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RECOVERING THE LAPLACIAN FROM CENTERED MEANS ON BALLS AND SPHERES OF FIXED RADIUS

Abstract. Various issues related to restrictions on radii in mean-value formulas are well-known in the theory of harmonic functions. In particular, using the Brown-Schreiber-Taylor theorem on spectral synthesis for motion-invariant subspaces in $C(\mathbb{R}^n)$, one can obtain the following strengthening of the classical mean-value theorem for harmonic functions: if a continuous function on $\mathbb{R}^n$ satisfies the mean-value equations for all balls and spheres of a fixed radius r, then it is harmonic on $\mathbb{R}^n$. In connection with this result, the following problem arises: recover the Laplacian from the deviation of a function from its average values on balls and spheres of a fixed radius. The aim of this work is to solve this problem. The article uses methods of harmonic analysis, as well as the theory of entire and special functions. The key step in the proof of the main result is expansion of the Dirac delta function in terms of a system of radial distributions supported in a fixed ball, biorthogonal to some system of spherical functions. A similar approach can be used to invert a number of convolution operators with compactly supported radial distributions.

Key words: *harmonic functions, one-radius theorems, radial distributions, Fourier-Bessel transform*

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1. Introduction. Let $n \geqslant 2$ be a fixed natural number, f be a continuous function on the Euclidean space $\mathbb{R}^n$, $S_r(x)$ be a sphere in $\mathbb{R}^n$ with center x and radius r, and $d\sigma$ be an area element on $S_r(x)$. The difference

$$(\Omega_r f)(x) \coloneqq \frac{\Gamma(n/2)}{2\pi^{n/2} r^{n-1}} \int\limits_{S_r(x)} f(y)d\sigma(y) - f(x)$$

is called the centered mean of the function f on the sphere $S_r(x)$. The Laplace operator of a function $f \in C^2(\mathbb{R}^n)$ satisfies the Blaschke equality

$$\Delta f(x) = \lim_{r\to 0} \frac{2n}{r^2}(\Omega_r f)(x) \tag{1}$$

(see, for example, [4], [18, Chap. 2, Sect. 4]). Similarly, if $(V_r f)(x)$ is the centered mean of a function f on the ball $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$, i.e.,

$$(V_r f)(x) = \frac{\Gamma((n+2)/2)}{\pi^{n/2} r^n} \int\limits_{B_r(x)} f(y)dy - f(x),$$

then the Privalov formula [17]

$$\Delta f(x) = \lim_{r\to 0} \frac{2n+4}{r^2}(V_r f)(x), \quad f \in C^2(\mathbb{R}^n) \tag{2}$$

is valid. Note that for a real analytic function f on $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and all sufficiently small $r > 0$, the following more general expansions, due to Pizzetti [15], Nicolesco [14] and Poritsky [16] hold:

$$\int\limits_{S_r(x)} f(y)d\sigma(y) = 2\pi^{n/2} r^{n-1} \sum_{k=0}^{\infty} \frac{(\Delta^k f)(x)}{k!\,\Gamma\left(k+\frac{n}{2}\right)} \left(\frac{r}{2}\right)^{2k},$$

$$\int\limits_{B_r(x)} f(y)dy = \pi^{n/2} r^{n} \sum_{k=0}^{\infty} \frac{(\Delta^k f)(x)}{k!\,\Gamma\left(k+1+\frac{n}{2}\right)} \left(\frac{r}{2}\right)^{2k}.$$

Formula (1) (respectively, (2)) allows one to reconstruct Δf using an infinite number of functions $\Omega_r f$ (respectively, $V_r f$). For a fixed $r > 0$, the kernel of the operator $f \to \left(\Omega_r f, V_r f\right)$ is invariant under translations and rotations of the space $\mathbb{R}^n$. Such invariant subspaces in $C(\mathbb{R}^n)$ can be characterized by the following Brown-Schreiber-Taylor theorem [5] on spectral synthesis:

Theorem 1. *Every closed translation-invariant rotation-invariant subspace E in $C(\mathbb{R}^n)$ is spanned by the polynomial-exponential functions it contains, i.e., functions from E of the form*

$$f(x) = p(x)e^{i(x_1\zeta_1+\ldots+x_n\zeta_n)}, \quad x = (x_1,\ldots,x_n) \in \mathbb{R}^n,$$

where p is a polynomial and $\zeta_j \in \mathbb{C}$, $1 \leqslant j \leqslant n$.

Using Theorem 1, one can obtain the following strengthening of the classical mean-value theorem for harmonic functions (see the proof of Theorem 4.6 in [5] and also [19] for a generalization of the statement).

Theorem 2. *Let r be a fixed positive number, $f \in C(\mathbb{R}^n)$, and assume that*

$$(\Omega_r f)(x) = (V_r f)(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Then the function f is harmonic in $\mathbb{R}^n$.

In connection with Theorem 2, the problem of finding Δf from only two functions $\Omega_r f$ and $V_r f$ arises. The purpose of this article is to solve this problem. It is closely related to the inversion problems of the classical Pompeiu transform (see [2, Sect. 3]). For example, in [11] a formula was found for reconstructing a function $f \in C^1(\mathbb{R}^n)$ from the spherical means

$$\int\limits_{S_r(x)} f(y)d\sigma(y), \qquad \int\limits_{S_r(x)} \frac{\partial f}{\partial \mathbf{n}}(y)d\sigma(y),$$

where $\dfrac{\partial}{\partial \mathbf{n}}$ means differentiation along the outward normal to $S_r(x)$. In [1], the well-known Zalcman problem [26, Sect. 8] about the inversion of the operator

$$f \to \Big(\int\limits_{S_{r_1}(x)} f d\sigma, \int\limits_{S_{r_2}(x)} f d\sigma \Big), \quad f \in C(\mathbb{R}^n)$$

under natural conditions on r_1/r_2 was studied (see also [3], [22], where the case of rank-one Riemannian symmetric spaces of noncompact type was considered). In paper [23], the problem of finding a function $f \in C(\mathbb{R}^n)$ by its known integrals

$$\int\limits_{S_r(x)} f(y)d\sigma(y), \qquad \int\limits_{B_r(x)} f(y)dy$$

is solved, and an answer is given to a similar question for two-point homogeneous spaces. All of these problems can be interpreted in terms of the general deconvolution problem, which has attracted attention of many authors (see [6], [25] and references therein). We also note that various questions related to restrictions on radii in mean-value formulas are well-known in the theory of harmonic functions (see [8], [12], [13, Sect. 3]).

The formulation of the main result and its discussion is given in Sect. 2 (see Theorem 3 below). Section 3 contains the necessary auxiliary statements. The proof of Theorem 3 is obtained in Sect. 4. Our constructions are based on the development of the ideas proposed in [21], [25]. For other methods and results related to recovering from spherical means, see [1], [11], [24, Part 2, Chap. 3] and the bibliography there.

2. Statement of the main result. In the sequel, as usual, $\mathbb{C}^n$ is a n-dimensional complex space with a Hermitian scalar product

$$(\zeta,\varsigma)=\sum_{j=1}^{n}\zeta_j\,\overline{\varsigma}_j,\quad \zeta=(\zeta_1,\ldots,\zeta_n),\quad \varsigma=(\varsigma_1,\ldots,\varsigma_n),$$

$\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are the spaces of distributions and compactly supported distributions on $\mathbb{R}^n$, respectively.

The Fourier-Laplace transform of a distribution $T\in\mathcal{E}'(\mathbb{R}^n)$ is the entire function

$$\widehat{T}(\zeta)=\langle T(x),e^{-i(\zeta,x)}\rangle,\quad \zeta\in\mathbb{C}^n.$$

In this case, $\widehat{T}$ grows on $\mathbb{R}^n$ not faster than a polynomial and

$$\langle\widehat{T},\psi\rangle=\langle T,\widehat{\psi}\rangle,\quad \psi\in\mathcal{S}(\mathbb{R}^n),\tag{3}$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions from $C^\infty(\mathbb{R}^n)$ (see [10, Chap. 7]).

If $T_1,T_2\in\mathcal{D}'(\mathbb{R}^n)$ and at least one of these distributions has compact support, then their convolution T_1*T_2 is a distribution in $\mathcal{D}'(\mathbb{R}^n)$, acting according to the rule

$$\langle T_1*T_2,\varphi\rangle=\langle T_2(y),\langle T_1(x),\varphi(x+y)\rangle\rangle,\quad \varphi\in\mathcal{D}(\mathbb{R}^n),\tag{4}$$

where $\mathcal{D}(\mathbb{R}^n)$ is the space of finite infinitely differentiable functions on $\mathbb{R}^n$. For $T_1,T_2\in\mathcal{E}'(\mathbb{R}^n)$, the Borel formula

$$\widehat{T_1*T_2}=\widehat{T}_1\,\widehat{T}_2.\tag{5}$$

is valid.

Let $\mathcal{E}'_\natural(\mathbb{R}^n)$ be the space of radial (invariant under rotations of the space $\mathbb{R}^n$) distributions in $\mathcal{E}'(\mathbb{R}^n)$, $n\geqslant 2$. The simplest example of a distribution in the class $\mathcal{E}'_\natural(\mathbb{R}^n)$ is the Dirac delta function δ with support at the zero. We put

$$\mathbf{I}_\nu(z)=\frac{J_\nu(z)}{z^\nu},\quad \nu\in\mathbb{C},$$

where J_ν is the Bessel function of the first kind of order ν. The spherical transform $\widetilde{T}$ of a distribution $T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ is defined by

$$\widetilde{T}(z) = \langle T, \varphi_z \rangle, \quad z \in \mathbb{C},$$

where φ_z is a spherical function on $\mathbb{R}^n$, i.e.,

$$\varphi_z(x) = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)\mathbf{I}_{\frac{n}{2}-1}(z|x|), \quad x \in \mathbb{R}^n$$

(see [9, Chap. 4]). The function φ_z is uniquely determined by the following conditions:

1) φ_z is radial and $\varphi_z(0) = 1$;
2) φ_z satisfies the Helmholtz differential equation

$$\Delta(\varphi_z) + z^2\varphi_z = 0. \tag{6}$$

Note that $\widetilde{T}$ is an even entire function of exponential type and the Fourier transform $\widehat{T}$ is expressed in terms of $\widetilde{T}$ by

$$\widehat{T}(\zeta) = \widetilde{T}\left(\sqrt{\zeta_1^2 + \ldots + \zeta_n^2}\right), \quad \zeta \in \mathbb{C}^n. \tag{7}$$

The set of all zeros of the function $\widetilde{T}$ that lie in the half-plane $\operatorname{Re} z \geqslant 0$ and do not belong to the negative part of the imaginary axis are denoted by $\mathcal{Z}_+(\widetilde{T})$, i.e.,

$$\mathcal{Z}_+(\widetilde{T}) = \{z \in \mathbb{C} \colon \widetilde{T}(z) = 0, \operatorname{Re} z \geqslant 0,\, iz \notin (0, +\infty)\}. \tag{8}$$

Let χ_r be the indicator of the ball $B_r = \{x \in \mathbb{R}^n \colon |x| < r\}$, σ_r be the surface delta function concentrated on the sphere $|x| = r$, i.e.,

$$\langle \sigma_r, \varphi \rangle = \int\limits_{S_r} \varphi(x) d\sigma(x), \quad \varphi \in C(\mathbb{R}^n).$$

Set

$$X_r(x) = \frac{1}{2\pi}\left(\ln\frac{r}{|x|}\right)\chi_r(x),\, Y_r(x) = X_r(x) + \frac{1}{4\pi r^2}(|x|^2 - r^2)\chi_r(x), \quad \text{if } n = 2,$$

$$X_r(x) = \frac{\Gamma(n/2)}{2(n-2)\pi^{n/2}}\left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right)\chi_r(x),$$

$$Y_r(x) = X_r(x) + \frac{\Gamma(n/2)}{4\pi^{n/2}r^n}(|x|^2 - r^2)\chi_r(x),$$

if $n \geqslant 3$,

$$\mathcal{A}_r = \delta - \frac{\Gamma(n/2)}{2\pi^{n/2}r^{n-1}}\,\sigma_r, \quad \mathcal{T}_r = \delta - \frac{\Gamma((n+2)/2)}{\pi^{n/2}r^n}\,\chi_r.$$

For these distributions, we have (see [24, Part 2, Ch. 3, formula (3.90)] and equalities (17), (41), (42) below):

$$\widetilde{\mathcal{A}}_r(z) = \vartheta_{n-2}(rz),\ \widetilde{X}_r(z) = \frac{\vartheta_{n-2}(rz)}{z^2},\ \widetilde{\mathcal{T}}_r(z) = \vartheta_n(rz),\ \widetilde{Y}_r(z) = \frac{\vartheta_n(rz)}{z^2}, \tag{9}$$

where

$$\vartheta_n(z) = 1 - 2^{\frac{n}{2}}\Gamma\left(\frac{n+2}{2}\right)\mathbf{I}_{\frac{n}{2}}(z). \tag{10}$$

We need some results about the zeros of $\vartheta_n(z)$ obtained in [20]. It follows from the general facts of the theory of entire functions that $\vartheta_n(z)$ has infinitely many zeros. In this case, all zeros except $z = 0$ are simple, and $z = 0$ is a zero of multiplicity 2. Note also that $\vartheta_n(z)$ has no real and purely imaginary zeros except $z = 0$. We denote by $\Upsilon_n = \{z_{n,1}, z_{n,2}, \ldots\}$ the sequence of all zeros of the function $\vartheta_n(z)$ in the half-plane $\operatorname{Re} z > 0$, numbered in ascending order of the module (if the modules are equal, then the numbering is arbitrary). The asymptotic equalities

$$|J_{n/2+1}(z_{n,k})| = \frac{|z_{n,k}/2|^{n/2}}{\pi\Gamma((n+2)/2)} + O(|z_{n,k}|^{n/2-1}), \quad k \to \infty, \tag{11}$$

$$\sqrt{\pi}|z_{n,k}|^{(n+1)/2} = 2^{(n-1)/2}\Gamma((n+2)/2)e^{|\operatorname{Im} z_{n,k}|} + O(|z_{n,k}|^{(n-1)/2}), \quad k \to \infty. \tag{12}$$

are valid. In addition, for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty}\frac{1}{|z_{n,k}|^{1+\varepsilon}} < +\infty. \tag{13}$$

Using the above-listed properties of zeros of the function $\vartheta_n(z)$ and relation (9), one can obtain the corresponding information about the sets $\mathcal{Z}_+(\widetilde{X}_r)$ and $\mathcal{Z}_+(\widetilde{Y}_r)$ (see (8)). In particular, all zeros of $\mathcal{Z}_+(\widetilde{X}_r)$ and $\mathcal{Z}_+(\widetilde{Y}_r)$ are simple,

$$\mathcal{Z}_+(\widetilde{X}_r) = \left\{\frac{z_{n-2,1}}{r}, \frac{z_{n-2,2}}{r}, \ldots\right\}, \quad \mathcal{Z}_+(\widetilde{Y}_r) = \left\{\frac{z_{n,1}}{r}, \frac{z_{n,2}}{r}, \ldots\right\}, \tag{14}$$

and $\mathcal{Z}_+(\widetilde{X}_r) \cap \mathcal{Z}_+(\widetilde{Y}_r) = \varnothing$ (see Lemma 3 below).

For $\lambda \in \mathcal{Z}_+(\widetilde{X}_r)$, $\mu \in \mathcal{Z}_+(\widetilde{Y}_r)$, we define the functions

$$X_r^\lambda(x) = \frac{\Gamma(\frac{n}{2})\chi_r(x)}{4\pi^{\frac{n}{2}-1}\lambda^{4-n}}\left(\mathbf{N}_{\frac{n}{2}-1}(\lambda r)\mathbf{I}_{\frac{n}{2}-1}(\lambda|x|) - \mathbf{I}_{\frac{n}{2}-1}(\lambda r)\mathbf{N}_{\frac{n}{2}-1}(\lambda|x|)\right) - \frac{X_r(x)}{\lambda^2},$$

$$Y_r^\mu(x) = \frac{n\Gamma(\frac{n}{2})\chi_r(x)}{4\pi^{\frac{n}{2}-1}\mu^{4-n}}\Big(\mathbf{N}_{\frac{n}{2}}(\mu r)\mathbf{I}_{\frac{n}{2}-1}(\mu|x|) - \mathbf{I}_{\frac{n}{2}}(\mu r)\mathbf{N}_{\frac{n}{2}-1}(\mu|x|) + \\ + \frac{2}{\pi(\mu r)^n}\Big) - \frac{Y_r(x)}{\mu^2},$$

where $\mathbf{N}_\nu(z) = N_\nu(z)/z^\nu$, N_ν is the Bessel function of the second kind of order ν (the Neumann function).

Let

$$a(z) = (z+1)(z+4), \quad b(z) = (z-1)(z-4), \tag{15}$$

$$\Theta_{1,r} = a(\Delta)X_r, \quad \Theta_{2,r} = b(\Delta)Y_r. \tag{16}$$

Then, by virtue of the formula

$$\widetilde{p(\Delta)f}(z) = p(-z^2)\widetilde{f}(z) \quad (p \quad \text{is an algebraic polynomial}), \tag{17}$$

and the equalities in (9), we obtain

$$\widetilde{\Theta}_{1,r}(z) = a(-z^2)\frac{\vartheta_{n-2}(rz)}{z^2}, \quad \widetilde{\Theta}_{2,r}(z) = b(-z^2)\frac{\vartheta_n(rz)}{z^2}, \tag{18}$$

$$\begin{aligned} \mathcal{Z}_+\left(\widetilde{\Theta}_{1,r}\right) &= \left\{\frac{z_{n-2,1}}{r}, \frac{z_{n-2,2}}{r}, \dots\right\} \cup \{1,2\}, \\ \mathcal{Z}_+\left(\widetilde{\Theta}_{2,r}\right) &= \left\{\frac{z_{n,1}}{r}, \frac{z_{n,2}}{r}, \dots\right\} \cup \{i,2i\}, \end{aligned} \tag{19}$$

and all zeros of $\widetilde{\Theta}_{1,r}$ and $\widetilde{\Theta}_{2,r}$ are simple. In addition,

$$\mathcal{Z}_+\left(\widetilde{\Theta}_{1,r}\right) \cap \mathcal{Z}_+\left(\widetilde{\Theta}_{2,r}\right) = \varnothing \tag{20}$$

(see Lemma 3 below).

For $\lambda \in \mathcal{Z}_+\left(\widetilde{\Theta}_{1,r}\right)$ $\left(\text{respectively, } \mu \in \mathcal{Z}_+\left(\widetilde{\Theta}_{2,r}\right)\right)$, we put

$$\Theta_{1,r}^\lambda = a(\Delta)X_r^\lambda \quad \left(\Theta_{2,r}^\mu = b(\Delta)Y_r^\mu\right), \tag{21}$$

if $\lambda \in \mathcal{Z}_+(\widetilde{X}_r)$ $\left(\mu \in \mathcal{Z}_+(\widetilde{Y}_r)\right)$, and

$$\Theta_{1,r}^\lambda = c_\lambda(\Delta)X_r \quad \left(\Theta_{2,r}^\mu = d_\mu(\Delta)Y_r\right), \tag{22}$$

if $a(-\lambda^2) = 0$ $\big(b(-\mu^2) = 0\big)$, where

$$c_\lambda(z) = -\frac{a(z)}{z+\lambda^2} \quad \Big(d_\mu(z) = -\frac{b(z)}{z+\mu^2}\Big). \tag{23}$$

Our main result is

Theorem 3. *Let $f \in \mathcal{D}'(\mathbb{R}^n)$, $n \geqslant 2$. Then*

$$\Delta f = \sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})} \sum_{\mu\in\mathcal{Z}_+(\widetilde{\Theta}_{2,r})} \frac{4\lambda\mu}{(\lambda^2-\mu^2)\widetilde{\Theta}'_{1,r}(\lambda)\widetilde{\Theta}'_{2,r}(\mu)} \Big(a(\Delta)(f*\mathcal{A}_r)*\Theta^{\mu}_{2,r} - \\ - b(\Delta)(f*\mathcal{T}_r)*\Theta^{\lambda}_{1,r}\Big), \tag{24}$$

where the series (24) *converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$.*

Using the definition of convolution, it is not difficult to obtain the equalities

$$f*\mathcal{A}_r = -\Omega_r f, \quad f*\mathcal{T}_r = -V_r f, \quad f \in C(\mathbb{R}^n).$$

Thus, Theorem 3 provides a solution to the problem formulated above (see (15), (18), (19), (21)–(23)). The key step in the proof of the main result is the expansion of the Dirac delta function in terms of a system of radial distributions supported in $\overline{B}_r$, biorthogonal to some system of spherical functions (see the proof of Lemma 7 in Sect. 3 below). A similar approach can be used to invert a number of convolution operators with radial distributions in $\mathcal{E}'(\mathbb{R}^n)$. Other methods in this direction have been developed in [1], [3], [11], [24, Part 2, Chap. 3]. However, the constructions that arise in this case are more cumbersome and less explicit.

3. Auxiliary assertions. First we prove the following simple statement:

Lemma 1. *Let $g\colon \mathbb{C} \to \mathbb{C}$ be an even entire function and $g(\lambda) = 0$ for some $\lambda \in \mathbb{C}$. Then*

$$\left|\frac{\lambda g(z)}{z^2-\lambda^2}\right| \leqslant \max_{|\zeta-z|\leqslant 2} |g(\zeta)|, \quad z \in \mathbb{C}, \tag{25}$$

where for $z = \pm\lambda$ the left-hand side in (25) *is extended by continuity.*

Proof. We have

$$\left|\frac{2\lambda g(z)}{z^2-\lambda^2}\right| = \left|\frac{g(z)}{z-\lambda} - \frac{g(z)}{z+\lambda}\right| \leqslant \left|\frac{g(z)}{z-\lambda}\right| + \left|\frac{g(z)}{z+\lambda}\right|. \tag{26}$$

Let us estimate the first term in the right-hand side of (26).

If $|z-\lambda|>1$, then

$$\left|\frac{g(z)}{z-\lambda}\right| \leqslant |g(z)| \leqslant \max_{|\zeta-z|\leqslant 2} |g(\zeta)|. \tag{27}$$

Assume that $|z-\lambda|\leqslant 1$. Then, applying the maximum-modulus principle to the entire function $\dfrac{g(\zeta)}{\zeta-\lambda}$, we obtain

$$\left|\frac{g(z)}{z-\lambda}\right| \leqslant \max_{|\zeta-\lambda|\leqslant 1} \left|\frac{g(\zeta)}{\zeta-\lambda}\right| = \max_{|\zeta-\lambda|=1} |g(\zeta)|.$$

Bearing in mind that the circle $|\zeta-\lambda|=1$ is contained in the disc $|\zeta-z|\leqslant 2$, we arrive at the estimate

$$\left|\frac{g(z)}{z-\lambda}\right| \leqslant \max_{|\zeta-z|\leqslant 2} |g(\zeta)|, \tag{28}$$

which is valid for all $z\in\mathbb{C}$ (see (27)).

Similarly,

$$\left|\frac{g(z)}{z+\lambda}\right| \leqslant \max_{|\zeta-z|\leqslant 2} |g(\zeta)|, \quad z\in\mathbb{C}, \tag{29}$$

because $g(-\lambda)=0$. By (28), (29), and (26) the required assertion follows. □

Let us now give some properties of the functions $\mathbf{I}_\nu$, which will be needed later.

Lemma 2. *1) When $\nu>-1/2$, $z\in\mathbb{C}$, the inequality*

$$\left|\frac{2^\nu\Gamma(\nu+1)\mathbf{I}_\nu(z)-1}{z^2}\right| \leqslant e^{|\operatorname{Im} z|} \tag{30}$$

takes place.

2) If $\nu\in\mathbb{R}$, then

$$|\mathbf{I}_\nu(z)| \sim \frac{1}{\sqrt{2\pi}} \frac{e^{|\operatorname{Im} z|}}{|z|^{\nu+\frac{1}{2}}}, \quad \operatorname{Im} z\to\infty. \tag{31}$$

3) Let $z\in\mathbb{C}\backslash(-\infty,0]$. Then

$$\Delta(\mathbf{N}_{\frac{n}{2}-1}(z|x|)) + z^2\mathbf{N}_{\frac{n}{2}-1}(z|x|) = 0, \tag{32}$$

$$\mathbf{I}_\nu(z)\mathbf{N}_{\nu-1}(z) - \mathbf{I}_{\nu-1}(z)\mathbf{N}_\nu(z) = \frac{2}{\pi z^{2\nu}}. \tag{33}$$

Proof. 1) By the Poisson integral representation [7, Chap. 7, Sect. 7.12, formula (8)] we have

$$\mathbf{I}_\nu(z) = \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^1 \cos(uz)(1-u^2)^{\nu-\frac{1}{2}}du.$$

Therefore,

$$|\mathbf{I}_\nu(z)| \leqslant \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^1 e^{u|\operatorname{Im} z|}(1-u^2)^{\nu-\frac{1}{2}}du \leqslant$$
$$\leqslant \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\frac{1}{2}\mathrm{B}\left(\frac{1}{2},\nu+\frac{1}{2}\right)e^{|\operatorname{Im} z|} = \frac{e^{|\operatorname{Im} z|}}{2^\nu\Gamma(\nu+1)}.$$

In particular,

$$\left|\frac{\sin z}{z}\right| = \left(\frac{\pi}{2}\right)^{1/2}\left|\mathbf{I}_{1/2}(z)\right| \leqslant e^{|\operatorname{Im} z|},$$
$$\left|\frac{\cos z - 1}{z^2}\right| = \frac{1}{2}\left|\frac{\sin(z/2)}{z/2}\right|^2 \leqslant \frac{e^{|\operatorname{Im} z|}}{2}.$$

From here, we get

$$\left|\frac{2^\nu\Gamma(\nu+1)\mathbf{I}_\nu(z)-1}{z^2}\right| = 2^\nu\Gamma(\nu+1)\left|\frac{\mathbf{I}_\nu(z)-\mathbf{I}_\nu(0)}{z^2}\right| =$$
$$= \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\left|\int_0^1 \frac{\cos(uz)-1}{z^2}(1-u^2)^{\nu-\frac{1}{2}}du\right| \leqslant$$
$$\leqslant \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}e^{|\operatorname{Im} z|}\int_0^1 (1-u^2)^{\nu-\frac{1}{2}}du = e^{|\operatorname{Im} z|},$$

which is the required result.

2) The asymptotic expansion of Bessel functions [7, Chap. 7, Sect. 7.13.1, formula (3)] implies the equality

$$\mathbf{I}_\nu(z) = \sqrt{\frac{2}{\pi}}\,z^{-\nu-\frac{1}{2}}\left(\cos\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)+O\left(\frac{e^{|\operatorname{Im} z|}}{|z|}\right)\right),\ z\to\infty,\ -\pi<\arg z<\pi. \tag{34}$$

Considering
$$|\cos w| \sim \frac{e^{|\operatorname{Im} w|}}{2}, \quad \operatorname{Im} w \to \infty,$$
by (34) we obtain (31).

3) The Neumann function $N_\nu(z)$ satisfies the Bessel differential equation
$$z^2\frac{d^2N_\nu(z)}{dz^2} + z\frac{dN_\nu(z)}{dz} + (z^2-\nu^2)N_\nu(z) = 0$$
(see [7, Chap. 7, Sect. 7.2.1, formula (1)]). Using this equality and the formula
$$\Delta\big(f(|x|)\big) = f''(|x|) + \frac{n-1}{|x|}f'(|x|),$$
we arrive at (32). The relation (33) is a form of writing the well-known Lommel-Hankel formula (see, for example, [21, Chap. 7, formula (7.6)]). □

Lemma 3. *For any $r>0$, the functions $\widetilde{X}_r$ and $\widetilde{Y}_r$ do not have common zeros.*

Proof. Assume that $\widetilde{X}_r(\lambda) = \widetilde{Y}_r(\lambda) = 0$. Then, from (9), the equalities
$$2^{\frac{n-2}{2}}\Gamma\Big(\frac{n}{2}\Big)\mathbf{I}_{\frac{n-2}{2}}(\mu) = 1, \quad 2^{\frac{n}{2}}\Gamma\Big(\frac{n+2}{2}\Big)\mathbf{I}_{\frac{n}{2}}(\mu) = 1$$
follow, where $\mu = r\lambda$. Hence, we find
$$\mu J_{\frac{n-2}{2}}(\mu) = nJ_{\frac{n}{2}}(\mu).$$
Using this equality and identity
$$2\nu J_\nu(z) = z(J_{\nu-1}(z) + J_{\nu+1}(z))$$
(see [7, Chap. 7, Sect. 7.2.8, formula (56)]), we have $\mu J_{\frac{n+2}{2}}(\mu) = 0$. Now, taking into account that all zeros of the function $J_{\frac{n+2}{2}}$ are real (see [7, Chap. 7, Sect. 7.9]), we obtain $\lambda = \mu/r \in \mathbb{R}$. This contradicts the properties of the zeros of the function ϑ_n given in Sect. 2. □

Lemma 4. *The equalities*
$$\Delta(X_r^\lambda) + \lambda^2 X_r^\lambda = -X_r, \quad \lambda \in \mathcal{Z}_+(\widetilde{X}_r), \tag{35}$$
$$\Delta(Y_r^\mu) + \mu^2 Y_r^\mu = -Y_r, \quad \mu \in \mathcal{Z}_+(\widetilde{Y}_r) \tag{36}$$
hold.

Proof. Assume that $\mu \in \mathbb{C}\setminus(-\infty, 0]$. For any function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\big\langle(\Delta + \mu^2)(\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\chi_r(x)), \varphi\big\rangle = \big\langle\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\chi_r(x), (\Delta + \mu^2)\varphi\big\rangle =$$

$$= \lim_{\varepsilon\to+0} \int\limits_{\varepsilon\leqslant|x|\leqslant r} \mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\Delta\varphi(x)dx + \mu^2 \int\limits_{|x|\leqslant r} \mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\varphi(x)dx.$$

We apply Green's formula

$$\int\limits_G (v\Delta u - u\Delta v)dx = \int\limits_{\partial G} \Big(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\Big)d\sigma, \tag{37}$$

to the integral under the sign of the limit, where $\dfrac{\partial}{\partial \mathbf{n}}$ is the differentiation operator in the direction of the external normal. Then

$$\int\limits_{\varepsilon\leqslant|x|\leqslant r} \mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\Delta\varphi(x)dx = \int\limits_{\varepsilon\leqslant|x|\leqslant r} \varphi(x)\Delta(\mathbf{N}_{\frac{n}{2}-1}(\mu|x|))dx+$$

$$+ \int\limits_{|x|=r} \Big(\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\frac{\partial\varphi}{\partial \mathbf{n}}(x) - \varphi(x)\frac{\partial}{\partial \mathbf{n}}\big(\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\big)\Big)d\sigma(x)-$$

$$- \int\limits_{|x|=\varepsilon} \Big(\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\frac{\partial\varphi}{\partial \mathbf{n}}(x) - \varphi(x)\frac{\partial}{\partial \mathbf{n}}\big(\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\big)\Big)d\sigma(x).$$

Now, using (32), (37), and formulas

$$\mathbf{N}'_\nu(z) = -z\mathbf{N}_{\nu+1}(z), \quad \frac{\partial}{\partial \mathbf{n}}\big(f(|x|)\big) = f'(|x|), \quad \mathbf{n} = \frac{x}{|x|}$$

(see [7, Chap. 7, Sect. 7.2.8]), we find

$$\int\limits_{\varepsilon\leqslant|x|\leqslant r} \mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\Delta\varphi(x)dx = -\mu^2 \int\limits_{\varepsilon\leqslant|x|\leqslant r} \mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\varphi(x)dx+$$

$$+\mathbf{N}_{\frac{n}{2}-1}(\mu r)\langle\Delta\chi_r, \varphi\rangle + \mu^2 r\mathbf{N}_{\frac{n}{2}}(\mu r)\langle\sigma_r, \varphi\rangle - \mathbf{N}_{\frac{n}{2}-1}(\mu\varepsilon) \int\limits_{|x|\leqslant\varepsilon} \Delta\varphi(x)dx-$$

$$-\mu^2\varepsilon\mathbf{N}_{\frac{n}{2}}(\mu\varepsilon) \int\limits_{|x|=\varepsilon} \varphi(x)d\sigma(x). \tag{38}$$

Since

$$\lim_{\varepsilon\to+0}(\mu\varepsilon)^n\mathbf{N}_{\frac{n}{2}}(\mu\varepsilon)=-\frac{2^{n/2}\Gamma(n/2)}{\pi}$$

(see [7, Chap. 7, Sect. 7.2, formulas (2), (4), (32)]), equality (38) and the mean-value theorem for the integral show that

$$\lim_{\varepsilon\to+0}\int\limits_{\varepsilon\leqslant|x|\leqslant r}\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\Delta\varphi(x)dx=-\mu^2\int\limits_{|x|\leqslant r}\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\varphi(x)dx+$$

$$+\mathbf{N}_{\frac{n}{2}-1}(\mu r)\langle\Delta\chi_r,\varphi\rangle+\mu^2r\mathbf{N}_{\frac{n}{2}}(\mu r)\langle\sigma_r,\varphi\rangle+2^{n/2+1}\pi^{n/2-1}\mu^{2-n}\varphi(0).$$

Thus,

$$(\Delta+\mu^2)\big(\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)\chi_r(x)\big)=\mathbf{N}_{\frac{n}{2}-1}(\mu r)\Delta\chi_r+$$
$$+\mu^2r\mathbf{N}_{\frac{n}{2}}(\mu r)\sigma_r+2^{n/2+1}\pi^{n/2-1}\mu^{2-n}\delta,\quad \mu\in\mathbb{C}\backslash(-\infty,0].$$

Hence,

$$(\Delta+\mu^2)\big((\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)-\mathbf{N}_{\frac{n}{2}-1}(\mu r))\chi_r(x)\big)=\mu^2r\mathbf{N}_{\frac{n}{2}}(\mu r)\sigma_r-$$
$$-\mu^2\mathbf{N}_{\frac{n}{2}-1}(\mu r)\chi_r+2^{n/2+1}\pi^{n/2-1}\mu^{2-n}\delta,\quad \mu\in\mathbb{C}\backslash(-\infty,0]. \tag{39}$$

By the similar reasoning one can verify the correctness of the equalities

$$(\Delta+\mu^2)\big((\mathbf{I}_{\frac{n}{2}-1}(\mu|x|)-\mathbf{I}_{\frac{n}{2}-1}(\mu r))\chi_r(x)\big)=$$
$$=\mu^2r\mathbf{I}_{\frac{n}{2}}(\mu r)\sigma_r-\mu^2\mathbf{I}_{\frac{n}{2}-1}(\mu r)\chi_r,\quad \mu\in\mathbb{C}, \tag{40}$$

$$(\Delta+\mu^2)Y_r=-\mathcal{T}_r+\mu^2Y_r=\mu^2Y_r-\delta+\frac{\Gamma((n+2)/2)}{\pi^{n/2}r^n}\chi_r. \tag{41}$$

Relations (39), (40), and (33) imply the representation

$$(\Delta+\mu^2)\left(\left(\mathbf{N}_{\frac{n}{2}}(\mu r)\mathbf{I}_{\frac{n}{2}-1}(\mu|x|)-\mathbf{I}_{\frac{n}{2}}(\mu r)\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)+\frac{2}{\pi(\mu r)^n}\right)\chi_r(x)\right)=$$

$$=\frac{2\mu^{2-n}}{\pi r^n}\chi_r-2^{n/2+1}\pi^{n/2-1}\mu^{2-n}\mathbf{I}_{\frac{n}{2}}(\mu r)\delta.$$

Therefore, by virtue of (41) and (9), we have

$$(\Delta+\mu^2)\left(\left(\mathbf{N}_{\frac{n}{2}}(\mu r)\mathbf{I}_{\frac{n}{2}-1}(\mu|x|)-\mathbf{I}_{\frac{n}{2}}(\mu r)\mathbf{N}_{\frac{n}{2}-1}(\mu|x|)+\frac{2}{\pi(\mu r)^n}\right)\chi_r(x)-\right.$$

$$-\frac{2\pi^{\frac{n}{2}-1}\mu^{2-n}Y_r}{\Gamma((n+2)/2)}\Bigg) = \frac{2\pi^{\frac{n}{2}-1}\mu^{2-n}}{\Gamma((n+2)/2)}\vartheta_n(\mu r)\delta - \frac{2\pi^{\frac{n}{2}-1}\mu^{4-n}Y_r}{\Gamma((n+2)/2)}, \ \mu \in \mathbb{C}\backslash(-\infty, 0].$$

For $\lambda \in \mathcal{Z}_+(\widetilde{\Phi}_r)$, this equality can be written as (36).

Similarly, we find

$$(\Delta + \lambda^2)X_r = -\mathcal{A}_r + \lambda^2 X_r = \lambda^2 X_r - \delta + \frac{\Gamma(n/2)}{2\pi^{n/2}r^{n-1}}\sigma_r, \tag{42}$$

$$(\Delta + \lambda^2)\left(\left(\mathbf{N}_{\frac{n}{2}-1}(\lambda r)\mathbf{I}_{\frac{n}{2}-1}(\lambda|x|) - \mathbf{I}_{\frac{n}{2}-1}(\lambda r)\mathbf{N}_{\frac{n}{2}-1}(\lambda|x|)\right)\chi_r(x)\right) =$$
$$= \frac{2r^{1-n}\lambda^{2-n}}{\pi}\sigma_r - 2^{n/2+1}\pi^{n/2-1}\lambda^{2-n}\mathbf{I}_{\frac{n}{2}-1}(\lambda r)\delta,$$

$$(\Delta + \lambda^2)\Bigg(\left(\mathbf{N}_{\frac{n}{2}-1}(\lambda r)\mathbf{I}_{\frac{n}{2}-1}(\lambda|x|) - \mathbf{I}_{\frac{n}{2}-1}(\lambda r)\mathbf{N}_{\frac{n}{2}-1}(\lambda|x|)\right)\chi_r(x) -$$
$$-\frac{4\pi^{\frac{n}{2}-1}\lambda^{2-n}X_r}{\Gamma(n/2)}\Bigg) = \frac{4\pi^{\frac{n}{2}-1}\lambda^{2-n}}{\Gamma(n/2)}\vartheta_{n-2}(\lambda r)\delta - \frac{4\pi^{\frac{n}{2}-1}\lambda^{4-n}X_r}{\Gamma(n/2)}, \quad \lambda \in \mathbb{C}\backslash(-\infty, 0].$$

For $\lambda \in \mathcal{Z}_+(\widetilde{X}_r)$, this equality is equivalent to (35). □

Remark 1. *From (17) and the injectivity of the spherical transform it follows that for distributions $U, T \in \mathcal{E}'_\natural(\mathbb{R}^n)$ and $\lambda \in \mathcal{Z}_+(\widetilde{T})$:*

$$\Delta U + \lambda^2 U = -T \quad \Leftrightarrow \quad \widetilde{U}(z) = \frac{\widetilde{T}(z)}{z^2 - \lambda^2}. \tag{43}$$

So, relations (35) and (36) imply the equalities

$$\widetilde{X_r^\lambda}(z) = \frac{\widetilde{X}_r(z)}{z^2 - \lambda^2}, \quad \lambda \in \mathcal{Z}_+(\widetilde{X}_r), \quad \widetilde{Y_r^\mu}(z) = \frac{\widetilde{Y}_r(z)}{z^2 - \mu^2}, \quad \mu \in \mathcal{Z}_+(\widetilde{Y}_r). \tag{44}$$

Lemma 5. *Let $\lambda \in \mathcal{Z}_+\big(\widetilde{\Theta}_{1,r}\big)$, $\mu \in \mathcal{Z}_+\big(\widetilde{\Theta}_{2,r}\big)$. Then*

$$\widetilde{\Theta_{1,r}^\lambda}(z) = \frac{\widetilde{\Theta}_{1,r}(z)}{z^2 - \lambda^2}, \quad \widetilde{\Theta_{2,r}^\mu}(z) = \frac{\widetilde{\Theta}_{2,r}(z)}{z^2 - \mu^2}. \tag{45}$$

Proof. Formulas in (45) easily follow from (17) and Remark 1. Indeed, if $\lambda \in \mathcal{Z}_+\big(\widetilde{X}_r\big)$, then, due to (21), (17), (44), and (18), we have

$$\widetilde{\Theta_{1,r}^\lambda}(z) = a(-z^2)\widetilde{X_r^\lambda}(z) = \frac{a(-z^2)\widetilde{X}_r(z)}{z^2 - \lambda^2} = \frac{\widetilde{\Theta}_{1,r}(z)}{z^2 - \lambda^2}.$$

Similarly, if $a(-\lambda^2) = 0$, then

$$\widetilde{\Theta^{\lambda}_{1,r}}(z) = c_\lambda(-z^2)\widetilde{X}_r(z) = \frac{a(-z^2)\widetilde{X}_r(z)}{z^2-\lambda^2} = \frac{\widetilde{\Theta}_{1,r}(z)}{z^2-\lambda^2}$$

(see (22), (23), (17), and (18)). The second equality in (45) is proved in exactly the same way. □

Lemma 6. *For any* $r > 0$,

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})} \frac{1}{|\widetilde{\Theta}_{1,r}'(\lambda)|} < +\infty, \qquad \sum_{\mu\in\mathcal{Z}_+(\widetilde{\Theta}_{2,r})} \frac{1}{|\widetilde{\Theta}_{2,r}'(\mu)|} < +\infty. \tag{46}$$

Proof. From (10) and formula

$$\mathbf{I}_\nu'(z) = -z\mathbf{I}_{\nu+1}(z)$$

(see [7, Chap. 7, Sect. 7.2.8]) it follows that

$$\vartheta_n'(z) = 2^{n/2}\Gamma((n+2)/2)z\mathbf{I}_{n/2+1}(z). \tag{47}$$

Using (18) and (47), we find

$$\widetilde{\Theta}_{2,r}'(z) = 2^{n/2}\Gamma((n+2)/2)r^2z^{-1}b(-z^2)\mathbf{I}_{n/2+1}(rz)-$$

$$-2\vartheta_n(rz)\big(z^{-1}b'(-z^2) + z^{-3}b(-z^2)\big).$$

Now, from (14) we have

$$\sum_{\mu\in\mathcal{Z}_+(\widetilde{Y}_r)} \frac{1}{|\widetilde{\Theta}_{2,r}'(\mu)|} = \sum_{k=1}^{\infty} \frac{|z_{n,k}|}{2^{n/2}\Gamma((n+2)/2)r^3|b(-z_{n,k}^2/r^2)||\mathbf{I}_{\frac{n}{2}+1}(z_{n,k})|}.$$

This series is comparable with the convergent series

$$\sum_{k=1}^{\infty} \frac{1}{|z_{n,k}|^2}$$

(see (11), (13) and (15)). Hence, we obtain the convergence of the second series in (46). The convergence of the first series in (46) is proved similarly. □

Lemma 7. *Let*

$$\Psi_{1,r}^{\lambda} = \frac{2\lambda}{\widetilde{\Theta}_{1,r}'(\lambda)}\,\Theta_{1,r}^{\lambda},\ \lambda \in \mathcal{Z}_{+}\big(\widetilde{\Theta}_{1,r}\big),\ \Psi_{2,r}^{\mu} = \frac{2\mu}{\widetilde{\Theta}_{2,r}'(\mu)}\,\Theta_{2,r}^{\mu},\ \mu \in \mathcal{Z}_{+}\big(\widetilde{\Theta}_{2,r}\big). \tag{48}$$

Then

$$\sum_{\lambda\in\mathcal{Z}_{+}(\widetilde{\Theta}_{1,r})} \Psi_{1,r}^{\lambda} = \sum_{\mu\in\mathcal{Z}_{+}(\widetilde{\Theta}_{2,r})} \Psi_{2,r}^{\mu} = \delta, \tag{49}$$

where the series in (49) converge unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$.

Proof. For an arbitrary function $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we define the function $\psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\psi(y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \varphi(x)e^{i(x,y)}dx, \quad y \in \mathbb{R}^n.$$

Then (see (3), (7), and (45))

$$\big\langle \Psi_{1,r}^{\lambda}, \varphi\big\rangle = \big\langle \Psi_{1,r}^{\lambda}, \widehat{\psi}\big\rangle = \big\langle \widehat{\Psi_{1,r}^{\lambda}}, \psi\big\rangle =$$

$$= \int_{\mathbb{R}^n} \psi(x)\widetilde{\Psi_{1,r}^{\lambda}}(|x|)dx = \frac{2}{\widetilde{\Theta}_{1,r}'(\lambda)}\int_{\mathbb{R}^n} \psi(x)\frac{\lambda\widetilde{\Theta}_{1,r}(|x|)}{|x|^2-\lambda^2}dx.$$

Using this representation and Lemma 1, we get

$$\big|\langle \Psi_{1,r}^{\lambda}, \varphi\rangle\big| \leqslant \frac{2}{\big|\widetilde{\Theta}_{1,r}'(\lambda)\big|}\int_{\mathbb{R}^n} |\psi(x)| \max_{|\zeta-|x||\leqslant 2}\big|\widetilde{\Theta}_{1,r}(\zeta)\big|dx.$$

From (18), (9) and (30) we have

$$\max_{|\zeta-|x||\leqslant 2}\big|\widetilde{\Theta}_{1,r}(\zeta)\big| = r^2 \max_{|\zeta-|x||\leqslant 2}\big|a(-\zeta^2)\big|e^{r|\mathrm{Im}\zeta|} \leqslant r^2\, e^{2r} \max_{|\zeta-|x||\leqslant 2}\big|a(-\zeta^2)\big|.$$

Therefore,

$$\big|\langle \Psi_{1,r}^{\lambda}, \varphi\rangle\big| \leqslant \frac{2r^2\, e^{2r}}{\big|\widetilde{\Theta}_{1,r}'(\lambda)\big|}\int_{\mathbb{R}^n} |\psi(x)| \max_{|\zeta-|x||\leqslant 2}\big|a(-\zeta^2)\big|dx. \tag{50}$$

This inequality and Lemma 6 show that the first series in (49) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$ to some distribution f supported in

$\overline{B}_r$. By Lemma 5, the spherical transform of this distribution satisfies the equality

$$\widetilde{f}(z)=\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\widetilde{\Psi^{\lambda}_{1,r}}(z)=\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\frac{2\lambda}{\widetilde{\Theta}_{1,r}^{\,'}(\lambda)}\frac{\widetilde{\Theta}_{1,r}(z)}{z^2-\lambda^2}. \tag{51}$$

In this case, if $\xi\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})$, then

$$\widetilde{f}(\xi)=\frac{2\xi}{\widetilde{\Theta}_{1,r}^{\,'}(\xi)}\lim_{z\to\xi}\frac{\widetilde{\Theta}_{1,r}(z)}{z^2-\xi^2}=1. \tag{52}$$

Next, since $\widetilde{f}(z)-1$ and $\widetilde{\Theta}_{1,r}(z)$ are even entire functions of the exponential type, then, by virtue of (52) and the simplicity of the zeros of $\widetilde{\Theta}_{1,r}$, their ratio

$$H(z)=\frac{\widetilde{f}(z)-1}{\widetilde{\Theta}_{1,r}(z)}$$

is an entire function of at most the first order. In view of equality (12), there exists $R>0$, such that $|\arg\lambda|<\pi/12$ for $\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})$, $|\lambda|\geqslant R$. Therefore, for $\operatorname{Im}z=\pm\operatorname{Re}z$, $|z|>R$, the function H is evaluated as follows:

$$|H(z)|\leqslant\frac{|\widetilde{f}(z)|}{|\widetilde{\Theta}_{1,r}(z)|}+\frac{1}{|\widetilde{\Theta}_{1,r}(z)|}\leqslant$$

$$\leqslant\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\frac{1}{|\widetilde{\Theta}_{1,r}^{\,'}(\lambda)|}\left(\frac{1}{|z-\lambda|}+\frac{1}{|z+\lambda|}\right)+\frac{1}{|\widetilde{\Theta}_{1,r}(z)|}\leqslant$$

$$\leqslant\sum_{\substack{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})\\|\lambda|<R}}\frac{1}{|\widetilde{\Theta}_{1,r}^{\,'}(\lambda)|}\left(\frac{1}{|z-\lambda|}+\frac{1}{|z+\lambda|}\right)+\frac{4}{|z|}\sum_{\substack{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})\\|\lambda|\geqslant R}}\frac{1}{|\widetilde{\Theta}_{1,r}^{\,'}(\lambda)|}+\frac{1}{|\widetilde{\Theta}_{1,r}(z)|}.$$

It can be seen from this estimate and relations (46) and (31) that

$$\lim_{\substack{z\to\infty\\ \operatorname{Im}z=\pm\operatorname{Re}z}}H(z)=0. \tag{53}$$

Then, according to the Phragmén-Lindelof principle, H is bounded on $\mathbb{C}$. Now it follows from (53) and Liouville's theorem that $H=0$. Hence $\widetilde{f}=1$, i.e., $f=\delta$. Similarly, we obtain that the second series in (49) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$ to the delta function δ. Thus, Lemma 7 is proved. $\square$

Lemma 8. *Let* $\lambda \in \mathcal{Z}_+(\widetilde{\Theta}_{1,r})$, $\mu \in \mathcal{Z}_+(\widetilde{\Theta}_{2,r})$. *Then*

$$(\lambda^2-\mu^2)\Psi^{\lambda}_{1,r} * \Psi^{\mu}_{2,r} = \frac{4\lambda\mu}{\widetilde{\Theta}'_{1,r}(\lambda)\widetilde{\Theta}'_{2,r}(\mu)}\left(\Theta_{2,r} * \Theta^{\lambda}_{1,r} - \Theta_{1,r} * \Theta^{\mu}_{2,r}\right). \tag{54}$$

Proof. By (45), (43) and (48) we have

$$(\Delta+\lambda^2)\left(\Psi^{\lambda}_{1,r}\right) = -\frac{2\lambda}{\widetilde{\Theta}'_{1,r}(\lambda)}\Theta_{1,r}, \tag{55}$$

$$(\Delta+\mu^2)\left(\Psi^{\mu}_{2,r}\right) = -\frac{2\mu}{\widetilde{\Theta}'_{2,r}(\mu)}\Theta_{2,r}. \tag{56}$$

From (55), (48), and the permutation of the differentiation operator with convolution, we obtain

$$(\Delta+\lambda^2)\left(\Psi^{\lambda}_{1,r} * \Psi^{\mu}_{2,r}\right) = \frac{-4\lambda\mu}{\widetilde{\Theta}'_{1,r}(\lambda)\widetilde{\Theta}'_{2,r}(\mu)}\,\Theta_{1,r} * \Theta^{\mu}_{2,r}.$$

Similarly, it follows from (56) that

$$-(\Delta+\mu^2)\left(\Psi^{\lambda}_{1,r} * \Psi^{\mu}_{2,r}\right) = \frac{4\lambda\mu}{\widetilde{\Theta}'_{1,r}(\lambda)\widetilde{\Theta}'_{2,r}(\mu)}\,\Theta_{2,r} * \Theta^{\lambda}_{1,r}.$$

Adding the last two equalities, we arrive at relation (54). □

4. Proof of Theorem 3.

We claim that

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\ \sum_{\mu\in\mathcal{Z}_+(\widetilde{\Theta}_{2,r})} \Psi^{\lambda}_{1,r} * \Psi^{\mu}_{2,r} = \delta, \tag{57}$$

where the series in (57) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n)$, and $\varphi = \widehat{\psi}$. For $\lambda \in \mathcal{Z}_+(\widetilde{\Theta}_{1,r})$, $\mu \in \mathcal{Z}_+(\widetilde{\Theta}_{2,r})$, we have (see (5) and the proof of estimate (50)):

$$\left|\left\langle \Psi^{\lambda}_{1,r} * \Psi^{\mu}_{2,r}, \varphi\right\rangle\right| = \left|\left\langle \Psi^{\lambda}_{1,r} * \Psi^{\mu}_{2,r}, \widehat{\psi}\right\rangle\right| = \left|\left\langle \widehat{\Psi^{\lambda}_{1,r}}\,\widehat{\Psi^{\mu}_{2,r}}, \psi\right\rangle\right| =$$

$$= \left|\int_{\mathbb{R}^n} \psi(x)\widetilde{\Psi^{\lambda}_{1,r}}(|x|)\widetilde{\Psi^{\mu}_{2,r}}(|x|)dx\right| =$$

$$= \frac{4}{\left|\widetilde{\Theta}_{1,r}'(\lambda)\widetilde{\Theta}_{2,r}'(\mu)\right|}\left|\int\limits_{\mathbb{R}^n}\psi(x)\frac{\lambda\widetilde{\Theta}_{1,r}(|x|)}{|x|^2-\lambda^2}\,\frac{\mu\widetilde{\Theta}_{2,r}(|x|)}{|x|^2-\mu^2}dx\right| \leqslant$$

$$\leqslant \frac{4r^4e^{4r}}{\left|\widetilde{\Theta}_{1,r}'(\lambda)\widetilde{\Theta}_{2,r}'(\mu)\right|}\int\limits_{\mathbb{R}^n}|\psi(x)|\max_{|\zeta-|x||\leqslant 2}\left|a(-\zeta^2)\right|\max_{|\zeta-|x||\leqslant 2}\left|b(-\zeta^2)\right|dx.$$

Hence, from (46) it follows that

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\left(\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Theta}_{2,r})}\left|\left\langle\Psi_{1,r}^{\lambda}*\Psi_{2,r}^{\mu},\varphi\right\rangle\right|\right)<\infty.$$

Therefore, the series in (57) converges unconditionally in the space $\mathcal{D}'(\mathbb{R}^n)$. In addition (see (4), (49)),

$$\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Theta}_{2,r})}\left\langle\Psi_{1,r}^{\lambda}*\Psi_{2,r}^{\mu},\varphi\right\rangle=$$

$$=\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\left(\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Theta}_{2,r})}\left\langle\Psi_{2,r}^{\mu}(y),\left\langle\Psi_{1,r}^{\lambda}(x),\varphi(x+y)\right\rangle\right\rangle\right)=$$

$$=\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\left\langle\Psi_{1,r}^{\lambda}(x),\varphi(x)\right\rangle=\varphi(0),$$

which proves (57).

Convolving both parts of (57) with Δf and taking into account the separate continuity of the convolution of $f\in\mathcal{D}'(\mathbb{R}^n)$ with $g\in\mathcal{E}'(\mathbb{R}^n)$, (54), and (20), we find

$$\Delta f=\sum_{\lambda\in\mathcal{Z}_+(\widetilde{\Theta}_{1,r})}\sum_{\mu\in\mathcal{Z}_+(\widetilde{\Theta}_{2,r})}\frac{4\lambda\mu}{(\lambda^2-\mu^2)\widetilde{\Theta}_{1,r}'(\lambda)\widetilde{\Theta}_{2,r}'(\mu)}\big(\Delta f*\Theta_{2,r}*\Theta_{1,r}^{\lambda}- -\Delta f*\Theta_{1,r}*\Theta_{2,r}^{\mu}\big). \tag{58}$$

Finally, using (58), (16), (41), (42) and the commutativity of the convolution operator with the differentiation operator, we arrive at formula (24). Thus, Theorem 3 is proved. □

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