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THE WEAK DROP PROPERTY AND THE DE LA VALLÉE POUSSIN THEOREM

Abstract. We prove that a closed bounded convex set is uniformly integrable if and only if it has the weak drop property. We extract the weakly compact subsets of the Henstock integrable functions on the H-Orlicz spaces with the weak drop property via de la Vallée Poussin Theorem.

Key words: *Young's function, weak drop property, H-Orlicz spaces*

2020 Mathematical Subject Classification: $46A55$, $46B20$

1. Introduction and preliminaries. The weak drop property was introduced by Giles, Sims, and Yorke (see [6]). Rolewicz [18] introduced the sequential approach to the drop property. Qiu established the following fact: in a locally convex space, every weakly sequentially compact convex set possesses the weak drop property, and every weakly compact convex set possesses the quasi-weak drop condition (see [19]). Bhayo et al. [3] introduced convexity and concavity of a function with the identric and Alzer mean. The class of spaces with norm having the drop property resides between the classes of nearly uniformly convex and nearly strictly convex Banach spaces (see [4]). For details of the drop property, one can see [12], [19] and reference therein. It can be traced to as early as 1915, the publications of de-la Vallée Poussin, although it was the Banach space research of 1920's that formally gave birth to what were later called the Orlicz spaces, first proposed by Z. W. Birnbaum and W. Orlicz. Later this space was further studied by Orlicz himself. Orlicz spaces are generalizations of L^p spaces. Further, Luxemburg developed the theory of Orlicz spaces. Details of this theory can be found in [1], [13], [17].

Kurzweil first proposed a solution to the primitives problem in 1957, and then Henstock did the same in 1963. It is a generalized form of the Riemann integral, commonly known as the Henstock-Kurzweil integral

(HK-integral); Kurzweil and Henstock each proposed this generalization independently. The Henstock-Kurzweil integral is similar to the Riemann integral, but is stronger than the Lebesgue integral. Additionally, it is well known that the HK-integral can resolve the issue of primitives in the real line. The limit of Riemann sums over the appropriate integration domain partitions is referred to as the HK-valued integrals. The HK-integral has a constructive definition. Within the HK-integral, a gauge-like positive function is employed to assess a partition's fineness, rather than a constant as in the Riemann integral: this is the fundamental distinction between the two definitions. One can see [5], [7], [8], [10] and references therein for details of Henstock-Kurzweil integrals.

Hazarika and Kalita [7] introduced the H-Orlicz space with non-absolute integrable functions. They proved that C_0^∞ is dense in the H-Orlicz space, but is not dense in the classical Orlicz space. It is known that if a function f is bounded with compact support, then the following statements are equivalent:
(a) f is Henstock–Kurzweil integrable,
(b) f is Lebesgue integrable,
(c) f is Lebesgue measurable.
In general, every Henstock-Kurzweil integrable function is measurable, and f is Lebesgue integrable if and only if both f and $|f|$ are Henstock-Kurzweil integrable. This means that the Henstock-Kurzweil integral can be thought of as a "non-absolutely convergent Lebesgue integral". This fact relies on compact support in H-Orlicz space. Of course, we are working with H-Orlicz spaces without compact support. In this case, H-Orlicz spaces are not equivalent to Classical Orlicz spaces.

Let f be a function defined on $[0, 1]$ that has values in a Banach space X. If f has an integrable majorant and X is separable, Caponetti et al. demonstrated in [5] that the limit set $I_{HK}(f)$ of Henstock-Kurzweil integral sums is non-empty and convex. They provided a detailed explanation of the limit set in the same setting.

In several contexts, the de la Vallée-Poussin Theorem has been taken into account. For instance, in the space of measures with values in a countably additive Banach space. The de la Vallée-Poussin Theorem is used to characterise the strongly measurable vector functions that are Pettis integrable through the compactness of a certain set of scalar functions in a specific space of Orlicz, as well as the countably additivity of the Dunford integral of vector functions (see [2]). The characterization of the countable additivity of the Henstock-Dunford integrable functions with the help of de la Vallée Poussin Theorem was discussed by Kalita and Hazarika in [8]. Recently, Volosivets [20] has studied de la Vallée-Poussin's mean in weighted Orlicz spaces.

The paper is organized as follows: In Section 1, the basic concepts and terminology are introduced together with some definitions and results. In Section 2, we prove that a closed bounded convex set has weak drop property only if it also possesses uniform integrability: Theorem 5. In Section 3, several properties of H-Orlicz spaces with drop property are discussed. In Section 4, with the help of Theorem 5, we find relatively weakly compact subset in H-Orlicz space via weak drop property.

Definition 1. [6] *Let $\mathfrak{C}$ be a closed convex set. A drop $D(x, \mathfrak{C})$ determined by a point $x \notin \mathfrak{C}$ is the convex hull of the set $\{x\} \cup \mathfrak{C}$. The set $\mathfrak{C}$ is said to have drop property if for every non empty closed set $\mathfrak{A}$ disjoint from $\mathfrak{C}$, there exists a point $a \in \mathfrak{A}$, such that $D(a, \mathfrak{C}) \cap \mathfrak{A} = \{a\}$.*

Next we recall the concept of weakly sequentially closed set below.

Definition 2. [11] *A set $\mathfrak{A}$ is weakly sequentially closed if every $\mathfrak{A}$-valued weakly convergent sequence has its limit in $\mathfrak{A}$.*

Definition 3. [18] *A closed convex set $\mathfrak{C}$ is said to have the weak drop property if for every weakly sequentially closed set $\mathfrak{A}$ disjoint from $\mathfrak{C}$ there is an $a \in \mathfrak{A}$, such that $D(a, \mathfrak{C}) \cap A = \{a\}$.*

Recalling a closed bounded convex set $\mathfrak{C}$, a sequence $\{x_n\}$ in $X \setminus \mathfrak{C}$, such that $x_{n+1} \in D(x_n, \mathfrak{C})$ $\forall n \in \mathbb{N}$ is called a stream. It is known that the norm $\|\cdot\|$ has drop property if and only if each stream in $X \setminus \overline{B}(X)$ contains a convergent subsequence (see [18]). Giles et al. in [6] showed that the norm $\|\cdot\|$ has weak drop property if and only if each stream in $X \setminus \overline{B}(X)$ contains a weakly convergent subsequence.

Definition 4. [14, Definition 2.5] *A subset $\mathfrak{F}$ of $L^1(\Omega)$ is said to be uniformly integrable if $\mathfrak{F}$ is a bounded subset of $L^1(\Omega)$, such that*

$$\lim_{c \to \infty} \sup_{f \in \mathfrak{F}} \int_{\{|f| \geqslant c\}} |f| d\mu = 0.$$

Recall de la Vallée Poussin Theorem from [17], as follows:

Theorem 1. *Let $\mathfrak{F}$ be a family of scalar measurable functions on a finite measure space (Ω, Σ, μ). Then the following conditions are equivalent:*

1) $\mathfrak{F} = \{f_\alpha\colon\ \alpha \text{ is in an index set}\}$ *is uniformly integrable.*
2) *There exists a convex function* $\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$, *such that* $\theta(0) = 0$, $\theta(-x) = \theta(x)$ *and* $\theta(x)/x \to \infty$ *as* $x \to \infty$, *in terms of which* $\sup\limits_{\alpha} \int\limits_{\Omega} \theta(f_\alpha) s\mu < \infty$.

Definition 5. [1, Definition 1.1] *Let* $m\colon \mathbb{R}^+ \to \mathbb{R}^+$ *be a non-decreasing right-continuous and non-negative function satisfying*

$$m(0) = 0, \text{ and } \lim_{t\to\infty} m(t) = \infty.$$

A function $\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is called an* N*-function if there is a function "*m*" satisfying the assumptions above and such that*

$$\theta(u) = \int\limits_0^{|u|} m(t)dt.$$

Evidently, θ is an N-function if it is continuous, convex, even, satisfies

$$\lim_{u\to\infty} \frac{\theta(u)}{u} = \infty \text{ and } \lim_{u\to 0} \frac{\theta(u)}{u} = 0.$$

For example, $\theta_p(x) = x^p$; $p > 1$. One can see [9] and references therein for details of N-functions.

Definition 6. ([17, Definition 7, p 28], [1, Definition 1.5, 1.6])

(a) *An* N*-function* θ *is said to satisfy* Δ' *condition if there is a* $k > 0$, *so that*

$$\theta(xy) \leqslant k\theta(x)\theta(y) \text{ for large values of } x \text{ and } y.$$

(b) *An* N*-function* θ *is said to satisfy* Δ_2 *condition if there is a* $k > 0$, *so that*

$$\theta(2x) \leqslant k\theta(x) \text{ for large values of } x.$$

Theorem 2. [18] *Every closed bounded convex set with weak drop property is weakly compact.*

Remark. *All Young's functions are convex. It is easy to claim that every closed bounded set of Young's functions with weak drop property is weakly compact. Also, every bounded closed convex set of* N*-function with weak drop property is weakly compact.*

Henstock-Kurzweil integral is defined as follows:

Definition 7. [8] *A function* $f\colon [a,b] \to \mathbb{R}$ *is said to be Henstock-Kurzweil integrable on a set* $[a,b]$ *if there is an element* $I_{[a,b]} \in \mathbb{R}$*, such that for every* $\epsilon > 0$ *there is a positive function* δ *(called gauge) on* $[a,b]$ *with*

$$\left|S(f,P) - I_{[a,b]}\right| < \epsilon$$

whenever $P = \left\{a = x_0 \leqslant x_1 \leqslant x_2 \ldots \leqslant x_n = b\right\}$ *is a* δ*-fine partition of* $[a,b]$ *and* $S(f,P) = \sum\limits_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ *is a Riemann sum.*

Through out the article, the space of Henstock-Kurzweil integrable functions is denoted by $HK(\mu)$. This space is not complete with the Alexiewicz semi-norm

$$\|f\|_A = \sup_{[a,b]\subset\mathbb{R}} |\int_a^b f|,$$

where the integral is in the sense of Henstock-Kurzweil.

Theorem 3. [7, Theorem 1.1] *If* $f\colon I_0 \subseteq [0,1] \to \mathbb{R}$ *is a measurable function with the gauge* $\delta\colon I_0 \to \mathbb{R}$*, then* $g = \theta(f)\colon [0,1] \to \mathbb{R}^+$ *is Henstock-Kurzweil integrable.*

Let (X, Σ, μ) be a measure space and X be a Banach space. The Orlicz space $L_*^\theta(X, \Sigma, \mu)$ or $L_*^\theta(\mu)$ is defined as follows: if $\overline{\theta}(f) = (L)\int \theta(f)d\mu$, then

$$L^\theta(\mu) = \left\{f \text{ measurable}\colon \overline{\theta}(f) < \infty\right\}. \tag{1}$$

The space $L^\theta(\mu)$ is absolutely convex, i.e., if f, $g \in L^\theta(\mu)$ and α, β are scalars, such that $|\alpha| + |\beta| \leqslant 1$, then $\alpha f + \beta g \in L^\theta(\mu)$. Also, if $\theta \in \Delta_2$, then $L^\theta(\mu)$ is a linear space (see [17, Theorem 2]). If $\theta\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a Young function, then it can be represented as

$$\theta(x) = \int\limits_0^x \phi(x)dx,\ x \in \mathbb{R}^+,$$

where $\phi(0) = 0$, $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing left-continuous, and if $\phi(x) = +\infty$ for $x \geqslant a$, $x \geqslant a > 0$, then $\theta(x) = +\infty$ for $x \geqslant a$. The function ϕ is called the complementary function of θ (see [17, Corollary 2]). If ϕ denotes the complementary in Young's sense function to θ, and

$$L_*^\theta(\mu) = \left\{f \text{ measurable}\colon\ \left|(L)\int fgd\mu\right| < \infty \text{ for all } g \in L^\phi(\mu)\right\},$$

the collection $L^{\theta}_{*}(\mu)$ is then a linear space. The collection $\Big(L^{\theta}_{*}(\mu), \|\cdot\|_{L^{\theta}_{*}}\Big)$ is a Banach space called an Orlicz space, where

$$\|f\|_{L^{\theta}_{*}} = \sup\Big\{\Big|\int fgd\mu\Big| : \overline{\phi}(g) \leqslant 1\Big\}.$$

Moreover, let $\|\cdot\|_{(\theta)}$ be the Minkowski functional associated with the convex set $\{f \in L^{\theta}_{*}(\mu) : \overline{\theta}(f) \leqslant 1\}$. In the sequel, $\|\cdot\|_{(\theta)}$ is an equivalent norm on $L^{\theta}_{*}(\mu)$, called the Luxemberg norm. Indeed,

$$\|f\|_{(\theta)} \leqslant \|f\|_{L^{\theta}_{*}} \leqslant 2\|f\|_{(\theta)} \text{ for all } f \in L^{\theta}_{*}(\mu).$$

For details of Orlicz spaces, see [1], [9], [13], [17] and references therein. It is clear that Δ_2 conditions of the Young function play crucial role for the reflexivity of the Orlicz space (see [9], [13], [17]). Now, we recall reflexivity of Orlicz space.

Theorem 4. [1], [9] *The Orlicz space $L^{\theta}_{*}(\mu)$ is reflexive if and only if $\theta \in \Delta_2$.*

2. Uniform integrability and weak drop property. In this section, we prove that a closed and bounded convex set of $HK(\mu)$ is uniform integrable if and only if the set has weak drop property. Consider a finite collection $P = \Big\{(\Delta_i, t_i) : i = 1, 2, \ldots, p\Big\}$, where Δ_i are non-overlapping intervals in $[0, 1]$, $t_i \in \Delta_i$ and $\bigcup_{i=1}^{p} \Delta_i = [0, 1]$.

A strictly positive function δ on $I_0 \subset [0, 1]$ is called a gauge on I_0. Given a gauge δ on $[0, 1]$, a partition $P = \Big\{(\Delta_i, t_i) : i = 1, 2, \ldots, p\Big\}$ of $[0, 1]$ is called δ-fine if $\Delta_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for $i = 1, 2, \ldots, p$. Given a partition $P = \Big\{(\Delta_i, t_i) : i = 1, 2, \ldots, p\Big\}$ of $[0, 1]$, Henstock-Kurzweil integral sum $S(P, g) = \sum_{i=1}^{p} g(t_i)|\Delta_i|$.

Definition 8. [5] *Let $g : [0, 1] \to X, x \in X$. We say x is a Henstock-Kurzweil point of g, if for every $\epsilon > 0$ and for every gauge δ on $[0, 1]$ there is a δ-fine partition P of $[0, 1]$, such that $\|S(P, g) - x\| < \epsilon$. We denote $I_{HK}(g)$ the set of all Henstock-Kurzweil points of g.*

It can be seen from the Theorem 3 that we get a class of subsets of $HK(\mu)$ as follows:

$$\Big\{I_{HK}(g) : g = \int_{[0,1]} \theta(f) \in HK(\mu)\Big\}.$$

Let $\mathfrak{F} = \{I_{HK}(g)\colon g = \theta(f) \in HK(\mu)\}$ be a bounded closed subset of $HK(\mu)$. Clearly, $\mathfrak{F}$ is a convex set.

Definition 9. *Let $g\colon [0,1] \to X$ be an arbitrary function. We will say that a point $x \in X$ belongs to the weak limit set $WI_{HK}(g)$ of g, if for any weak neighborhood U of x and any partition P of $[0,1]$ there exists a finer partition $Q > P$, such that $S(Q,g) \in U$.*

For $\epsilon > 0$, we set

$$\nu(\mathfrak{F},\epsilon) = \sup\left\{(HK)\int_A |\xi| d\mu\colon \xi \in \mathfrak{F},\ A \in \Sigma,\ \mu(A) \leqslant \epsilon\right\}$$

and define the modulus of uniform integrability $\nu(\mathfrak{F})$ of $\mathfrak{F}$ by

$$\nu(\mathfrak{F}) = \lim_{\epsilon\to 0} \nu(\mathfrak{F},\epsilon) = \inf_{\epsilon>0} \nu(\mathfrak{F},\epsilon).$$

Lemma 1. *Let $\mathfrak{F}$ be a subset of $HK(\mu)$. Then the following are equivalent:*

1) *$\mathfrak{F}$ has weak drop property.*
2) *$\mathfrak{F}$ is weakly sequentially compact in $HK(\mu)$.*
3) *$\mathfrak{F}$ is a bounded subset of $HK(\mu)$ satisfying the two properties: $\nu(\mathfrak{F}) = 0$ and for every $\epsilon > 0$ there exists $[0,1]_\epsilon \in \Sigma$, such that $\mu([0,1]_\epsilon) < \infty$ and*

$$\sup_{\xi\in\mathfrak{F}}(HK) \int_{[0,1]\setminus[0,1]_\epsilon} |\xi| d\mu \leqslant \epsilon.$$

Proof. For (1) $\implies$ (2): Suppose $\mathfrak{F}$ has weak drop property. The [15, Proposition 4.4.7] states that $\mathfrak{F}$ is weakly compact. Clearly, Eberlein-Smulian theorem says that $\mathfrak{F}$ is weakly sequentially compact in $HK(\mu)$.

(2) $\implies$ (3) follows from [14, Theorem 2.3].

For (3) $\implies$ (1): Given $\mathfrak{F}$ is a bounded subset of $HK(\mu)$ satisfying the two properties: $\nu(\mathfrak{F}) = 0$ and for every $\epsilon > 0$ there exist $[0,1]_\epsilon \in \Sigma$, such that $\mu([0,1]_\epsilon) < \infty$ and $\sup\limits_{\xi\in\mathfrak{F}}(HK) \int_{[0,1]\setminus[0,1]_\epsilon} |\xi| d\mu \leqslant \epsilon$. Since $\nu(\mathfrak{F}) = 0$ and $\mathfrak{F}$ is a bounded subset of $HK(\mu)$, we can confirm from [14, Proposition 2.6] that $\mathfrak{F}$ is uniformly integrable. Hence, from Prokhorov's theorem (see [16, Theorem 1.12]) $\mathfrak{F} = \left[(HK)\underline{\int_{[0,1]}}\xi d\mu,\ (HK)\overline{\int_{[0,1]}}\xi d\mu\right]$ is weakly sequentially pre-compact. The definition of weakly sequentially pre-compact is: every sequence $\{\xi_n\}$ in $\mathfrak{F}$ has a subsequence $\{\xi_{n_k}\}$ that converges weakly to some $\xi \in HK(\mu)$.

Suppose $\mathfrak{F}$ does not have weak drop property. Then there exists a weakly sequentially closed set $\mathfrak{C}$ of $HK(\mu)$ disjoint from $\mathfrak{F}$, such that $c \in \mathfrak{C}$, $\inf\Big\{d(x,\mathfrak{F})\colon x \in \mathfrak{C} \cap D(c,\mathfrak{F})\Big\} = 0$. So, there exists a sequence $\{x_n\}$ in $\mathfrak{C}$, such that $x_{n+1} \in D(x_n,\mathfrak{F})$ and there exists a sequence $\{y_n\}$ in $\mathfrak{F}$, such that $\|x_n - y_n\| \to 0$. As a stream $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ weakly convergent to some x_0 and $\mathfrak{C}$ is weakly sequentially closed, $x_0 \in \mathfrak{C}$, but $\|x_{n_k} - y_{n_k}\| \to 0$. So, $\{y_{n_k}\}$ is weakly convergent to x_0. Since $\mathfrak{F}$ is closed and convex, in both cases $x_0 \in \mathfrak{F}$. This is a contradiction to the fact that $\mathfrak{C}$ & $\mathfrak{F}$ are disjoint. Hence, $\mathfrak{F}$ has weak drop property. □

Theorem 5. *A closed bounded convex subset $\mathfrak{F}$ of $HK(\mu)$ is uniformly integrable if and only if $\mathfrak{F}$ has weak drop property.*

Proof. The proof follows directly from Lemma 1. □

Proposition 1. *A subset $\mathfrak{F}$ of $HK(\mu)$ is bounded and uniformly integrable if and only if there is an N-function θ, such that*

$$\sup\Big\{I_{HK}(g)\colon g = \theta(f) \in HK(\mu)\Big\} < \infty.$$

3. H-Orlicz spaces and weak drop property. In this section, we discuss the reflexivity of the H-Orlicz spaces with the drop property. For an N-function θ and a measurable f define $g = (HK)\int_{[0,1]} \theta(f)d\mu$.

Let

$$H^\theta = \Big\{f \ \textit{measurable}\colon g < \infty\Big\}.$$

If ϕ is the complement of θ, we assume

$$H^\theta_* = \Big\{f \ \textit{measurable}\colon |(HK)\int f\mathsf{f}| < \infty \ \textit{forall}\ \mathsf{f} \in H^\phi\Big\}.$$

The collection H^θ_* is a linear space. For $f \in H^\theta_*$, define

$$\|f\|_\theta = \sup\Big\{|(HK)\int f\mathsf{f}d\mu| : \ \mathsf{g} \leqslant 1\Big\}, \ \ \mathsf{g} = (HK)\int_{[0,1]} \phi(\mathsf{f})d\mu.$$

Then $(H^\theta_*, \|\cdot\|_\theta)$ is a Banach space called an H-Orlicz space. Moreover, letting $\|\cdot\|_{(\theta)}$ be the Minkowski functional associated with the convex set

$\{f \in H_*^\theta \colon \mathsf{g} \leqslant 1\}$, we have the fact that $\|\cdot\|_{(\theta)}$ is an equivalent norm on H_*^θ, called the Luxemberg norm.

In fact, $\|f\|_{(\theta)} \leqslant \|f\|_\theta \leqslant 2\|f\|_{(\theta)}$, for all $f \in H_*^\theta$.

Theorem 6. *The H-Orlicz space $H_*^\theta(\mu)$ is reflexive if and only if $\theta \in \Delta_2$.*

Proof. The proof is similar to the proof [13, Theorem 5]. □

Theorem 7. *Let θ be a N-function and E_θ be the closure of the bounded functions in H_*^θ. Then the conjugate space $(E_\theta, \|\cdot\|_{(\theta)})$ is isometrically isomorphic to $(H_*^\phi, \|\cdot\|_{(\phi)})$, where ϕ is the complement of θ.*

Proof. Let $v_\theta \in H_*^\theta$; define $f \colon E_\theta \to \mathbb{R}$ by

$$f(u_\theta) = (HK)\int_{[0,1]} < v_{\theta(t)}, u_{\theta(t)} > d\mu.$$

Then f is well defined. Also u_θ, v_θ are limits of simple functions, so that $< v_\theta, u_\theta >$ is measurable. So,

$$\begin{aligned}(HK)\int_{[0,1]} | < v_\theta, u_\theta > |d\mu &\leqslant (HK)\int_{[0,1]} \|v_\theta(t)\|_{HK}.\|u_\theta(t)\|_{HK} d\mu \\ &\leqslant \|v_\theta\|_\theta.\|u_\theta\|_\phi.\end{aligned}$$

Consequently, $f(v_\theta) = (HK)\int_{[0,1]} < v_\theta(t), u_\theta(t) > d\mu$ is linear and $\|f\|_{HK} \leqslant \|v\|_\theta$. Consider $v_\theta = \sum_{i=1}^{\infty} x_i \chi_{\mathcal{E}_i}$, $x_i \in X$, $i = 1, 2, \ldots$, and $\{\mathcal{E}_i\}_{i=1}^{\infty}$ is a countable partition of $[0,1]$ with $\mu(\mathcal{E}_i) > 0$, $i = 1, 2 \ldots$. Now, for $\epsilon > 0$, $|v_\theta| \in H_*^\theta(\mu)$, so there exists a non-negative $h \in E_\phi(\mu) \subset HK(\mu)$, $0 < \|h\|_\phi \leqslant 1$, such that

$$\|v_\theta\|_\theta - \frac{\epsilon}{2} < (HK)\int\limits_{[0,1]} \|v_\theta(t)\|_{HK} h(t) d\mu.$$

If $\|x_i\| = 1$, so that

$$\|x_i^*\| - \frac{\epsilon}{2}\|h\|_{HK}^{-1} \leqslant x_i^*(x_i)$$

and $u_\theta = \sum_{i=1}^{\infty} x_i h \chi_{\mathcal{E}_i}$, then for $\alpha > 0$ we have

$$(HK)\int\limits_{[0,1]} \theta\big(\|\alpha u_\theta(t)\|\big) d\mu = \sum_{i=1}^{\infty} (HK)\int\limits_{\mathcal{E}_i} \theta(\|\alpha x_i h(t)\|) d\mu =$$

$$= \sum_{i=1}^{\infty} (HK) \int\limits_{\mathcal{E}_i} \theta(\alpha|h(t)|) d\mu = (HK) \int\limits_{[0,1]} \theta(\alpha|h(t)) d\mu < \infty.$$

Thus, $u_\theta \in E_\theta$ and

$$\|u_\theta\|_\theta = \inf \Big\{ \alpha > 0 \colon (HK) \int\limits_{[0,1]} \theta\Big(\frac{|u|}{\alpha}\Big) \leqslant 1 \Big\} =$$
$$= \inf \Big\{ \alpha > 0 \colon \sum_{i=1}^{\infty} (HK) \int\limits_{\mathcal{E}_i} \theta\Big(\frac{|h(t)|}{\alpha}\Big) d\mu \leqslant 1 \Big\} =$$
$$= \|h\|_\theta \leqslant 1.$$

Now,

$$f(u_\theta) =< v_\theta, u_\theta >= (HK) \int_{[0,1]} < v_\theta, u_\theta > d\mu =$$
$$= (HK) \int\limits_{[0,1]} h(t) \sum_{i=1}^{\infty} x_i^*(x_i) \chi_{\mathcal{E}_i} d\mu \geqslant$$
$$\geqslant (HK) \int\limits_{[0,1]} h(t) \sum_{i=1}^{\infty} \Big(\|x_i^*\| - \frac{\epsilon}{2} \|h\|_{HK}^{-1} \Big) d\mu =$$
$$= (HK) \int\limits_{[0,1]} h(t) \|u_\theta(t)\| d\mu - \frac{\epsilon}{2} . \frac{(HK) \int\limits_{[0,1]} h(t) d\mu}{\|h\|_{HK}} \geqslant$$
$$\geqslant \|v_\theta\|_\phi - \frac{\epsilon}{2} - \frac{\epsilon}{2}.$$

So, $\|f\|_{HK} \geqslant \|v_\theta\|_\phi$. Hence, $\|f\|_{HK} = \|v_\theta\|_\phi$ and $0 \leqslant \|f - f_n\|_{HK} \leqslant$ $\leqslant \|v_\theta - v_{\theta_n}\|_\phi \to 0$. Therefore, $\|f\|_{HK} = \lim\limits_{n\to\infty} \|f_n\| = \lim\limits_{n\to\infty} \|v_{\theta_n}\|_\phi = \|v_\theta\|_\phi$. So, H_*^ϕ is isometrically isomorphic to a subspace of E_θ. □

Theorem 8. *The H-Orlicz space $H_*^\theta(\mu)$ has weak drop property if and only if $\theta \in \Delta_2$.*

Proof. The H-Orlicz space $H_*^\theta(\mu)$ is reflexive if and only if $\theta \in \Delta_2$. We can conclude the complete proof with a similar way of the proof of the [6, Theorem 5]. □

4. Some consequences for relatively weakly compact subset in H-Orlicz spaces. In this section, we discuss the subsets of $HK(\mu)$ with weak drop property that are relatively weakly compact in $H_*^\theta(\mu)$.

Theorem 9. *A closed and bounded convex subset $\mathfrak{F}$ of $HK(\mu)$ has weak drop property if and only if there is an N-function θ with Δ' condition, such that $\mathfrak{F}$ is relatively weakly compact in $H_*^\theta(\mu)$.*

Proof. Suppose a closed and bounded convex subset $\mathfrak{F}$ of $HK(\mu)$ has weak drop property. From the Theorem 5, $\mathfrak{F}$ is uniform integrable.

Take an N-function $\theta \in \Delta'$ with $\theta(\theta(x)) \leqslant \theta_1(x)$, where θ_1 is an N-function for large values of x.

Then $\sup\left\{(HK)\int\limits_{[0,1]}\theta(\theta(\xi))d\xi\colon \xi \in \mathfrak{F}\right\} < \infty$. So, by de la Vallée Poussin's theorem, $\mathfrak{F} = \left\{I_{HK}(g)\colon g = \int\limits_{[0,1]}\theta(f)\right\}$ is uniformly integrable in $HK(\mu)$. The modulus of uniform integrability vanishes: $\nu(\mathfrak{F}) = 0$ in H_*^θ. Since $\mathfrak{F}$ has weak drop property, so, $\mathfrak{F}$ is relatively compact in $HK(\mu)$. It is also relatively compact in the topology of convergence in measure. Hence, $\mathfrak{F}$ is relatively compact in $H_*^\theta(\mu)$. □

Theorem 10. *Let $\theta \in \Delta_2$ and suppose that its complement ϕ satisfies $\lim\limits_{t\to\infty}\frac{\phi(ct)}{\phi(t)} = \infty$ for some $c > 0$; then there is a closed bounded convex set $\mathfrak{F} \subset H_*^\theta(\mu)$ that is relatively weakly compact if and only if $\nu(\mathfrak{F}) = 0$.*

Proof. Let us consider the closed bounded convex set $\mathfrak{F} \subset H_*^\theta(\mu)$ that is relatively compact. If possible, assume $\nu(\mathfrak{F}) \neq 0$; then there is an $\epsilon_0 > 0$, a sequence $(f_n) \subset \mathfrak{F}$, and a measurable sequence $(\mathcal{E}_n)$ with $\mu(\mathcal{E}_n) \to 0$, so $\|\chi_{\mathcal{E}_n} f_n\| > \epsilon_0$ for each $n \in \mathbb{N}$.

Using the Eberlein-Smulian theorem, we can have $f \in H_*^\theta$ and a subsequence (f_{n_k}) of (f_n), so that $f_{n_k} \to f$ weakly in $H_*^\theta(\mu)$. We can state that $\|\cdot\|_\theta$ has weak drop from [6, Theorem 5]. The Lemma 1 gives $\nu(\mathfrak{F}) = 0$.

Conversely, suppose $\nu(\mathfrak{F}) = 0$ and $\mathfrak{F} \subset E_\theta$. If (f_n) is a sequence in $\mathfrak{F}$, it is bounded in $HK(\mu)$-norm as (f_n) is bounded in H_*^θ-norm. Now, by Komlós's theorem, there is a subsequence (f_{n_k}) that has μ-a.e. convergent arithmetic means to f. When ϕ is the complement of θ, we have, for any measurable subset $\mathcal{E} \subset [0,1]$ and $g \in H_*^\theta$ with $\|g\|_\phi \leqslant 1$:

$$\nu(\mathfrak{F}) = |(HK)\int g\chi_{\mathcal{E}} f d\mu| \leqslant (HK)\int |g\chi_{\mathcal{E}} f| d\mu \leqslant$$

$$\leqslant \liminf_n (HK)\int |g\chi_{\mathcal{E}}\frac{1}{n}\sum_{k=1}^{n} f_{n_k}|d\mu \leqslant \sup_n \frac{1}{n}\sum_{k=1}^{n}(HK)\int |g\chi_{\mathcal{E}} f_{n_k}|d\mu \leqslant$$
$$\leqslant \sup_n \frac{1}{n}\sum_{k=1}^{n} \|g\|_{(\phi)}.\|\chi_{\mathcal{E}} f_{n_k}\|_\theta \leqslant \sup\left\{\|\chi_{\mathcal{E}} h\|_\theta \colon h\in\mathfrak{F}\right\} = 0$$

Thus,

$$\|\chi_{\mathcal{E}} f\|_\theta \leqslant \sup\left\{|(HK)\int g\chi_{\mathcal{E}} f d\mu| \colon \|g\|_{(\phi)}\leqslant 1\right\} \leqslant$$
$$\leqslant \sup\left\{\|\chi_{\mathcal{E}} h\|_\theta \colon h\in\mathfrak{F}\right\} = 0$$

So, $f \in H_*^\theta(\mu)$ with absolutely continuous norm. Assume $a_n = \frac{1}{n}\sum_{k=1}^{n} h_k$. Since the inclusion map: $H_*^\phi \to HK(\mu)$ is continuous, there is a $K > 0$, such that $\|g\|_{HK} \leqslant K\|g\|_\phi \; \forall \; g \in H_*^\phi$. Now for any $\epsilon > 0$ choose $\delta > 0$, so that $\sup\left\{|\chi_{\mathcal{E}} h\|_\theta \colon h\in\mathfrak{F}\right\} = 0$ whenever $\mu(\mathcal{E}) < \delta$. By Egorov's theorem, there is a measurable set $\mathcal{E}$ with $\mu([0,1]\setminus\mathcal{E}) < \delta$, so that $a_n \to f$ uniformly on $\mathcal{E}$. Choose $N \in \mathbb{N}$, so that $\|\chi_{\mathcal{E}}(a_n - f)\|_\infty \to 0$ whenever $n \geqslant N$. Then, for $g \in H_*^\phi$ with $\|g\|_\phi \leqslant 1$ and $n \geqslant N$, $|(HK)\int g(a_n - f)d\mu| \to 0$. So, $\mathfrak{F}$ is a Banach-Saks set in H_*^θ. Since $f_{n_k} \to f$ weakly in H_*^θ, so, by Eberlein Smulian theorem, $\mathfrak{F}$ is relatively weakly compact in H_*^θ. □

Conclusions. In this article, we have studied uniform integrability of a closed bounded convex sets with weak drop property. Relatively weakly compactness of some subsets of $HK(\mu)$ with weak drop property that are relatively weakly compact in $H_*^\theta(\mu)$ have been also discussed.

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