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Y. Habbachi

SECOND STRUCTURE RELATION FOR THE DUNKL-CLASSICAL ORTHOGONAL POLYNOMIALS

Abstract. In this paper, we characterize the Dunkl-classical orthogonal polynomials by a second structure relation.

Key words: *orthogonal polynomials, Dunkl-classical polynomials, regular forms, second structure relation*

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1. Introduction and Preliminary Results. Classical orthogonal polynomials (Hermite, Laguerre, Bessel, and Jacobi) are characterized by several properties: they satisfy Hahn's property (that the sequence of monic derivatives of the polynomial is again orthogonal (see [2], [11], [20], [22]); they are characterized as the polynomial eigenfunctions of a second-order homogeneous linear differential (or difference) hypergeometric operator with polynomial coefficients [6], [21], [22]; their corresponding linear forms satisfy a distribution equation of Pearson type (see [15], [19], [21]); they satisfy a first structure relation (the Al-Salam and Chihara property [2]) and can be characterized by the so-called Rodrigues formula (see, for instance, [11], [13]).

Another characterization was established by J. L. Geronimus in [15]; in particular, he proved that a classical sequence of monic orthogonal polynomials $\{P_n(x)\}_{n\geqslant 0}$ can be characterized by the fact that $P_n(x) = Q_n(x) + a_n Q_{n-1}(x) + b_n Q_{n-2}(x)$, where $Q_n(x) = \frac{1}{n+1} P'_{n+1}(x)$. This is the so called second structure relation for classical orthogonal polynomials (see also [20], [21], [23]).

In the recent years, many authors (see [7], [8], [9], [10], [17], [24]) have started to study Dunkl-classical orthogonal polynomials, as analogues of the Hahn definition of D-classical orthogonal polynomials [18]. Symmetric case was studied for the first time by Y. Ben Cheikh and his coworker [4]; in particular, they proved that the only symmetric Dunkl-classical orthogonal polynomials are the generalized Hermite polynomials and the generalized Gegenbauer polynomials. Later on, M. Sghaier [24] characterized the symmetric Dunkl-classical forms by a distributional equation of the Pearson type and he showed that the corresponding polynomials can be characterized by a second-order differential-difference equation in the space of polynomials. Another characterization called the first structure relation was established by L. Khériji et al [5].

Non-symmetric Dunkl-classical orthogonal polynomials have been studied in [7], [8], [9], [24]. In particular in [9] the authors showed that the unique non-symmetric Dunkl-classical linear form for $\mu \neq 0$ and $\mu > \frac{1}{2}$ is, up to a dilation, the perturbed generalized Gegenbauer linear form

$$\delta_1 - \frac{2\alpha}{1+2\mu+2\alpha}(x-1)^{-1}\mathcal{G}(\alpha, \mu - \frac{1}{2}).$$

where $n+\alpha \neq 0$, $2\mu + 2\alpha + 2n + 1 \neq 0$, $n \geqslant 0$ and $\mathcal{G}(\alpha, \mu - \frac{1}{2})$ is the generalized Gegenbauer form [1], [3].

The aim of this contribution is to give a new characterization of Dunkl-classical orthogonal polynomials.

We begin by reviewing some preliminary results needed for the sequel. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted by $\langle u, f \rangle$. In particular, we denote by $(u)_n = \langle u, x^n \rangle$, $n \geqslant 0$, the moments of u.

Let us define the following operations on $\mathcal{P}'$ [22]:

The left-multiplication of a linear form by a polynomial

$$\langle gu, f \rangle = \langle u, gf \rangle, \quad f, g \in \mathcal{P},\, u \in \mathcal{P}'.$$

The dilation of a linear form

$$\langle h_a u, f \rangle = \langle u, h_a f \rangle, \quad f \in \mathcal{P},\, u \in \mathcal{P}',\, a \in \mathbb{C} \setminus \{0\},$$

where

$$h_a f(x) = f(ax), \quad f \in \mathcal{P},\, a \in \mathbb{C} \setminus \{0\}.$$

The derivative of a linear form u is the linear form Du, such that

$$\langle Du, f \rangle = -\langle u, f' \rangle, \quad f \in \mathcal{P},\, u \in \mathcal{P}'.$$

Let $\{P_n\}_{n \geqslant 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n \geqslant 0$, and let $\{u_n\}_{n \geqslant 0}$ be its dual sequence, $u_n \in \mathcal{P}'$, defined by $\langle u_n, P_m \rangle = \delta_{n,m}$, $n, m \geqslant 0$.

The form u is called regular if there exists a sequence of polynomials $\{P_n\}_{n\geqslant 0}$, such that

$$\langle u, P_n P_m\rangle = r_n \delta_{n,m}, \quad n,\, m \geqslant 0,\, r_n \neq 0,\, n \geqslant 0.$$

The sequence $\{P_n\}_{n\geqslant 0}$ is then called orthogonal with respect to u. In this case, we have

$$u_n = r_n^{-1} P_n u_0,\, n \geqslant 0. \tag{1}$$

Let us recall the following result [20]:

Lemma 1. *Let $\{P_n\}_{n\geqslant 0}$ be a monic orthogonal polynomial sequence (MOPS, in short) with respect to u and let $\{u_n\}_{n\geqslant 0}$ be its dual sequence. If v is an element of $\mathcal{P}'$, then it can be expressed as*

$$v = \sum_{n=0}^{\infty} \alpha_n u_n,$$

where

$$\alpha_n = \langle v, P_n\rangle, \quad n = 0, 1, 2 \ldots.$$

Moreover, if v satisfies $\langle v, P_n\rangle = 0$ for $n \geqslant m$, then

$$v = \sum_{n=0}^{m-1} \alpha_n u_n.$$

According to the previous lemma, we have $u = \lambda u_0$, where $(u)_0 = \lambda \neq 0$. In what follows, all regular linear forms u will be taken normalized, i.e., $(u)_0 = 1$. Then $u = u_0$.

According to Favard's theorem, a MOPS $\{P_n\}_{n\geqslant 0}$ is characterized by the following three-term recurrence relation [11]:

$$\begin{aligned} &P_0(x) = 1,\; P_1(x) = x - \beta_0, \\ &P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x),\; n \geqslant 0, \end{aligned} \tag{2}$$

where

$$\beta_n = \frac{\langle u_0, xP_n^2\rangle}{\langle u_0, P_n^2\rangle} \in \mathbb{C}\,;\; \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2\rangle}{\langle u_0, P_n^2\rangle} \in \mathbb{C}\setminus\{0\},\; n \geqslant 0. \tag{3}$$

A form u is said to be symmetric if and only if $(u)_{2n+1} = 0, n \geqslant 0$, or, equivalently, in (2), $\beta_n = 0$, $n \geqslant 0$.

From (2), we have

$$P_2(x) = x^2 - (\beta_0 + \beta_1)x + \beta_0\beta_1 - \gamma_1. \tag{4}$$

As a consequence of the orthogonality of $\{P_n\}_{n\geqslant 0}$ with respect to u_0, we have

$$(u_0)_2 = \beta_0^2 + \gamma_1. \tag{5}$$

Let us introduce the Dunkl operator [14]:

$$T_\mu(f) = f' + 2\mu H_{-1}f,\ (H_{-1}f)(x) = \frac{f(x) - f(-x)}{2x}, \quad f \in \mathcal{P}, \mu \in \mathbb{C}.$$

By transposition, we define the operator T_μ from $\mathcal{P}'$ to $\mathcal{P}'$ as follows:

$$\langle T_\mu u, f\rangle = -\langle u, T_\mu f\rangle, \quad f \in \mathcal{P},\ u \in \mathcal{P}'.$$

In particular, this yields

$$(T_\mu u)_n = -\mu_n (u)_{n-1},\ n \geqslant 0,$$

with the convention $(u)_{-1} = 0$, where

$$\mu_n = n + 2\mu[n], \quad [n] = \frac{1 - (-1)^n}{2}, \quad n \geqslant 0.$$

Note that T_0 is the derivative operator D.

Using the previous definitions, we get the following formula [7]:

$$T_\mu(fu) = fT_\mu u + (T_\mu f)u + 2\mu(H_{-1}f)(h_{-1}u - u), \quad f \in \mathcal{P},\ u \in \mathcal{P}'. \tag{6}$$

In particular, if u is a symmetric linear form, then (6) becomes

$$T_\mu(fu) = fT_\mu u + (T_\mu f)u, \quad f \in \mathcal{P},\ u \in \mathcal{P}'. \tag{7}$$

Now, consider an MOPS $\{P_n\}_{n\geqslant 0}$ and let

$$P_n^{[1]}(x,\mu) = \frac{1}{\mu_{n+1}}(T_\mu P_{n+1})(x),\ \mu \neq -n - \frac{1}{2},\ n \geqslant 0.$$

Denoting by $\{u_n^{[1]}\}_{n\geqslant 0}$ the dual sequence of $\{P_n^{[1]}(\cdot,\mu)\}_{n\geqslant 0}$, the following result is proved in [24]:

$$T_\mu u_n^{[1]} = -\mu_{n+1} u_{n+1},\ n \geqslant 0. \tag{8}$$

Definition 1. *[5], [7], [24] An MOPS $\{P_n\}_{n\geqslant 0}$ is called Dunkl-classical or T_μ-classical if $P_n^{[1]}(\cdot,\mu)$ is also an MOPS. In this case, the form u_0 is called either a Dunkl-classical or a T_μ-classical form.*

Any symmetric Dunkl-classical polynomial sequence $\{P_n\}_{n\geqslant 0}$ can be characterized taking into account its orthogonality as well as one of the four difference equations:

- Second-order differential equation of the Bochner type [24]

$$\Phi(x)(T_\mu^2 P_{n+1})(x) - \Psi(x)(T_\mu P_{n+1})(x) + \lambda_n P_{n+1}(x) = 0,\ n \geqslant 0. \tag{9}$$

- First structure relation [5]

$$\begin{aligned}\Phi(x)P_n^{[1]}(x,\mu) = \sum_{k=n}^{n+t} \lambda_{n,k} P_k(x),\ n \geqslant 0,\ 0 \leqslant t = \deg \Phi \leqslant 2.\\ \lambda_{n,n} \neq 0,\ n \geqslant 0.\end{aligned} \tag{10}$$

- Rodrigues-type formula [25]

$$P_n u_0 = \vartheta_n T_\mu^n(\Phi^n u_0),\ n \geqslant 0. \tag{11}$$

- Its canonical form u_0 satisfies the Pearson differential equation [24]

$$\begin{aligned}T_\mu(\Phi u_0) + \Psi u_0 = 0,\\ \Psi'(0) - \tfrac{\Phi''(0)}{2}\mu_n \neq 0, n \geqslant 0,\end{aligned} \tag{12}$$

  where Φ is a monic polynomial of degree t, $0 \leqslant t \leqslant 2$, Ψ is a first degree polynomial, and $\{\lambda_{n,k}\}_{n\geqslant 0,\, n\leqslant k\leqslant n+t}$ and $\{\vartheta_n\}_{n\geqslant 0}$ are sequences of complex numbers, such that $\vartheta_n \neq 0, n \geqslant 0$.

Remark 1. *Under conditions of relations (9)–(12), the linear form $u_0^{[1]}$, corresponding to $\{P_n^{[1]}\}_{n\geqslant 0}$, is given by:*

$$u_0^{[1]} = (1+2\mu)^{-1}\gamma_1^{-1} K \Phi u_0, \tag{13}$$

where K is a non-zero constant chosen to make Φ monic, and Ψ is given by

$$\Psi(x) = K^{-1}(1+2\mu)^2 P_1(x). \tag{14}$$

On the other hand, some characterizations of non-symmetric Dunkl-classical orthogonal polynomials have been provided (see [7], [8], [10], [16], [17]).

2. Main Result. In this section, we prove the characterization theorem in both situations.

2.1. The symmetric case.

Theorem 1. *For any symmetric MOPS $\{P_n\}_{n\geqslant 0}$, the following statements are equivalent*

(a) *The sequence $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical.*

(b) *There exist an integer t, $0 \leqslant t \leqslant 2$, and a sequence of complex numbers $\{\lambda_{n,k}\}_{n\geqslant t,\, n-t\leqslant k\leqslant n}$, such that*

$$P_n(x) = \sum_{k=n-t}^{n} \lambda_{n,k} P_k^{[1]}(x,\mu), \quad n \geqslant t, \tag{15}$$

$$\lambda_{n,n} = 1, \quad n \geqslant t, \tag{16}$$

$$\frac{1+2\mu}{\lambda_{2,0}}\gamma_2 - \mu_n \neq 0, \quad n \geqslant 0 \text{ when } \ \lambda_{2,0} \neq 0. \tag{17}$$

Proof. $(a) \implies (b)$ Assume that $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical; then there exist polynomials Φ (monic), $\deg\Phi = t$, $0 \leqslant t \leqslant 2$, and Ψ, $\deg\Psi = 1$, such that the canonical regular form u_0 satisfies (12). Moreover, since P_n is a polynomial of degree n, then there exists a sequence of complex numbers $\{\lambda_{n,k}\}_{n\geqslant t}$, $0 \leqslant k \leqslant n$, such that

$$P_n(x) = \sum_{k=0}^{n} \lambda_{n,k} P_k^{[1]}(x,\mu), \quad n \geqslant t. \tag{18}$$

By comparing the degrees in the previous equation, we get

$$\lambda_{n,n} = 1, \quad n \geqslant t.$$

Therefore, (18) becomes

$$P_n(x) = P_n^{[1]}(x,\mu) + \sum_{k=0}^{n-1} \lambda_{n,k} P_k^{[1]}(x,\mu), \quad n \geqslant t. \tag{19}$$

Multiplying the last equation by $P_m^{[1]}(\cdot,\mu)$, $0 \leqslant m \leqslant n-1$, $n \geqslant 1$, and applying Φu_0, we get

$$\langle \Phi u_0, P_m^{[1]}(\cdot,\mu) P_n \rangle =$$

$$= \langle \Phi u_0, P_m^{[1]}(\cdot,\mu) P_n^{[1]}(\cdot,\mu)\rangle + \sum_{k=0}^{n-1} \lambda_{n,k} \langle \Phi u_0, P_m^{[1]}(\cdot,\mu) P_k^{[1]}(\cdot,\mu)\rangle =$$
$$= \lambda_{n,m} \langle \Phi u_0, (P_m^{[1]}(\cdot,\mu))^2\rangle, \quad n \geqslant 1.$$

Hence,

$$\lambda_{n,m} = \frac{\langle u_0, (\Phi P_m^{[1]}(\cdot,\mu)) P_n\rangle}{\langle \Phi u_0, (P_m^{[1]}(\cdot,\mu))^2\rangle}, \quad 0 \leqslant m \leqslant n-1,\ n \geqslant 1. \tag{20}$$

Since $\deg(\Phi P_m^{[1]}(\cdot,\mu)) = m+t$, the orthogonality of $\{P_n\}_{n\geqslant 0}$ leads to

$$\langle u_0, (\Phi P_m^{[1]}(\cdot,\mu)) P_n\rangle = 0, \quad 0 \leqslant m+t \leqslant n-1,\ n \geqslant 1.$$

So, we have

$$\lambda_{n,m} = 0, \quad 0 \leqslant m \leqslant n-t-1,\ n \geqslant 1.$$

Consequently, (19) becomes

$$P_n(x) = P_n^{[1]}(x,\mu) + \sum_{k=n-t}^{n-1} \lambda_{n,k} P_k^{[1]}(x,\mu), \quad n \geqslant t. \tag{21}$$

It remains to prove (17). Assume that $\lambda_{2,0} \neq 0$. From (21), where $n=2$, we have

$$P_2(x) = P_2^{[1]}(x,\mu) + \lambda_{2,0} P_0^{[1]}(x,\mu).$$

Therefore,

$$\langle u_0^{[1]}, P_2\rangle = \lambda_{2,0}.$$

But from (8) and the fact that Φ is monic, we have

$$\langle u_0^{[1]}, P_2\rangle = (1+2\mu)^{-1} \gamma_1^{-1} K r_2 = (1+2\mu)^{-1} \gamma_2 K.$$

Then

$$K = \frac{(1+2\mu)\lambda_{2,0}}{\gamma_2}. \tag{22}$$

Substitution of (22) in (14) gives

$$\Psi(x) = \frac{(1+2\mu)\gamma_2}{\lambda_{2,0}} P_1(x).$$

Therefore,

$$\Psi'(0) = \frac{(1+2\mu)\gamma_2}{\lambda_{2,0}}.$$

So, condition (17) becomes an immediate consequence of the second equality in (12). Thus the desired result (15)–(17).

$(b) \implies (a)$ Assume that there exist an integer t, $0 \leqslant t \leqslant 2$, and a sequence of complex numbers $\{\lambda_{n,k}\}_{n\geqslant t,\ n-t\leqslant k\leqslant n}$, such that (15), (16), and (17) hold.

Let $\{P_n\}_{n\geqslant 0}$ and $\{P_n^{[1]}(\cdot,\mu)\}_{n\geqslant 0}$ be sequences of monic polynomials with $\{u_n\}_{n\geqslant 0}$ and $\{u_n^{[1]}\}_{n\geqslant 0}$ be their respective dual sequences. Using (15) and (16) for $n \geqslant t+1$, we have

$$\langle u_0^{[1]}, P_n\rangle = \langle u_0^{[1]}, P_n^{[1]}(\cdot,\mu)\rangle + \sum_{k=n-t}^{n-1} \lambda_{n,k}\langle u_0^{[1]}, P_k^{[1]}(\cdot,\mu)\rangle = 0.$$

Thus, according to Lemma 1, there exist complex numbers α_i, $i \in \{0,\dots,t\}$, such that

$$u_0^{[1]} = \sum_{i=0}^{t} \alpha_i u_i,\ 0 \leqslant t \leqslant 2.$$

Or, equivalently,

$$u_0^{[1]} = \alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2. \tag{23}$$

On account of (1), the previous equation becomes

$$u_0^{[1]} = (\alpha_0 + \alpha_1 r_1^{-1} P_1 + \alpha_2 r_2^{-1} P_2)u_0.$$

Therefore, there exists a polynomial Φ, $\deg\Phi \leqslant 2$, such that

$$u_0^{[1]} = k\Phi u_0, \tag{24}$$

where

$$k\Phi = \alpha_0 + \alpha_1 r_1^{-1} P_1 + \alpha_2 r_2^{-1} P_2, \tag{25}$$

and the non-zero constant k is chosen to make Φ monic.

Moreover, Φ is an even polynomial. Indeed, since $P_1(x) = P_1^{[1]}(x,\mu) = x$, we have

$$0 = \langle u_0^{[1]}, P_1^{[1]}(\cdot,\mu)\rangle = k\Big(\alpha_0\langle u_0, P_1\rangle + \alpha_1 r_1^{-1}\langle u_0, P_1^2\rangle + \alpha_2 r_2^{-1}\langle u_0, P_1P_2\rangle\Big) = k\alpha_1.$$

Hence, $\alpha_1 = 0$.

Thus, taking into account (25) and the fact that $P_2(x) = x^2 - \gamma_1$, we can easily see that Φ is even.

On the other hand, putting $n = 0$ in (8), we obtain

$$T_\mu u_0^{[1]} = -(1+2\mu)u_1.$$

Substitution of (24) in the previous equation gives (12), with

$$\Psi(x) = k^{-1}\gamma_1^{-1}(1+2\mu)P_1(x).$$

To complete the proof, we will show that the second equality in (12) is fulfilled. Indeed, from (23) we have

$$\alpha_2 = \langle u_0^{[1]}, P_2 \rangle.$$

But from (15) and (16), where $n = 2$, we have

$$P_2(x) = P_2^{[1]}(x,\mu) + \lambda_{2,0} P_0^{[1]}(x,\mu).$$

Thus,

$$\alpha_2 = \lambda_{2,0}.$$

On the other hand, taking into account (23) and the fact that u_0 and $u_0^{[1]}$ are normalized, we get

$$\alpha_0 = 1.$$

Therefore, (25) becomes

$$k\Phi(x) = 1 + \lambda_{2,0} r_2^{-1} P_2(x). \tag{26}$$

So, we distinguish two cases: $\lambda_{2,0} = 0$ and $\lambda_{2,0} \neq 0$.

The first case: $\lambda_{2,0} = 0$. In this case, $\deg \Phi = 0$; then $\Phi''(0) = 0$ and, since Φ is monic, we get $k = 1$. Therefore,

$$\Psi'(0) - \frac{\Phi''(0)}{2}\mu_n = \Psi'(0) = \gamma_1^{-1}(1+2\mu) \neq 0,\ n \geqslant 0.$$

The second case: $\lambda_{2,0} \neq 0$. In this case, $\deg \Phi = 2$. But, since Φ is monic, $\dfrac{\Phi''(0)}{2} = 1$. Furthermore, identification of degrees in (26) gives

$$k = \lambda_{2,0} r_2^{-1}.$$

Therefore,

$$\Psi'(0) - \tfrac{\Phi''(0)}{2}\,\mu_n = \frac{(1+2\mu)}{\lambda_{2,0}}\,\gamma_2 - \mu_n \neq 0, \quad n \geqslant 0 \qquad (\text{by } (17)\ ).$$

So, according to relation (12), the sequence $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical. □

In the sequel, using the previous theorem, we will determine the second structure relation for the generalized Hermite polynomials and the generalized Gegenbauer polynomials.

Put $\Phi(x) = \frac{\Phi''(0)}{2}x^2 + \Phi(0)$ and $\Psi(x) = \Psi'(0)x$ and let $\{P_n\}_{n\geqslant 0}$ be a symmetric Dunkl-classical MOPS, such that its associated regular form u_0 satisfies (12). So, from (15)–(16) we have:

$$P_n(x) = P_n^{[1]}(x,\mu) + \lambda_{n,n-1}P_{n-1}^{[1]}(x,\mu) + \lambda_{n,n-2}P_{n-2}^{[1]}(x,\mu),\ n \geqslant t. \tag{27}$$

Since the sequences $\{P_n\}_{n\geqslant 0}$ and $\{P_n^{[1]}(\cdot,\mu)\}_{n\geqslant 0}$ are symmetric,

$$\lambda_{n,n-1} = 0, \quad n \geqslant t. \tag{28}$$

The coefficient $\lambda_{n,n-2}$ is given by

$$\lambda_{n,n-2} = \frac{\tfrac{\Phi''(0)}{2}\,\mu_{n-1}}{\Psi'(0) - \tfrac{\Phi''(0)}{2}\,\mu_{n-2}}\,\gamma_n, \quad n \geqslant t, \tag{29}$$

with the convention $\lambda_{0,n-2} = 0$. Indeed, from (20), we have

$$\lambda_{n,n-2} = \frac{\langle u_0, (\Phi P_{n-2}^{[1]}(\cdot,\mu))P_n\rangle}{\langle \Phi u_0, (P_{n-2}^{[1]}(\cdot,\mu))^2\rangle}, \quad n \geqslant t.$$

Writing

$$\Phi(x)P_{n-2}^{[1]}(x,\mu) = \frac{\Phi''(0)}{2}\,x^n + \text{lower degree terms.}$$

On the one hand, from the orthogonality of $\{P_n\}_{n\geqslant 0}$ with respect to u_0, we have

$$\langle u_0, (\Phi P_{n-2}^{[1]}(\cdot,\mu))P_n\rangle = \frac{\Phi''(0)}{2}\langle u_0, x^n P_n\rangle = \frac{\Phi''(0)}{2}\langle u_0, P_n^2\rangle, \quad n \geqslant t.$$

On the other hand, from (7) and the fact that Φu_0 is symmetric, we have

$$\langle \Phi u_0, (P^{[1]}_{n-2}(\cdot,\mu))^2\rangle = -\frac{1}{\mu_{n-1}}\langle T_\mu(P^{[1]}_{n-2}(\cdot,\mu)\Phi u_0), P_{n-1}\rangle =$$
$$= -\frac{1}{\mu_{n-1}}\langle T_\mu(P^{[1]}_{n-2}(\cdot,\mu))\Phi u_0 + P^{[1]}_{n-2}(\cdot,\mu)T_\mu(\Phi u_0), P_{n-1}\rangle.$$

Taking into account (12), we get

$$\langle \Phi u_0, (P^{[1]}_{n-2}(\cdot,\mu))^2\rangle = \frac{1}{\mu_{n-1}}\langle P^{[1]}_{n-2}(\cdot,\mu)\Psi u_0 - T_\mu(P^{[1]}_{n-2}(\cdot,\mu))\Phi u_0, P_{n-1}\rangle.$$

Hence, the orthogonality of $\{P_n\}_{n\geqslant 0}$ with respect to u_0 gives

$$\langle \Phi u_0, (P^{[1]}_{n-2}(\cdot,\mu))^2\rangle = \frac{\Psi'(0) - \frac{\Phi''(0)}{2}\,\mu_{n-2}}{\mu_{n-1}}\langle u_0, x^{n-1}P_{n-1}\rangle =$$
$$= \frac{\Psi'(0) - \frac{\Phi''(0)}{2}\,\mu_{n-2}}{\mu_{n-1}}\langle u_0, P^2_{n-1}\rangle, \quad n \geqslant t.$$

Consequently, from the second equality of (3) we deduce (29).

Substitution of (28) and (29) in (27) gives

$$P_n(x) = P^{[1]}_n(x,\mu) + \frac{\frac{\Phi''(0)}{2}\,\mu_{n-1}}{\Psi'(0) - \frac{\Phi''(0)}{2}\,\mu_{n-2}}\,\gamma_n\,P^{[1]}_{n-2}(x,\mu), \quad n \geqslant t. \tag{30}$$

Corollary.

1) *The generalized Hermite polynomial $\{H^{(\mu)}_n\}_{n\geqslant 0}$ is characterized by the following second structure relation:*

$$H^{(\mu)}_n(x) = (H^{(\mu)}_n)^{[1]}(x), \quad n \geqslant 0. \tag{31}$$

2) *The generalized Gegenbauer polynomial $\{S^{(\alpha,\mu-\frac{1}{2})}_n\}_{n\geqslant 0}$ is characterized by the following second structure relation:*

$$S^{(\alpha,\mu-\frac{1}{2})}_n(x) = (S^{(\alpha,\mu-\frac{1}{2})}_n)^{[1]}(x) - \\ - \frac{\mu_{n-1}\,\mu_n}{(2n+2\alpha+2\mu-1)(2n+2\alpha+2\mu+1)}\,(S^{(\alpha,\mu-\frac{1}{2})}_{n-2})^{[1]}(x), \quad n \geqslant 2. \tag{32}$$

Proof. 1) The sequence of generalized Hermite polynomials $\{H^{(\mu)}_n\}_{n\geqslant 0}$ satisfies (2) with (see [11]):

$$\beta_n = 0,\ \gamma_{n+1} = \frac{\mu_{n+1}}{2}, \quad n \geqslant 0, \tag{33}$$

where the regularity condition is

$$\mu \neq -n - \frac{1}{2}, \quad n \geqslant 0.$$

This sequence is Dunkl-classical and its associated form $\mathcal{H}(\mu)$ satisfies (12) with (see [7])

$$\Phi(x) = 1,\ \Psi(x) = 2x. \tag{34}$$

So, using (33) and (34) the proof of (31) is an immediate consequence of (30).

2) The sequence of generalized Gegenbauer polynomials $\{S_n^{(\alpha,\mu-\frac{1}{2})}\}_{n\geqslant 0}$ satisfies (2) with (see [11]):

$$\begin{aligned} &\beta_n = 0, \quad \gamma_{n+1} = \frac{(n+1+\delta_n)(n+1+2\alpha+\delta_n)}{(2n+2\alpha+2\mu+1)(2n+2\alpha+2\mu+3)}, \\ &\delta_n = \mu(1+(-1)^n), \quad n \geqslant 0, \end{aligned} \tag{35}$$

where the regularity conditions are

$$\alpha \neq -n,\ \beta \neq -n,\ \alpha+\beta \neq -n, \quad n \geqslant 1.$$

This sequence is Dunkl-classical and its associated form $\mathcal{G}(\alpha, \mu - \frac{1}{2})$ satisfies (12) with (see [7])

$$\Phi(x) = x^2 - 1,\ \Psi(x) = -2(\alpha+1)x. \tag{36}$$

Then, using (35) and (36), equation (32) is deduced from (30). □

Remark 2.

1) *From equation (31), we can recover again the following structure relation established by T. S. Chihara [12]:*

$$\begin{aligned} xD\mathcal{H}_{n+1}^{(\mu)}(x) = &-\mu(1+(-1)^n)\mathcal{H}_{n+1}^{(\mu)}(x) + \\ &+ \Big(n+1+\mu(1+(-1)^n)\Big)x\mathcal{H}_n^{(\mu)}(x), \quad n \geqslant 0. \end{aligned} \tag{37}$$

Indeed, using the definition of T_μ and the fact $\{H_n^{(\mu)}\}_{n\geqslant 0}$ is symmetric, equation (31) becomes

$$D\mathcal{H}_{n+1}^{(\mu)}(x) + \mu(1-(-1)^{n+1})\frac{\mathcal{H}_{n+1}^{(\mu)}(x)}{x} = \mu_{n+1}\mathcal{H}_n^{(\mu)}(x), \quad n \geqslant 0.$$

Therefore, multiplication of the last equation by x gives (37).

2) *The relation* (32) *can be written of the following form:*

$$S_n^{(\alpha,\mu-\frac{1}{2})}(x) = S_n^{(\alpha+1,\mu-\frac{1}{2})}(x) - \\ - \frac{\mu_n\,\mu_{n-1}}{(2n+2\alpha+2\mu-1)(2n+2\alpha+2\mu+1)} S_{n-2}^{(\alpha+1,\mu-\frac{1}{2})}(x), \quad n \geqslant 2.$$

This result is deduced from (32) *and the fact that* $T_\mu S_{n+1}^{(\alpha,\mu-\frac{1}{2})} = \mu_{n+1} S_n^{(\alpha+1,\mu-\frac{1}{2})}$ *(see [4]).*

2.2. The non-symmetric case. In this subsection, we will present a second structure relation for non-symmetric Dunkl-classical polynomial sequences. But first, let us recall the following result.

Theorem 2. *[7] Let* $\{P_n\}_{n\geqslant 0}$ *be a MPS orthogonal with respect to a linear form* u_0. *For* $\mu \neq 0$ *and* $\mu \neq \frac{1}{2}$, *the following statements are equivalent:*

(a) The sequence $\{P_n\}_{n\geqslant 0}$ *is Dunkl-classical.*

(b) There exist $K \in \mathbb{C}^*$ *and three polynomials* Φ *(monic),* B *and* Ψ *with* $\deg \Phi \leqslant 2$, $\deg B \leqslant 3$, *and* $\deg \Psi = 1$, *such that*

$$\Psi'(0) + \frac{K\Phi''(0)}{2(1-4\mu^2)}(4\mu^2[n]-n) + \frac{KB'''(0)}{3(1-4\mu^2)}\mu([n]-n) \neq 0, \tag{38}$$

and

$$T_\mu\Big(\Phi u_0 - 2\mu h_{-1}(\Phi u_0)\Big) + \frac{1-4\mu^2}{K}\Psi u_0 = 0, \tag{39}$$

with

$$x\Phi(x)u_0 = h_{-1}(B(x)u_0). \tag{40}$$

The authors [9] used Theorem 2 to classify all Dunkl-classical linear forms. In particular, they proved that the unique non-symmetric Dunkl-classical linear form for $\mu \neq 0$ and $\mu \neq \frac{1}{2}$ is, up a dilation, the perturbed generalized Gegenbauer linear form u_0, satisfying

$$T_\mu\Big((x^2-1)u_0\Big) - \frac{1+2\mu}{\beta_0}(x-\beta_0)u_0 = 0, \tag{41}$$

$$(x-1)u_0 = h_{-1}((x-1)u_0), \tag{42}$$

with the regularity conditions:

$$\beta_0 \notin \{0,1\},\ 1+2\mu+\beta_0(n-2\mu[n]) \neq 0, \quad n \geqslant 0. \tag{43}$$

Remark 3. *If $\{P_n\}_{n\geqslant 0}$ is a non-symmetric Dunkl-classical MOPS, then $\{P_n^{[1]}(\cdot,\mu)\}_{n\geqslant 0}$ is orthogonal with respect to [10]*

$$u_0^{[1]} = \frac{1}{\beta_0 - 1}(x^2-1)u_0, \quad \beta_0 \notin \{0,1\}. \tag{44}$$

Theorem 3. *Let $\{P_n(x)\}_{n\geqslant 0}$ be a MOPS fulfilling (2) with (43). Assume that its corresponding linear form u_0 satisfies (42). The following statements are equivalent:*

(a) The sequence $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical.

(b) $\{P_n\}_{n\geqslant 0}$ satisfies the second structure relation

$$P_n(x) = P_n^{[1]}(x,\mu) + \lambda_{n,n-1}P_{n-1}^{[1]}(x,\mu) + \lambda_{n,n-2}P_{n-2}^{[1]}(x,\mu), \quad n \geqslant 0, \tag{45}$$

where

$$\begin{aligned} \lambda_{n,n-1} &= \frac{\langle u_0, (x^2-1)P_{n-1}^{[1]}(\cdot,\mu)P_n\rangle}{\langle u_0, (x^2-1)(P_{n-1}^{[1]}(\cdot,\mu))^2\rangle}, \\ \lambda_{n,n-2} &= \frac{\langle u_0, (x^2-1)P_{n-2}^{[1]}(\cdot,\mu)P_n\rangle}{\langle u_0, (x^2-1)(P_{n-2}^{[1]}(\cdot,\mu))^2\rangle}, \quad n \geqslant 0. \end{aligned} \tag{46}$$

Proof. $(a) \implies (b)$ Suppose that $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical; then its canonical regular form u_0 satisfies (41)–(42). Moreover, since P_n is a polynomial of degree n, there exists a sequence of complex numbers $\{\lambda_{n,k}\}_{n\geqslant 0}$, $0 \leqslant k \leqslant n$, such that

$$P_n(x) = \sum_{k=0}^{n} \lambda_{n,k} P_k^{[1]}(x,\mu), \quad n \geqslant 0. \tag{47}$$

By comparing the degrees in the previous equation, we get

$$\lambda_{n,n} = 1, \quad n \geqslant 0.$$

Therefore, (47) becomes

$$P_n(x) = P_n^{[1]}(x,\mu) + \sum_{k=0}^{n-1} \lambda_{n,k} P_k^{[1]}(x,\mu), \quad n \geqslant 0. \tag{48}$$

It is clear that (45) holds for $n = 0$, where $\lambda_{0,-1} = \lambda_{0,-2} = 0$. For $n \geqslant 1$, multiplying the previous equation by $P_m^{[1]}(\cdot,\mu)$, $0 \leqslant m \leqslant n-1$, $n \geqslant 1$, and applying $(x^2-1)u_0$, we get

$$\langle (x^2-1)u_0, P_m^{[1]}(\cdot,\mu)P_n\rangle =$$
$$= \langle (x^2-1)u_0, P_m^{[1]}(\cdot,\mu)P_n^{[1]}(\cdot,\mu)\rangle + \sum_{k=0}^{n-1}\lambda_{n,k}\langle (x^2-1)u_0, P_m^{[1]}(\cdot,\mu)P_k^{[1]}(\cdot,\mu)\rangle =$$
$$= \lambda_{n,m}\langle (x^2-1)u_0, (P_m^{[1]}(\cdot,\mu))^2\rangle, \quad n \geqslant 1.$$

Hence,

$$\lambda_{n,m} = \frac{\langle u_0, ((x^2-1)P_m^{[1]}(\cdot,\mu))P_n\rangle}{\langle (x^2-1)u_0, (P_m^{[1]}(\cdot,\mu))^2\rangle}, \quad 0 \leqslant m \leqslant n-1, \quad n \geqslant 1. \tag{49}$$

Since $\deg((x^2-1)P_m^{[1]}(\cdot,\mu)) = m+2$, the orthogonality of $\{P_n\}_{n\geqslant 0}$ leads to

$$\langle u_0, ((x^2-1)P_m^{[1]}(\cdot,\mu))P_n\rangle = 0, \quad 0 \leqslant m+2 \leqslant n-1.$$

So, we have

$$\lambda_{n,m} = 0, \quad 0 \leqslant m \leqslant n-3, \quad n \geqslant 3.$$

Consequently, (48) becomes

$$P_n(x) = P_n^{[1]}(x,\mu) + \lambda_{n,n-1}P_{n-1}^{[1]}(x,\mu) + \lambda_{n,n-2}P_{n-2}^{[1]}(x,\mu), \quad n \geqslant 0$$

with the equalities in (46) are obtained by (49). Therefore, (45) holds.

(b) $\Longrightarrow$ *(a)* Let $\{P_n\}_{n\geqslant 0}$ and $\{P_n^{[1]}(\cdot,\mu)\}_{n\geqslant 0}$ be sequences of monic polynomials with $\{u_n\}_{n\geqslant 0}$ and $\{u_n^{[1]}\}_{n\geqslant 0}$ be their respective dual sequences. Suppose that $\{P_n\}_{n\geqslant 0}$ satisfies (45).

From (45) for $n \geqslant 3$, we have

$$\langle u_0^{[1]}, P_n\rangle =$$
$$= \langle u_0^{[1]}, P_n^{[1]}(\cdot,\mu)\rangle + \lambda_{n,n-1}\langle u_0^{[1]}, P_{n-1}^{[1]}(\cdot,\mu) + \lambda_{n,n-2}\langle u_0^{[1]}, P_{n-2}^{[1]}(\cdot,\mu)\rangle = 0.$$

Thus, according to Lemma 1, there exist complex numbers α_k, $0 \leqslant k \leqslant 2$, such that

$$u_0^{[1]} = \alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2. \tag{50}$$

On account of (1), the previous equation becomes

$$u_0^{[1]} = (\alpha_0 + \alpha_1 r_1^{-1}P_1 + \alpha_2 r_2^{-1}P_2)u_0. \tag{51}$$

Taking into account (50) and the fact that u_0 and $u_0^{[1]}$ are normalized, we get

$$\alpha_0 = 1. \tag{52}$$

According to (50), we have

$$\alpha_1 = \langle u_0^{[1]}, P_1 \rangle.$$

Making $n = 1$ in (45), we get

$$P_1(x) = P_1^{[1]}(x, \mu) + \lambda_{1,0} P_0^{[1]}(x, \mu).$$

Therefore,

$$\alpha_1 = \lambda_{1,0}.$$

Using the first equality in (46) for $n = 1$, we have

$$\lambda_{1,0} = \frac{\beta_0 + \beta_1}{(u_0)_2 - 1} r_1.$$

Then

$$\alpha_1 = \frac{\beta_0 + \beta_1}{(u_0)_2 - 1} r_1. \tag{53}$$

From (50), we have

$$\alpha_2 = \langle u_0^{[1]}, P_2 \rangle.$$

But from (45), where $n = 2$, we have

$$P_2(x) = P_2^{[1]}(x, \mu) + \lambda_{2,1} P_1^{[1]}(x, \mu) + \lambda_{2,0} P_0^{[1]}(x, \mu).$$

Thus,

$$\alpha_2 = \lambda_{2,0}.$$

On the other hand, from the second equality in (46) for $n = 2$, we deduce

$$\alpha_2 = \lambda_{2,0} = \frac{r_2}{(u_0)_2 - 1}. \tag{54}$$

Substitution of (52), (53) and (54) in (51) gives

$$u_0^{[1]} = \Big(1 + \frac{\beta_0 + \beta_1}{(u_0)_2 - 1} P_1 + \frac{1}{(u_0)_2 - 1} P_2\Big) u_0.$$

Using (4)–(5) and the fact that $P_1(x) = x - \beta_0$, the last equation becomes

$$u_0^{[1]} = \frac{1}{\beta_0^2 + \gamma_1 - 1}(x^2 - 1) u_0. \tag{55}$$

But from (42), it is easy to see that

$$(u_0)_2 = (u_0)_1.$$

Since $(u_0)_1 = \beta_0$, from (5) we have

$$\gamma_1 = \beta_0 - \beta_0^2. \tag{56}$$

Therefore, equation (55) becomes

$$u_0^{[1]} = \frac{1}{\beta_0 - 1}(x^2 - 1)u_0. \tag{57}$$

For $n = 0$ in (8), we obtain

$$T_\mu u_0^{[1]} = -(1 + 2\mu)\gamma_1^{-1}(x - \beta_0)u_0. \tag{58}$$

Substitution of (56) and (57) in (58) gives (41).
So, according to Theorem 2, the sequence $\{P_n\}_{n\geqslant 0}$ is Dunkl-classical. □

Remark 4. *Theorems 1 and 3 are the main results of this paper. From them, we carry out the complete study of the Dunkl-classical orthogonal polynomials.*

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Y. Habbachi
Laboratory of Mathematics and Applications,
Faculty of Sciences, Gabès University,
Omar Ibn Khattab Street 6029 Gabes, Tunisia
E-mail: E-mail: Yahiahabbachi@gmail.com