this implies that

$$\phi(\varepsilon) = 0 \text{ or } \lim_{k\to\infty} \psi(M(x_{2m_k}, x_{2n_k+1})) = 0;$$

so we get $\varepsilon = \lim_{k\to\infty} M(x_{2m_k},\ x_{2n_k+1}) = 0$: a contradiction. Thus, $\{x_{2n}\}$ is a G-Cauchy sequence in (X, G). So, the sequence $\{x_n\}$ is a G-Cauchy sequence in (X, G). Since (X, G) is complete, there exists $u \in$

X, such that $\{x_n\}$ is G-convergent to u. Therefore, the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are G-convergent to u. Since $\{x_{2n}\} \subseteq A$ and A are closed, $u \in A$. Also, $\{x_{2n+1}\} \subseteq B$ and B are closed, so $u \in B$. Now, we may assume that f is continuous. So, we have $fu = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} x_{2n+1} = u$. By uniqueness of the limit, we have $fu = u$.
Since $u \preceq u$, from (1) we have:

$$\phi(G(u,gu,gu)) = \phi(G(fu,gfu,gu)) \leqslant F(\phi(M(u,u)),\psi(M(u,u))), \tag{13}$$

where

$$M(u,u) = \max\Big\{G(u,fu,u), \frac{G(fu,fu,u)[1+G(u,u,gu)]}{[1+G(u,fu,u)]}, \frac{G(gu,gu,u)[1+G(fu,fu,u)]}{[1+G(u,fu,u)]}\Big\} = G(u,gu,gu).$$

Using (13), we obtain

$$\phi(G(u,gu,gu)) \leqslant F(\phi(G(u,gu,gu)),\psi(G(u,gu,gu))).$$

Since F is a $\mathcal{C}$-class function, we have

$$\phi(G(u,gu,gu)) = 0 \text{ or } \psi(G(u,gu,gu)) = 0.$$

This implies $G(u,gu,gu) = 0$. Hence, $gu = u$. Thus, u is a common fixed point of f and g in $A \bigcap B$. $\square$

The following example shows that the condition (iii) defined in Theorem 1 is more general than the condition (iii) of Theorem 2.1 in [9].

Example 8. Let $X = \{0,1\}$ and define $G\colon X\times X\times X \to [0,\infty)$ as

$$G(0,0,0) = G(1,1,1) = 0,\ G(0,0,1) = 1 \text{ and } G(0,1,1) = 2.$$

Then the function G is a G-metric on X.
Take $A = B = \{0,1\}$, and $x \preceq y$ if and only if $x \leqslant y$. Define the mappings $f,g\colon X \to X$ as follows:

$$f(0) = 1, f(1) = 0 \text{ and } g(0) = 0,\ g(1) = 1.$$

Let $\phi, \psi\colon [0,\infty) \to [0,\infty)$ and $F\colon [0,\infty)\times[0,\infty) \to \mathbb{R}$ be defined by $\phi(t) = t/2$, $\psi(t) = t$ and $F(s,t) = s/(1+t)$, for all $s,t \in [0,\infty)$.
For $x = 0$, $y = 1$,

$$M(0,1) = \max\Big\{G(0,f0,1), \frac{G(f0,f0,1)[1+G(0,0,g1)]}{1+G(0,f0,1)}, \frac{G(g1,g1,1)[1+G(f0,f0,0)]}{1+G(0,f0,1)}\Big\} = \max\{2,0\} = 2.$$

Now,

$$F(\phi(M(0,1)),\psi(M(0,1))) = F(\phi(2),\psi(2)) = F(1,2) = \frac{1}{3} \geqslant \geqslant \phi(G(f0,gf0,g1)) = \phi(0).$$

For $x = 1$, $y = 0$:

$$M(1,0) = \max\Big\{G(1,f1,0), \frac{G(f1,f1,0)[1+G(1,1,g0)]}{1+G(1,f1,0)}, \frac{G(g0,g0,0)[1+G(f1,f1,1)]}{1+G(1,f1,0)}\Big\} = \max\{1,0\} = 1.$$

Now,

$$F(\phi(M(1,0)),\psi(M(1,0))) = F(\phi(1),\psi(1)) = F(\frac{1}{2},1) = \frac{1}{4} \geqslant \geqslant \phi(G(f1,gf1,g0)) = \phi(0).$$

Hence, the condition (iii) of Theorem 1 is satisfied.
But $\phi(G(0,f0,1)) - \psi(G(0,f0,1)) = \phi(2) - \psi(2) = 1 - 2 = -1 \leqslant 0$. This shows that the condition (iii) of Theorem 2.1 in [9] does not hold.

In Theorem 1, if we replace $\psi \in \Psi$ with $\psi \in \Phi$ and take $M(x,y) = G(x,fx,y)$, $M'(x,y) = G(x,gx,y)$ and $F(s,t) = s - t$, then we get Theorem 2.1 of [9], as a particular case.
Now, the following example validates Theorem 1.

Example 9. Let $X = [0,1/2]$ and let $f,g\colon X \to X$ be given as $f(x) = \dfrac{x^2}{1+x}$ and $g(x) = \dfrac{x}{2}$. Take $A = [0,1/2]$ and $B = [0,1/2]$.

Define the function $G\colon X \times X \times X \to [0, \infty)$ as

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Clearly, G is a complete G-metric on X. We introduce a relation on X by $x \preceq y$ if and only if $y \leqslant x$. Also, define the functions $F\colon [0, \infty)^2 \to \mathbb{R}$ by $F(s, t) = s - t$ and $\phi, \psi\colon [0, \infty) \to [0,\infty)$ by $\phi(t) = 2t$ and $\psi(t) = \dfrac{t}{1 + 2t}$.
Note that $f(A) = [0, 1/4] \subseteq B$ and $g(B) = [0, 1/2] \subseteq A$.
To prove (i), given $x \in X$,

$$gfx = \frac{x^2}{2(1 + x)}.$$

Since $x \in [0, 1/2]$, $\dfrac{x^2}{2(1 + x)} < \dfrac{x^2}{(1 + x)}$. Thus, $gfx \leqslant fx$ and, hence, $fx \preceq gfx$ for all $x \in X$.
To prove (iii), given $x \in A$ and $y \in B$ with $y \geqslant x$. Then

$$G(fx, gfx, gy) = \max\left\{\frac{x^2}{(1 + x)}, \frac{x^2}{2(1 + x)}, \frac{y}{2}\right\} = \frac{y}{2}$$

and

$$M(x, y) = \max\left\{y, \frac{y(1 + x)}{(1 + y)}, \frac{y(1 + \frac{y}{2})}{(1 + y)}\right\} = y.$$

Since

$$\frac{2y}{2} \leqslant 2y - \frac{y}{(1 + 2y)},$$

we have

$$\phi(G(fx, gfx, fy)) \leqslant F(\phi(M(x, y)), \psi(M(x, y))).$$

Hence, all the conditions of Theorem 1 are satisfied. Notice that 0 is the unique common fixed point of f and g.

Corollary 1. *Let $\preceq$ be an ordered relation in a set X. Let (X, G) be a complete G-metric space and $X = A \cup B$, where A and B are nonempty closed subsets of X. Let f be a continuous self map on X that satisfies the following conditions:*

(1) $fx \preceq f^2x, \forall x \in X$.

(2) $f(A) \subseteq B$ *and* $f(B) \subseteq A$.

(3) There exist two functions $\phi \in \Phi, \psi \in \Psi$, *such that*

$$\phi(G(fx, f^2x, fy)) \leqslant F(\phi(M(x,y)), \psi(M(x,y))) \tag{14}$$

holds for all comparative elements $x, y \in X$, *where* F *is a* $\mathcal{C}$*-class function,*

$$M(x,y) = \max\left\{G(x, fx, y), \frac{G(fx, fx, y)[1 + G(x, x, fy)]}{1 + G(x, fx, y)}, \frac{G(fy, fy, y)[1 + G(fx, fx, x)]}{1 + G(x, fx, y)}\right\}.$$

Then f *has a fixed point in* $A \cap B$.

Proof. The proof follows from Theorem 1 by taking $g = f$. □

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