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INTEGRAL INEQUALITIES OF SIMPSON TYPE VIA WEIGHTED INTEGRALS

Abstract. In this work, we use weighted integrals to obtain new integral inequalities of the Simpson type for the class of (h, m, s)-convex functions of the second type. In the work we show that the obtained results include some known from the literature, as particular cases.

Key words: *convex fuction, inequality of Simpson, weighted integral operator, (h, m, s)-convex function, Hadamard-type inequality, Hölder inequality, power mean inequality*

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1. Introduction. The concept of convexity is one of the important concepts of a number of applied disciplines, such as computational mathematics, optimization theory, the theory of inequalities, and many more.

The peculiarity of this concept is that it is associated with an estimate of the mean value of a function given on an interval.

To improve and extend this estimate, a number of convexity classes have been defined in the literature. The study [22] presents a fairly wide range of convexity classes.

One of the most famous inequalities used to estimate the mean value of a convex function on an interval is the double Hermite-Hadamard inequality:

$$f\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \leqslant \frac{1}{\upsilon_2-\upsilon_1}\int\limits_{\upsilon_1}^{\upsilon_2} f(\tau)d\tau \leqslant \frac{f(\upsilon_1)+f(\upsilon_2)}{2}.$$

In the case of concavity of the function, the inequalities have the opposite sign.

More information on the Hermite-Hadamard inequality and other extensions can be seen in [5], [9], [20], [21] and references therein.

Along with the Hermite-Hadamard-type inequality, the Simpson-type inequality is well-known and is provided in the literature as follows:

If $f \in C^4((\upsilon_1, \upsilon_2))$ and $\|f^{(4)}\|_\infty := \sup\limits_{x\in(\upsilon_1,\upsilon_2)} |f^{(4)}(x)| < \infty$, then

$$\left|\frac{\upsilon_2-\upsilon_1}{3}\left[\frac{f(\upsilon_1)+f(\upsilon_2)}{2}+2f\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right]-\int\limits_{\upsilon_1}^{\upsilon_2} f(\tau)d\tau\right| \leqslant \frac{(\upsilon_2-\upsilon_1)^5}{2880}\left\|f^{(4)}\right\|_\infty.$$

Much research in the recent years has been devoted to Simpson-type inequalities.

In the study [1], Alomari and Hussain, and [2] Bayraktar have obtained some Simpson-type inequalities for quasi-convex and r-convex functions in terms of the third derivatives. In [4], Bayraktar et al., several new generalized integral inequalities of the Hadamard- and Simpson-type were obtained for the functions whose first and third derivatives are convex or satisfy the Lipschitz condition. Dragomir et al. in [7] obtained new Simpson-type inequalities and gave their application in the quadrature formula. In article [8], Du and et al. established new inequalities of Simpson type for extended (s, m)-convex functions under certain conditions. In [9], authors introduce the concepts of m-invex set, generalized (s, m)-preinvex function, and explicitly (s, m)-preinvex function, and established new Hadamard-Simpson-type integral inequalities. Du et al. in [10] obtained estimation-type results related to $k-$fractional integral operators for functions with preinvex absolute value of the derivative. In [13], Hua et al. introduced the concept of a «strongly s-convex function» and obtained some new Simpson-type inequalities for strongly s-convex functions. Hussain and Qaisar [14] established new inequalities of Simpson-type for the functions whose third derivatives in absolute values are preinvex and prequasi-convex. Hsu et al. [15] established some extended Simpson-type inequalities for differentiable convex and concave functions that are connected with Hermite-Hadamard inequality. Kashuri et al., in the study [17], using a new identity, established Simpson-type inequalities for differentiable $s-$convex functions in the second sense.

Other refinements and generalizations of this inequality can be seen in [18], [19] and references therein.

In [3] the authors presented the following definitions:

Definition 1. *Let* $h\colon [0,1] \to (0,1]$ *and* $f\colon J = [0,+\infty) \to [0,+\infty)$.

If inequality

$$f(\tau\xi + m(1-\tau)\varsigma) \leqslant h^s(\tau)f(\xi) + m(1-h^s(\tau))f(\varsigma)$$

is fulfilled for all $\xi, \varsigma \in J$ *and* $\tau \in [0,1]$, *where* $m \in [0,1]$, $s \in [-1,1]$, *then a function* ψ *is called a* (h,m,s)*-convex modified of the first type on* J.

Definition 2. *Let* $h\colon [0,1] \to (0,1]$ *and* $f\colon J = [0,+\infty) \to [0,+\infty)$. *If inequality*

$$f(\tau\xi + m(1-\tau)\varsigma) \leqslant h^s(\tau)f(\xi) + m(1-h(\tau))^s f(\varsigma)$$

is fulfilled for all $\xi, \varsigma \in J$ *and* $\tau \in [0,1]$, *where* $m \in [0,1]$, $s \in [-1,1]$, *then a function* ψ *is called a* (h,m,s)*-convex modified of the second type on* J.

Remark 1. *From Definitions 1 and 2, we can obtain many known convex notions, for example: the classic convex (with* $h(\varsigma) = \varsigma$, $m = s = 1$*), the* m*-convex (with* $h(\varsigma) = \varsigma$, $s = 1$*), the* s*-convex (with* $h(\varsigma) = \varsigma$, $m = 1$*), among other.*

The use of fractional operators in different fields of application sciences is known to everyone. Fractional operators frequently used in the literature are Riemann-Liouville, Katugampola, and Caputo operators (for example see [12], [6], [16], [23]). However, the researchers did not settle for the classic fractional operators and defined new generalized fractional operators.

Let us give the definition of weighted integral operators, which is the basis of this work.

Definition 3. *Let* $f \in L([\upsilon_1, \upsilon_2]), w \in C([0,1])$ *and* $w\colon [0,1] \to \mathbb{R}$, *with* w' *piecewise continuous on* I. *Then the weighted fractional integrals are defined by (right and left, respectively):*

$$J^w_{\upsilon_1^+} f(r) = \int_{\upsilon_1}^{r} w'\left(\frac{r-\sigma}{r-\upsilon_1}\right) f(\sigma)d\sigma \quad \text{and} \quad J^w_{\upsilon_2^-} f(r) = \int_{r}^{\upsilon_2} w'\left(\frac{\sigma-r}{\upsilon_2-r}\right) f(\sigma)d\sigma,$$

with $\upsilon_1 < r \leqslant \upsilon_2$.

Remark 2. *It is obvious that by choosing the expression for the function* $w(\varsigma)$ *one can obtain various well-known integral operators.*

For example:

(a) *if we take* $w(t) = \frac{(r-\upsilon_1)^{\alpha-1}}{\alpha\Gamma(\alpha)}t^\alpha$ *and* $w(t) = \frac{(\upsilon_2-r)^{\alpha-1}}{\alpha\Gamma(\alpha)}t^\alpha$, *then we get the Riemann-Liouville integrals (right and left, respectively);*

(b) *if we take*

$$w'\left(\frac{r-t}{r-\upsilon_1}\right) = \frac{(r-\upsilon_1)^{\alpha-1}\rho^{1-\alpha}}{\Gamma(\alpha)}t^{\rho-1}(r^\rho - t^\rho)^{\alpha-1}$$

and

$$w'\left(\frac{t-r}{\upsilon_2-r}\right) = \frac{(\upsilon_2-r)^{\alpha-1}\rho^{1-\alpha}}{\Gamma(\alpha)}t^{\rho-1}(t^\rho - r^\rho)^{\alpha-1},$$

then we get Katugampola integrals (right and left, respectively).

The main purpose of this paper is to establish several integral inequalities of the Simpson type using the Definition 3 of the weighted integral.

2. Results. The following result will be fundamental to our work.

Lemma 1. *Let* $0 < m \leqslant 1$; $f\colon [\upsilon_1 m, \upsilon_2] \to \mathbb{R}$ *be a differentiable function,* $\upsilon_1 < \upsilon_2$ *with* $\upsilon_1 \in \mathbb{R}$, $\upsilon_2 > 0$. *If* $f \in L^1([\upsilon_1 m, \upsilon_2])$ *and* $w' \geqslant 0$, *then the following equality*

$$\begin{aligned}\frac{\varrho+2}{x-\upsilon_1 m}&\left[w(1)f\left(\frac{2x+\upsilon_1\varrho m}{\varrho+2}\right) - w(0)f\left(\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}\right)\right]+\\ &+\frac{\varrho+2}{\upsilon_2-x}\left[w(1)f\left(\frac{2x+\varrho\upsilon_2}{\varrho+2}\right) - w(0)f\left(\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}\right)\right]-\\ &-\left(\frac{\varrho+2}{x-\upsilon_1 m}\right)^2 J^w_{\frac{2x+\upsilon_1\varrho m}{\varrho+2}-}f\left(\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}\right)-\\ &-\left(\frac{\varrho+2}{\upsilon_2-x}\right)^2 J^w_{\frac{2x+\varrho\upsilon_2}{\varrho+2}+}f\left(\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}\right) =\\ =\int_0^1 w(\varsigma)&\left[f'\left(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m + \frac{1+\varsigma}{\varrho+2}x\right) - f'\left(\frac{1+\varsigma}{\varrho+2}x + \frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\right)\right]d\varsigma\end{aligned}$$

holds with $\varrho \in \mathbb{N}$ *and* $x = (1-r)\upsilon_1 m + r\upsilon_2$, *for* $r \in [0,1]$.

Proof. By the properties, we have:

$$I = \int_0^1 w(\varsigma)\left[f'\left(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m+\frac{1+\varsigma}{\varrho+2}x\right)-f'\left(\frac{1+\varsigma}{\varrho+2}x+\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\right)\right]d\varsigma =$$

$$= \int_0^1 w(\varsigma) f'\Big(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m + \frac{1+\varsigma}{\varrho+2}x\Big)d\varsigma -$$

$$- \int_0^1 w(\varsigma) f'\Big(\frac{1+\varsigma}{\varrho+2}x + \frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\Big)d\varsigma = I_1 - I_2.$$

Integrating by parts and changing the variable in I_1, we state that

$$I_1 = \int_0^1 w(\varsigma) f'\Big(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m + \frac{1+\varsigma}{\varrho+2}x\Big)d\varsigma =$$

$$= \frac{\varrho+2}{x-\upsilon_1 m}\Big[w(1) f\Big(\frac{2x+\upsilon_1\varrho m}{\varrho+2}\Big) - w(0) f\left(\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}\right)\Big] -$$

$$- \Big(\frac{\varrho+2}{x-\upsilon_1 m}\Big)^2 \int_{\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}^{\frac{2x+\upsilon_1\varrho m}{\varrho+2}} w'\Big[\frac{z-\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}{\frac{x-\upsilon_1 m}{\varrho+2}}\Big] f(z)dz.$$

Since $\dfrac{x-\upsilon_1 m}{\varrho+2} = \dfrac{2x+\upsilon_1\varrho m}{\varrho+2} - \dfrac{x+\upsilon_1(\varrho+1)m}{\varrho+2}$, we finally get, for I_1:

$$I_1 = \int_0^1 w(\varsigma) f'\Big(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m + \frac{1+\varsigma}{\varrho+2}x\Big)d\varsigma =$$

$$= \frac{\varrho+2}{x-\upsilon_1 m}\Big[w(1) f\Big(\frac{2x+\upsilon_1\varrho m}{\varrho+2}\Big) - w(0) f\Big(\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}\Big)\Big] -$$

$$- \Big(\frac{\varrho+2}{x-\upsilon_1 m}\Big)^2 \int_{\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}^{\frac{2x+\upsilon_1\varrho m}{\varrho+2}} w'\left[\frac{z-\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}{\frac{2x+\upsilon_1\varrho m}{\varrho+2}-\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}\right] f(z)dz =$$

$$= \frac{\varrho+2}{x-\upsilon_1 m}\Big[w(1) f\Big(\frac{2x+\upsilon_1\varrho m}{\varrho+2}\Big) - w(0) f\Big(\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}\Big)\Big] -$$

$$- \Big(\frac{\varrho+2}{x-\upsilon_1 m}\Big)^2 J^w_{\frac{2x+\upsilon_1\varrho m}{\varrho+2}-} f\Big(\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}\Big). \quad (1)$$

Similarly, for I_2, we obtain:

$$I_2 = -\frac{\varrho+2}{\upsilon_2 - x}\Big[w(1)f\Big(\frac{2x+\varrho\upsilon_2}{\varrho+2}\Big) - w(0)f\left(\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}\right)\Big]+ \\ +\left(\frac{\varrho+2}{\upsilon_2-x}\right)^2 J^w_{\frac{2x+\varrho\upsilon_2}{\varrho+2}+} f\Big(\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}\Big). \quad (2)$$

Subtracting (2) form (1), we obtain the desired equality. This completes the proof. □

Remark 3. *Consider the previous result taking $\varrho = 0$, $w(\varsigma) = \frac{\varsigma^\alpha}{2} - \frac{1}{5}$; then Lemma 2.1 of [18] is easily obtained.*

Remark 4. *In the case $\varrho = 0$, $w(\varsigma) = \varsigma$, $m = 1$, taking $x = \upsilon_2$ in I_1 and $x = \upsilon_1$ in I_2, we obtain Lemma 1 from [4]*

$$\frac{f(\upsilon_1)+f(\upsilon_2)}{2} - \frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} f(z)dz =$$

$$= \frac{\upsilon_2-\upsilon_1}{4}\int_0^1 \varsigma\Big[f'\Big(\frac{1-\varsigma}{2}\upsilon_1 + \frac{1+\varsigma}{2}\upsilon_2\Big) - f'\Big(\frac{1+\varsigma}{2}\upsilon_1 + \frac{1-\varsigma}{2}\upsilon_2\Big)\Big]d\varsigma.$$

Remark 5. *Multiplying both sides of (1) by $\left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^2$ and both sides of (2) by $\left(\frac{\upsilon_2-x}{\varrho+2}\right)^2$, we get, respectively:*

$$\Big(\frac{x-\upsilon_1 m}{\varrho+2}\Big)^2 I_1 = \frac{x-\upsilon_1 m}{\varrho+2}\Big[w(1)f\Big(\frac{2x+\upsilon_1\varrho m}{\varrho+2}\Big) - w(0)f\Big(\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}\Big)\Big] - \\ - J^w_{\frac{2x+\upsilon_1\varrho m}{\varrho+2}-} f\left(\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}\right) = \\ = \left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^2 \int_0^1 w(\varsigma) f'\Big(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m + \frac{1+\varsigma}{\varrho+2}x\Big)d\varsigma,$$

$$\Big(\frac{\upsilon_2-x}{\varrho+2}\Big)^2 I_2 = \frac{\upsilon_2-x}{\varrho+2}\Big[w(1)f\Big(\frac{2x+\varrho\upsilon_2}{\varrho+2}\Big) - w(0)f\Big(\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}\Big)\Big] - \\ - J^w_{\frac{2x+\varrho\upsilon_2}{\varrho+2}+} f\left(\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}\right) =$$

$$= \left(\frac{\upsilon_2 - x}{\varrho + 2}\right)^2 \int_0^1 w(\varsigma) f'\left(\frac{1+\varsigma}{\varrho+2}x + \frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\right) d\varsigma.$$

Let us define

$$L = \left(\frac{x - \upsilon_1 m}{\varrho + 2}\right)^2 \cdot I_1 - \left(\frac{\upsilon_2 - x}{\varrho + 2}\right)^2 \cdot I_2 =$$
$$= \frac{x - \upsilon_1 m}{\varrho + 2}\left[w(1) f\left(\frac{2x + \upsilon_1 \varrho m}{\varrho + 2}\right) - w(0) f\left(\frac{x + \upsilon_1(\varrho+1)m}{\varrho + 2}\right)\right] +$$
$$+ \frac{\upsilon_2 - x}{\varrho + 2}\left[w(1) f\left(\frac{2x + \varrho \upsilon_2}{\varrho + 2}\right) - w(0) f\left(\frac{x + (\varrho+1)\upsilon_2}{\varrho + 2}\right)\right] -$$
$$- J^w_{\frac{2x+\upsilon_1 \varrho m}{\varrho+2}-} f\left(\frac{x + \upsilon_1(\varrho+1)m}{\varrho+2}\right) - J^w_{\frac{2x+\varrho\upsilon_2}{\varrho+2}+} f\left(\frac{x + (\varrho+1)\upsilon_2}{\varrho+2}\right).$$

From this result, we obtain different Simpson-type inequalities, which are generalizations of several ones reported in the literature.

Based on Remark 5, we prove the following theorem:

Theorem 1. *Let $0 < m \leqslant 1$ and $f : [\upsilon_1 m, \upsilon_2] \to \mathbb{R}$ be a differentiable function, $\upsilon_1 < \upsilon_2$ with $\upsilon_1 \in \mathbb{R}$, $\upsilon_2 > 0$. If $f \in L^1([\upsilon_1 m, \upsilon_2])$ is bounded, we have:*

$$|L| \leqslant \left(\frac{\upsilon_2 - \upsilon_1 m}{\varrho + 2}\right)^2 \|f'\|_\infty \int_0^1 w(\varsigma) d\varsigma,$$

where $\|f'\|_\infty = \sup_{\varsigma \in [\upsilon_1 m, \upsilon_2]} |f'(\varsigma)|$.

Proof. Use the Remark 5 and the absolute value properties to get:

$$|L| = \left| \left(\frac{x - \upsilon_1 m}{\varrho + 2}\right)^2 \int_0^1 w(\varsigma) f'\left(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m + \frac{1+\varsigma}{\varrho+2}x\right) d\varsigma + \right.$$
$$\left. + \left(\frac{\upsilon_2 - x}{\varrho + 2}\right)^2 \int_0^1 w(\varsigma) f'\left(\frac{1+\varsigma}{\varrho+2}x + \frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\right) d\varsigma \right| \leqslant$$
$$\leqslant \left(\frac{x - \upsilon_1 m}{\varrho + 2}\right)^2 \int_0^1 w(\varsigma) \left| f'\left(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m + \frac{1+\varsigma}{\varrho+2}x\right)\right| d\varsigma +$$

$$+\left(\frac{\upsilon_2-x}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\left|f'\left(\frac{1+\varsigma}{\varrho+2}x+\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\right)\right|d\varsigma\leqslant$$

$$\leqslant\left(\frac{\upsilon_2-\upsilon_1 m}{\varrho+2}\right)^2\|f'\|_\infty\int_0^1 w(\varsigma)d\varsigma.$$

Thus, we have got the desired result. □

Remark 6. *If we take* $w(\varsigma)=\dfrac{\varsigma^\alpha}{2}-\dfrac{1}{5}$ *and* $\varrho=0$, *we have the theorem analogous to Theorem 3.1 of [18].*

Theorem 2. *Let* $0<m\leqslant 1$ *and* $f\colon[\upsilon_1 m,\upsilon_2]\to\mathbb{R}$ *be a differentiable function,* $\upsilon_1<\upsilon_2$ *with* $\upsilon_1\in\mathbb{R}$, $\upsilon_2>0$. *If* $f'\in L^1([\upsilon_1 m,\upsilon_2])$, *then, for* $x\in[\upsilon_1 m,\upsilon_2]$, *we have:*

$$|L|\leqslant \mathbf{B}\cdot w(0)\,\|f'\|_1\,,$$

where $\mathbf{B}=\sup\limits_{x\in[\upsilon_1 m,\upsilon_2]}\left\{\dfrac{x-\upsilon_1 m}{\varrho+2},\dfrac{\upsilon_2-x}{\varrho+2}\right\}$ *and* $\|f'\|_1=\displaystyle\int_{\upsilon_1 m}^{\upsilon_2}|f'(x)|\,dx<\infty.$

Proof. From Remark 5, after changing variables, we obtain:

$$|L|\leqslant\left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\left|f'\left(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m+\frac{1+\varsigma}{\varrho+2}x\right)\right|d\varsigma+$$

$$+\left(\frac{\upsilon_2-x}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\left|f'\left(\frac{1+\varsigma}{\varrho+2}x+\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\right)\right|d\varsigma=$$

$$=\frac{x-\upsilon_1 m}{\varrho+2}\int_{\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}^{\frac{2x+\upsilon_1\varrho m}{\varrho+2}} w\left(\frac{z-\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}{\frac{x-\upsilon_1 m}{\varrho+2}}\right)|f'(z)|\,dz+$$

$$+\frac{\upsilon_2-x}{\varrho+2}\int_{\frac{2x+\varrho\upsilon_2}{\varrho+2}}^{\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}} w\left(\frac{\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}-z}{\frac{\upsilon_2-x}{\varrho+2}}\right)|f'(z)|\,dz\leqslant$$

$$\leqslant \mathbf{B}\cdot\left\{\int_{\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}^{\frac{2x+\upsilon_1\varrho m}{\varrho+2}} w\left(\frac{z-\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}{\frac{x-\upsilon_1 m}{\varrho+2}}\right)|f'(z)|dz+\int_{\frac{2x+\varrho\upsilon_2}{\varrho+2}}^{\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}} w\left(\frac{\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}-z}{\frac{\upsilon_2-x}{\varrho+2}}\right)|f'(z)|dz\right\}\leqslant$$

$$\leqslant \mathbf{B}\cdot w(0)\int_{\upsilon_1 m}^{\upsilon_2}\left|f'(z)\right|dz.$$

Therefore, the proof is finished. □

Remark 7. *If we take* $w(\varsigma)=\frac{\varsigma^{\alpha}}{2}-\frac{1}{5}$ *and* $\varrho=0$*, we have the Theorem 3.2 of [18].*

Theorem 3. *Let* $0<m\leqslant 1$ *and* $f:[\upsilon_1 m,\upsilon_2]\to\mathbb{R}$ *be a differentiable function,* $\upsilon_1<\upsilon_2$ *with* $\upsilon_1\in\mathbb{R}$*,* $\upsilon_2>0$*. If* $f'\in L^q([\upsilon_1 m,\upsilon_2])$*, with* $1<q$*,* $p<\infty$*, and* $\frac{1}{p}+\frac{1}{q}=1$*, we have:*

$$|L|\leqslant \mathbf{M}\cdot\left(\int_0^1|w(\varsigma)|^p\,d\varsigma\right)^{\frac{1}{p}}\left\|f'\right\|_q,$$

where $\mathbf{M}=\sup_{x\in[\upsilon_1 m,\,\upsilon_2]}\left\{\left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^{2-\frac{1}{q}},\left(\frac{\upsilon_2-x}{\varrho+2}\right)^{2-\frac{1}{q}}\right\}$, $\left\|f'\right\|_q=\left(\int_{\upsilon_1 m}^{\upsilon_2}|f'(x)|^q\,dx\right)^{\frac{1}{q}}$.

Proof. Using the Remark 5 and Hölder's inequality, we have:

$$|L|\leqslant\left(\frac{\upsilon_2-\upsilon_1 m}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\left|f'\left(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m+\frac{1+\varsigma}{\varrho+2}x\right)\right|d\varsigma+ \tag{3}$$

$$+\left(\frac{\upsilon_2-\upsilon_1 m}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\left|f'\left(\frac{1+\varsigma}{\varrho+2}x+\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\right)\right|d\varsigma\leqslant$$

$$\leqslant\left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^2\left(\int_0^1|w(\varsigma)|^p\,d\varsigma\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m+\frac{1+\varsigma}{\varrho+2}x\right)\right|^q d\varsigma\right)^{\frac{1}{q}}+$$

$$+\left(\frac{\upsilon_2-x}{\varrho+2}\right)^2\left(\int_0^1|w(\varsigma)|^p\,d\varsigma\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+\varsigma}{\varrho+2}x+\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_2\right)\right|^q d\varsigma\right)^{\frac{1}{q}};$$

rearranging, we get

$$|L| \leqslant \left(\int_0^1 |w(\varsigma)|^p \, d\varsigma \right)^{\frac{1}{p}} \left[\frac{(x - \upsilon_1 m)^{2-\frac{1}{q}}}{(\varrho + 2)^{2-\frac{1}{q}}} \left(\int_{\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}^{\frac{2x+\upsilon_1 \varrho m}{\varrho+2}} |f'(z)|^q \, dz \right)^{\frac{1}{q}} + \right.$$

$$\left. + \frac{(\upsilon_2 - x)^{2-\frac{1}{q}}}{(\varrho + 2)^{2-\frac{1}{q}}} \left(\int_{\frac{2x+\varrho\upsilon_2}{\varrho+2}}^{\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}} |f'(z)|^q \, dz \right)^{\frac{1}{q}} \right] \leqslant$$

$$\leqslant \mathbf{M} \cdot \left(\int_0^1 |w(\varsigma)|^p \, d\varsigma \right)^{\frac{1}{p}} \left[\left(\int_{\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}^{\frac{2x+\upsilon_1 \varrho m}{\varrho+2}} |f'(z)|^q \, dz \right)^{\frac{1}{q}} + \left(\int_{\frac{2x+\varrho\upsilon_2}{\varrho+2}}^{\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}} |f'(z)|^q \, dz \right)^{\frac{1}{q}} \right] \leqslant$$

$$\leqslant \mathbf{M} \cdot \left(\int_0^1 |w(\varsigma)|^p \, d\varsigma \right)^{\frac{1}{p}} \left[\left(\int_{\frac{x+\upsilon_1(\varrho+1)m}{\varrho+2}}^{\frac{2x+\varrho\upsilon_2}{\varrho+2}} |f'(z)|^q \, dz \right)^{\frac{1}{q}} + \left(\int_{\frac{2x+\varrho\upsilon_2}{\varrho+2}}^{\frac{x+(\varrho+1)\upsilon_2}{\varrho+2}} |f'(z)|^q \, dz \right)^{\frac{1}{q}} \right].$$

Thus, we have obtained the desired inequality. □

Remark 8. *if we take* $w(\varsigma) = \dfrac{\varsigma^\alpha}{2} - \dfrac{1}{5}$ *and* $\varrho = 0$, *we have the theorem analogous to Theorem 3.3 of [18].*

Theorem 4. *Let* $0 < m \leqslant 1$; $f : \left[\upsilon_1 m, \dfrac{\upsilon_2}{m}\right] \to \mathbb{R}$ *be a differentiable function,* $0 \leqslant \upsilon_1 < \upsilon_2$, *such that* $f' \in L^1\left(\left[\upsilon_1 m, \dfrac{\upsilon_2}{m}\right]\right)$. *If* $|f'|^q$ *is a* (h, m, s)*-convex modified of the second type, for* $q > 1$ *with* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *we have*

$$|L| \leqslant \mathbf{M} \cdot \left(\int_0^1 |w(\varsigma)|^p \, d\varsigma \right)^{\frac{1}{p}} \left[\left\{ m\mathbf{H}_1 |f'(\upsilon_1)|^q + \mathbf{H}_2 |f(x)|^q \right\}^{\frac{1}{q}} + \right. \tag{4}$$

$$\left. + \left\{ \mathbf{H}_2 |f'(x)|^q + m\mathbf{H}_1 \left| f\left(\frac{\upsilon_2}{m}\right) \right|^q \right\}^{\frac{1}{q}} \right],$$

$\mathbf{M}$ *as before and*

$$\mathbf{H}_1 = \int_0^1 \left(1 - h\left(\frac{1+\varsigma}{\varrho+2}\right)\right)^s d\varsigma, \quad \mathbf{H}_2 = \int_0^1 h^s\left(\frac{1+\varsigma}{\varrho+2}\right) d\varsigma.$$

Proof. From the inequality (3), since $\frac{\varrho+1-\varsigma}{\varrho+2}=1-\frac{1+\varsigma}{\varrho+2}$, we can write:

$$|L|\leqslant\Big(\frac{x-\upsilon_1 m}{\varrho+2}\Big)^2\Big(\int_0^1|w(\varsigma)|^p\,d\varsigma\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'\Big(\Big(1-\frac{1+\varsigma}{\varrho+2}\Big)\upsilon_1 m+\frac{1+\varsigma}{\varrho+2}x\Big)\Big|^q d\varsigma\Big)^{\frac{1}{q}}+$$
$$+\Big(\frac{\upsilon_2-x}{\varrho+2}\Big)^2\Big(\int_0^1|w(\varsigma)|^p d\varsigma\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'\Big(\frac{1+\varsigma}{\varrho+2}x+\Big(1-\frac{1+\varsigma}{\varrho+2}\Big)\upsilon_2\Big)\Big|^q d\varsigma\Big)^{\frac{1}{q}}.$$

By using the fact that $|f'|^q$ is (h,m,s)-convex modified of the second type, we get

$$|L|\leqslant\Big(\int_0^1|w(\varsigma)|^p\,d\varsigma\Big)^{\frac{1}{p}}\Big\{\Big(\frac{x-\upsilon_1 m}{\varrho+2}\Big)^2\times$$
$$\times\Big[m\,|f'(\upsilon_1)|^q\int_0^1\Big(1-h\Big(\frac{1+\varsigma}{\varrho+2}\Big)\Big)^s d\varsigma+|f\,(x)|^q\int_0^1 h^s\Big(\frac{1+\varsigma}{\varrho+2}\Big)d\varsigma\Big]+$$
$$+\Big(\frac{\upsilon_2-x}{\varrho+2}\Big)^2\Big[\,|f\,(x)|^q\int_0^1 h^s\Big(\frac{1+\varsigma}{\varrho+2}\Big)d\varsigma+$$
$$+\,m\Big|f'(\frac{\upsilon_2}{m})\Big|^q\int_0^1\Big(1-h\Big(\frac{1+\varsigma}{\varrho+2}\Big)\Big)^s d\varsigma\Big]\Big\}.$$

Hence, taking into account the notation, we obtain (4). □

Corollary 1. *Considering Theorem 4, we have the following cases:*

1) *Putting $x=\upsilon_1 m$, we have:*

$$|L|=\frac{\upsilon_2-\upsilon_1 m}{\varrho+2}\Big[w(1)f\Big(\frac{2ma+\varrho\upsilon_2}{\varrho+2}\Big)-w(0)f\Big(\frac{\upsilon_1 m+(\varrho+1)\upsilon_2}{\varrho+2}\Big)\Big]-$$
$$-\,J^{w}_{\frac{2ma+\varrho\upsilon_2}{\varrho+2}+}f\Big(\frac{\upsilon_1 m+(\varrho+1)\upsilon_2}{\varrho+2}\Big)\leqslant$$
$$\leqslant\Big(\frac{\upsilon_2-\upsilon_1 m}{\varrho+2}\Big)^2\Big(\int_0^1|w(\varsigma)|^p d\varsigma\Big)^{\frac{1}{p}}\Big\{\mathbf{H}_2\,|f'(\upsilon_1 m)|^q+m\mathbf{H}_1\Big|f\,\Big(\frac{\upsilon_2}{m}\Big)\Big|^q\Big\}^{\frac{1}{q}}.\quad(5)$$

2) *Putting* $x = \dfrac{\upsilon_1 m + \upsilon_2}{2}$, *we have*

$$|L| \leqslant \left(\frac{\upsilon_2 - \upsilon_1 m}{\varrho + 2}\right)^2 \left(\int_0^1 |w(\varsigma)|^p \, d\varsigma\right)^{\frac{1}{p}} \times$$
$$\times \left[\left(m\mathbf{H}_1 \left|f'(\upsilon_1)\right|^q + \mathbf{H}_2 \left|f'\left(\frac{\upsilon_1 m + \upsilon_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right] +$$
$$+ \left(\frac{\upsilon_2 - \upsilon_1 m}{\varrho + 2}\right)^2 \left(\int_0^1 |w(\varsigma)|^p \, d\varsigma\right)^{\frac{1}{p}} \times$$
$$\times \left[\left(\mathbf{H}_2 \left|f'\left(\frac{\upsilon_1 m + \upsilon_2}{2}\right)\right|^q + m\mathbf{H}_1 \left|f'\left(\frac{\upsilon_2}{m}\right)\right|^q\right)^{\frac{1}{q}}\right]. \quad (6)$$

3) *Putting* $x = \upsilon_2$, *we have*

$$|L| = \frac{\upsilon_2 - \upsilon_1 m}{\varrho + 2}\left[w(1) f\left(\frac{\varrho m \upsilon_1 + 2\upsilon_2}{\varrho + 2}\right) - w(0) f\left(\frac{(\varrho+1)\upsilon_1 m + \upsilon_2}{\varrho + 2}\right)\right] -$$
$$- J^w_{\frac{\varrho m \upsilon_1 + 2\upsilon_2}{\varrho + 2}-} f\left(\frac{(\varrho+1)\upsilon_1 m + \upsilon_2}{\varrho + 2}\right) \leqslant$$
$$\leqslant \left(\frac{\upsilon_2 - \upsilon_1 m}{\varrho + 2}\right)^2 \left(\int_0^1 |w(\varsigma)|^p d\varsigma\right)^{\frac{1}{p}} (m\mathbf{H}_1 |f'(\upsilon_1)|^q + \mathbf{H}_2 |f'(\upsilon_2)|^q)^{\frac{1}{q}}. \quad (7)$$

Corollary 2. *Combining the inequalities (5) and (7), it follows that*

$$\left|\frac{\upsilon_2 - \upsilon_1 m}{\varrho + 2}\left[\begin{array}{c} w(1) f\left(\frac{2ma + \varrho \upsilon_2}{\varrho + 2}\right) - w(0) f\left(\frac{\upsilon_1 m + (\varrho+1)\upsilon_2}{\varrho + 2}\right) \\ + w(1) f\left(\frac{\varrho m \upsilon_1 + 2\upsilon_2}{\varrho + 2}\right) - w(0) f\left(\frac{(\varrho+1)\upsilon_1 m + \upsilon_2}{\varrho + 2}\right) \end{array}\right] - \right. \quad (8)$$
$$\left. - J^w_{\frac{2ma + \varrho \upsilon_2}{\varrho + 2}+} f\left(\frac{\upsilon_1 m + (\varrho+1)\upsilon_2}{\varrho + 2}\right) - J^w_{\frac{\varrho m \upsilon_1 + 2\upsilon_2}{\varrho + 2}-} f\left(\frac{(\varrho+1)\upsilon_1 m + \upsilon_2}{\varrho + 2}\right)\right| \leqslant$$
$$\leqslant \left(\frac{\upsilon_2 - \upsilon_1 m}{\varrho + 2}\right)^2 \left(\int_0^1 |w(\varsigma)|^p \, d\varsigma\right)^{\frac{1}{p}} \cdot \mathbf{R}$$

with

$$\mathbf{R} = \left(\mathbf{H}_2 \left|f'(\upsilon_1 m)\right|^q + m\mathbf{H}_1 \left|f\left(\frac{\upsilon_2}{m}\right)\right|^q\right)^{\frac{1}{q}} + \left(m\mathbf{H}_1 \left|f'(\upsilon_1)\right|^q + \mathbf{H}_2 \left|f'(\upsilon_2)\right|^q\right)^{\frac{1}{q}}.$$

Remark 9. *Putting $w(\varsigma)=\varsigma$, $\varrho=0$, $s=m=1$, and $h(\varsigma)=\varsigma$ from (8), we obtain:*

$$\left|\frac{f(\upsilon_1)+f(\upsilon_2)}{2}-\frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2}f(x)dx\right|\leqslant$$
$$\leqslant\frac{\upsilon_2-\upsilon_1}{4^{1+\frac{1}{q}}}\frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(3\left|f'(\upsilon_1)\right|^q+\left|f'(\upsilon_2)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(\upsilon_1)\right|^q+3\left|f'(\upsilon_2)\right|^q\right)^{\frac{1}{q}}\right].$$

Theorem 5. *Let $0<m\leqslant 1$; $f:\left[\upsilon_1 m,\frac{\upsilon_2}{m}\right]\to R$ be a differentiable function, $0\leqslant\upsilon_1<\upsilon_2$, such that $f\in L^1\left(\left[\upsilon_1 m,\frac{\upsilon_2}{m}\right]\right)$. If $|f'|^q$ is a (h,m,s)-convex modified of the second type, with $m\in(0,1]$ for $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for $x\in[\upsilon_1 m,\upsilon_2]$ we have*

$$|L|\leqslant\left(\int_0^1|w(\varsigma)|^p\,d\varsigma\right)^{\frac{1}{p}}\left\{\left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^2\cdot\mathbf{E}_1+\left(\frac{\upsilon_2-x}{\varrho+2}\right)^2\cdot\mathbf{E}_2\right\}$$

with

$$\mathbf{E}_1=\left[m\left|f'\left(\upsilon_1\right)\right|^q\mathbf{H}_1+\left|f'\left(x\right)\right|^q\mathbf{H}_2\right]^{\frac{1}{q}},$$
$$\mathbf{E}_2=\left[\left|f'(x)\right|^q\mathbf{H}_2+m\left|f'\left(\frac{\upsilon_2}{m}\right)\right|^q\mathbf{H}_1\right]^{\frac{1}{q}},$$

where $\mathbf{H}_1$ and $\mathbf{H}_2$ are defined above in Theorem 4.

Proof. Using the Remark 5 and the fact that $|f'|^q$ is (h,m,s)-convex modified of the second type, we have

$$|L|\leqslant\left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\left|f'\left(\frac{\varrho+1-\varsigma}{\varrho+2}\upsilon_1 m+\frac{1+\varsigma}{\varrho+2}x\right)\right|d\varsigma+$$
$$+\left(\frac{\upsilon_2-x}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\left|f'\left(\frac{1+\varsigma}{\varrho+2}x+\frac{\varrho+1-\varsigma}{\varrho+2}\frac{\upsilon_2}{m}\right)\right|d\varsigma=$$
$$=\left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^2\int_0^1 w(\varsigma)\left|f'\left(\left(1-\frac{1+\varsigma}{\varrho+2}\right)\upsilon_1 m+\frac{1+\varsigma}{\varrho+2}x\right)\right|d\varsigma+$$

$$+\left(\frac{\upsilon_2-x}{\varrho+2}\right)^2\int\limits_0^1 w(\varsigma)\left|f'\left(\frac{1+\varsigma}{\varrho+2}x+m\left(1-\frac{1+\varsigma}{\varrho+2}\right)\frac{\upsilon_2}{m}\right)\right|d\varsigma \leqslant$$

$$\leqslant\left(\frac{x-\upsilon_1 m}{\varrho+2}\right)^2\left(\int\limits_0^1|w(\varsigma)|^p d\varsigma\right)^{\frac{1}{p}}\left\{m|f'(\upsilon_1)|\int\limits_0^1\left(1-h\left(\frac{1+\varsigma}{\varrho+2}\right)\right)^s d\varsigma+\right.$$

$$\left.+|f'(x)|^q\int\limits_0^1 h^s\left(\frac{1+\varsigma}{\varrho+2}\right)d\varsigma\right\}^{\frac{1}{q}}+$$

$$+\left(\frac{\upsilon_2-x}{\varrho+2}\right)^2\left(\int\limits_0^1|w(\varsigma)|^p\,d\varsigma\right)^{\frac{1}{p}}\left\{|f'(x)|\int\limits_0^1 h^s\left(\frac{1+\varsigma}{\varrho+2}\right)d\varsigma+\right.$$

$$\left.+m\left|f'\left(\frac{\upsilon_2}{m}\right)\right|\int\limits_0^1\left(1-h\left(\frac{1+\varsigma}{\varrho+2}\right)\right)^s d\varsigma\right\}^{\frac{1}{q}}.$$

In this way, we obtain the desired inequality. □

Remark 10. *If we take* $w(\varsigma)=\dfrac{\varsigma^\alpha}{2}-\dfrac{1}{5}$, $\varrho=0$ *and* $h(\varsigma)=\varsigma$, *we have Theorem 3.7 of [18].*

Corollary 3. *Taking* $x=\dfrac{\upsilon_1 m+\upsilon_2}{2}$ *in Theorem 5, we have the following inequality:*

$$|L|\leqslant\frac{1}{2}\left(\frac{\upsilon_2-\upsilon_1 m}{\varrho+2}\right)^2\left(\int\limits_0^1|w(\varsigma)|^p\,d\varsigma\right)^{\frac{1}{p}}(\mathbf{E}_1+\mathbf{E}_2)\,.$$

3. Applications to special means. For $0<\upsilon_1<\upsilon_2$, we have the well-known mean values, as follows:

(i) The arithmetic mean, $A(\upsilon_1,\upsilon_2)=\dfrac{\upsilon_1+\upsilon_2}{2}$.

(ii) The k-logarithmic mean, $L_k(\upsilon_1,\upsilon_2)=\left[\frac{\upsilon_2^{k+1}-\upsilon_1^{k+1}}{(k+1)(\upsilon_2-\upsilon_1)}\right]^{\frac{1}{k}}$, $k\in\mathbb{Z}\neq\{0,-1\}$.

From (6), we obtain, with $w(\varsigma)=\varsigma$, $m=1$, $\varrho=0$, $h(z)=z$ the following inequality:

$$|L|=\left|\frac{\upsilon_2-\upsilon_1}{20}\left\{3f\Big(\frac{\upsilon_1+\upsilon_2}{2}\Big)+f\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)+f\Big(\frac{\upsilon_1+3\upsilon_2}{4}\Big)\right\}-\int_{\frac{3\upsilon_1+\upsilon_2}{4}}^{\frac{\upsilon_1+3\upsilon_2}{4}} f(\tau)d\tau\right|\leqslant$$

$$\leqslant\Big(\frac{\upsilon_2-\upsilon_1}{2}\Big)^2\frac{1}{(p+1)^{\frac{1}{p}}}\left\{\Big[\frac{1}{2^s(s+1)}\Big(|f'(\upsilon_1)|^q+\Big|f'\Big(\frac{\upsilon_1+\upsilon_2}{2}\Big)\Big|^q\Big)\Big]^{\frac{1}{q}}+\right.$$

$$\left.+\Big[\frac{2^{s+1}-1}{2^s(s+1)}\Big(\Big|f'\Big(\frac{\upsilon_1+\upsilon_2}{2}\Big)\Big|^q+|f'(\upsilon_2)|^q\Big)\Big]^{\frac{1}{q}}\right\}.$$

If we additionally consider particular cases of the function $f(x)$, we can obtain different estimates for the previous means. For example, if $f(\tau)=\frac{\tau^{\frac{s}{q}+1}}{\frac{s}{q}+1}$, we have

$$|L|=\left|\frac{\upsilon_2-\upsilon_1}{20}\left\{3\frac{\Big(\frac{\upsilon_1+\upsilon_2}{2}\Big)^{\frac{s}{q}+1}}{\frac{s}{q}+1}+\frac{\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)^{\frac{s}{q}+1}}{\frac{s}{q}+1}+\frac{\Big(\frac{\upsilon_1+3\upsilon_2}{4}\Big)^{\frac{s}{q}+1}}{\frac{s}{q}+1}\right\}-\right.$$

$$\left.-\frac{\Big(\frac{\upsilon_1+3\upsilon_2}{4}\Big)^{\frac{s}{q}+2}-\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)^{\frac{s}{q}+2}}{\Big(\frac{s}{q}+1\Big)\Big(\frac{s}{q}+2\Big)}\right|=$$

$$=\left|\frac{\upsilon_2-\upsilon_1}{20}\left\{3\frac{\Big(\frac{\upsilon_1+\upsilon_2}{2}\Big)^{\frac{s}{q}+1}}{\frac{s}{q}+1}+\frac{\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)^{\frac{s}{q}+1}}{\frac{s}{q}+1}+\frac{\Big(\frac{\upsilon_1+3\upsilon_2}{4}\Big)^{\frac{s}{q}+1}}{\frac{s}{q}+1}\right\}-\right.$$

$$\left.-\frac{\Big(\frac{\upsilon_1+3\upsilon_2}{4}\Big)^{\frac{s}{q}+2}-\Big(\frac{3\upsilon_1+\upsilon_2}{4}\Big)^{\frac{s}{q}+2}}{\Big(\frac{s}{q}+2\Big)\frac{\upsilon_2-\upsilon_1}{2}}\cdot\frac{\upsilon_2-\upsilon_1}{2\Big(\frac{s}{q}+1\Big)}\right|=$$

$$=\frac{\upsilon_2-\upsilon_1}{2}\left|\frac{q}{10(s+q)}\Big[3A^{\frac{s}{q}+1}(\upsilon_1,\upsilon_2)+A^{\frac{s}{q}+1}\Big(\frac{3\upsilon_1}{2},\frac{\upsilon_2}{2}\Big)+A^{\frac{s}{q}+1}\Big(\frac{\upsilon_1}{2},\frac{3\upsilon_2}{2}\Big)\Big]-\right.$$

$$\left.-\frac{q}{s+q}L^{\frac{s}{q}+1}_{\frac{s}{q}+1}\Big(\frac{\upsilon_1+3\upsilon_2}{4},\frac{3\upsilon_1+\upsilon_2}{4}\Big)\right|,$$

and since

$$f'(x)=x^{\frac{s}{q}},\ |f'(\upsilon_1)|^q=|\upsilon_1|^s,\ |f'(\upsilon_2)|^q=|\upsilon_2|^s,\Big|f'\Big(\frac{\upsilon_1+\upsilon_2}{2}\Big)\Big|^q=\Big|\frac{\upsilon_1+\upsilon_2}{2}\Big|^s,$$

we get

$$\frac{\Big(\frac{\upsilon_2-\upsilon_1}{2}\Big)^2}{(p+1)^{\frac{1}{p}}}\left\{\left[\frac{\Big(|\upsilon_1|^s+\Big|\frac{\upsilon_1+\upsilon_2}{2}\Big|^s\Big)}{2^s(s+1)}\right]^{\frac{1}{q}}+\left[\frac{(2^{s+1}-1)\Big(\Big|\frac{\upsilon_1+\upsilon_2}{2}\Big|^s+|\upsilon_2|^s\Big)}{2^s(s+1)}\right]^{\frac{1}{q}}\right\}=$$

$$= \frac{(\upsilon_2-\upsilon_1)^2}{4(p+1)^{\frac{1}{p}}\left[2^s(s+1)\right]^{\frac{1}{q}}}\left\{\left[\left(|\upsilon_1|^s+|A(\upsilon_1,\upsilon_2)|^s\right)\right]^{\frac{1}{q}}+\right.$$
$$\left.+\left[\left(2^{s+1}-1\right)\left(|A(\upsilon_1,\upsilon_2)|^s+|\upsilon_2|^s\right)\right]^{\frac{1}{q}}\right\}.$$

Thus, we get

$$\frac{\upsilon_2-\upsilon_1}{2\left(\frac{s}{q}+1\right)}\left|\frac{1}{10}\left[3A^{\frac{s}{q}+1}(\upsilon_1,\upsilon_2)+A^{\frac{s}{q}+1}\left(\frac{3\upsilon_1}{2},\frac{\upsilon_2}{2}\right)+A^{\frac{s}{q}+1}\left(\frac{\upsilon_1}{2},\frac{3\upsilon_2}{2}\right)\right]-\right.$$
$$\left.-L_{\frac{s}{q}+1}^{\frac{s}{q}+1}\left(\frac{\upsilon_1+3\upsilon_2}{4},\frac{3\upsilon_1+\upsilon_2}{4}\right)\right|\leqslant$$
$$\leqslant\frac{(\upsilon_2-\upsilon_1)^2}{4(p+1)^{\frac{1}{p}}\left[2^s(s+1)\right]^{\frac{1}{q}}}\left\{\left[\left(|\upsilon_1|^s+|A(\upsilon_1,\upsilon_2)|^s\right)\right]^{\frac{1}{q}}+\right.$$
$$\left.+\left[\left(2^{s+1}-1\right)\left(|A(\upsilon_1,\upsilon_2)|^s+|\upsilon_2|^s\right)\right]^{\frac{1}{q}}\right\}$$

or, since $p+1=\frac{2q-1}{q-1}$,

$$\left|\frac{1}{10}\left[A^{\frac{s}{q}+1}\left(\frac{3\upsilon_1}{2},\frac{\upsilon_2}{2}\right)+3A^{\frac{s}{q}+1}(\upsilon_1,\upsilon_2)+A^{\frac{s}{q}+1}\left(\frac{\upsilon_1}{2},\frac{3\upsilon_2}{2}\right)\right]-\right.$$
$$\left.-L_{\frac{s}{q}+1}^{\frac{s}{q}+1}\left(\frac{\upsilon_1+3\upsilon_2}{4},\frac{3\upsilon_1+\upsilon_2}{4}\right)\right|\leqslant$$
$$\leqslant\frac{(\upsilon_2-\upsilon_1)(s+q)}{2q}\left(\frac{2^s(s+1)(q-1)}{2q-1}\right)^{1-\frac{1}{q}}\times$$
$$\times\left\{\left[|\upsilon_1|^s+|A(\upsilon_1,\upsilon_2)|^s\right]^{\frac{1}{q}}+\left[\left(2^{s+1}-1\right)\left(|A(\upsilon_1,\upsilon_2)|^s+|\upsilon_2|^s\right)\right]^{\frac{1}{q}}\right\}.$$

Analogous inequality is in Proposition 4.1 of [18].

4. Conclusions. In this paper, we give a new definition of the weighted integral operators (Definition 3). Using this definition, we obtain some generalized integral inequalities. It should be emphasized that the results obtained are valid for various classes of convex functions defined on a closed interval of non-negative real numbers. For example, h-convex, s-convex, and (s,m)-convex in the second sense, m-convex, and P-convex functions.

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