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STRONG COUPLED FIXED POINTS AND APPLICATIONS TO FRACTAL GENERATIONS IN FUZZY METRIC SPACES

Abstract. In this paper, we establish a strong coupled fixed-point result for a fuzzy contractive coupling, defined between two subsets of a fuzzy metric space. The coupling is defined by combining the concepts of coupled fuzzy contractions and cyclic mappings. It is the main instrument in the paper. Uniqueness of the strong coupled fixed-point is also shown. There is a corollary and an illustrative example. An example shows that the main theorem properly contains the strong coupled fixed-point result as a corollary. We discuss an application, where construct a special type of Iterated Function System by utilizing a family of fuzzy contractive couplings; this finally leads to the generation of a strong coupled fractal set in fuzzy metric space. A fuzzy version of the Hausdorff distance between compact sets is utilized in the above process. The method of fractal generation is illustrated.

Key words: *fuzzy metric space, iterated function system, coupling, strong coupled fixed point, strong coupled fractal*

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1. Introduction. The ultimate goal of this research is to establish a method of generating strong coupled fractals in fuzzy metric spaces. This is done by an application of a strong coupled fixed-point result, also proved here. There are several inequivalent extensions of metric spaces into the domain of fuzzy mathematics, like those in [6], [9], [16], [20]. Out of these notions, the definition given in [9] has several salient features including the fact that the space is naturally endowed with a Hausdorff topology. This is one of the reasons why this version of the fuzzy metric space has been so extensively considered in fuzzy fixed-point theory.

We concern ourselves exclusively with the definition of the fuzzy metric space given by George and Veeramani [8], [9] and refer to it simply as 'fuzzy metric space'. Fuzzy fixed-point theory has experienced very extensive and versatile development over the last three decades. References [11], [12], [13], [14], [17] are some instances from this domain of study. Strong coupled fractals are already introduced in metric spaces in a recent paper by Choudhury et al. [4]. They are generated here through the construction of the Huchinsion-Barnsley operator [1] corresponding to a specialized Iterated Function System [1], [15] called Iterated Coupling System (ICS) [4]. It uses couplings between two subsets of the fuzzy metric space. Iterated Function Systems (IFS for short) are widely used as means of generating fractal sets in different spaces by involving contractions of various kinds. Some instances of these works are noted in [4], [21]. In the fuzzy domain, there are already works in this direction (e. g., like [28]), but it appears that there is still much to be explored. This is one of our motivations for starting this research.

Couplings are basic instruments in our paper. They are cyclic coupled mappings that have appeared in works like [4]. Couplings are defined by combining the ideas of coupled mappings and cyclic mappings. Both these types of mappings have featured prominently in the recent literature on fixed-point theory on metric and fuzzy metric spaces, along with several applications. Some instances of the former are [14], [23], while [18], [19] are works involving the latter-type mappings.

This paper is organized as follows. Firstly, we introduce a fuzzy contractive coupling in fuzzy metric spaces and obtain a strong coupled fixed-point result for them. The proof depends on the property of the t-norm that it is stronger than the product t-norm. Then a Fuzzy Iterated Coupling System is introduced with the help of these couplings. Finally, we utilize the fixed-point result to generate strong coupled fractals through the Hutchinson-Barnsley operator. In the sequel, we show that our strong coupled fixed-point result effectively generalizes an existing fuzzy coupled fixed point result. Both the fixed-point theorem and the process of fractal generation are illustrated with examples.

2. Preliminaries. Throughout the paper we use the following notation: $\mathbb{N}_n$ denotes the set of first n natural numbers; for any topological space (X, τ), the symbols $\mathcal{P}(X)$ and $\mathcal{K}(X)$ denote, respectively, the set of all non-empty subsets and the set of non-empty compact subsets.

Definition 1. [26] *A binary operator* $*\colon [0,1]^2 \to [0,1]$ *is called a t-norm,* *if the following conditions are satisfied:*

(i) $*$ *is associative and commutative;*

(ii) $a * 1 = a$ *for all* $a \in [0, 1]$;

(iii) $a * b \leqslant c * d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$ *for each* $a, b, c, d \in [0, 1]$.

Now let us define the fuzzy metric space. There are several non-equivalent definitions in literature; we use the simple definition by George et. al. [8]:

Definition 2. [8] *A 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary non-empty set,* $*$ *is a continuous t-norm, and* M *is a fuzzy set on* $X \times X \times (0, \infty)$, *satisfying the conditions* $(FM1)$ – $(FM5)$ *for all* $x, y, z \in X$ and $t, s > 0$;
$(FM1)$ $M(x, y, t) > 0$;
$(FM2)$ $M(x, y, t) = 1$ *iff* $x = y$;
$(FM3)$ $M(x, y, t) = M(y, x, t)$;
$(FM4)$ $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + s)$;
$(FM5)$ $M(x, y, .): (0, \infty) \to (0, 1]$ *is continuous.*

We also use the following condition:
$(FM6)$ for all $x, y \in X$, $\lim_{t\to\infty} M(x, y, t) = 1$.

Definition 3. [8] *Let* $(X, M, *)$ *be a fuzzy metric space. The open ball* $B(x, r, t)$ *with center* $x \in X$ *for* $t > 0$ *and* $0 < r < 1$ *is defined as* $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

It has been proved in [8] that the family $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$ *is a basis for a Hausdorff topology on* X. *The following notions of convergences are consistent with the topology described above.*

Definition 4. [8] *Let* $(X, M, *)$ *be a fuzzy metric space.*

(i) A sequence $\{x_n\}$ *in* X *is said to be convergent if there exists some* $x \in X$, *such that* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for all* $t > 0$.

(ii) A sequence $\{x_n\}$ *in* X *is said to be a Cauchy sequence if* $\lim_{m,n\to\infty} M(x_n, x_m, t) = 1$ *for all* $t > 0$.

Definition 5. [24] *Let* A *and* B *be two non-empty compact subsets of a fuzzy metric space* $(X, M, *)$. *The Hausdorff fuzzy metric* $\mathcal{H}_M$ on $\mathcal{K}(X)$ *is defined by*

$$\mathcal{H}_M(A, B, t) = \min\{\omega(A, B, t),\ \bar{\omega}(A, B, t)\}$$

where $\omega(A,B,t) = \inf_{a\in A} \sup_{b\in B} M(a,b,t)$ *and* $\bar{\omega}(A,B,t) = \inf_{b\in B} \sup_{a\in A} M(a,b,t)$ *and* $t > 0$.

Lemma 1. [24] *Let* $(X, M, *)$ *be a fuzzy metric space. Suppose* $\{A_i\}_{i=1}^m$, $\{B_i\}_{i=1}^m \subseteq \mathcal{K}(X)$, $A = \bigcup_{i=1}^m A_i$ *and* $B = \bigcup_{i=1}^m B_i$. *Then, for all* $t > 0$,

$$\mathcal{H}_M(A,B,t) \geqslant \min_{1\leqslant i\leqslant m} \mathcal{H}_M(A_i,B_i,t).$$

Theorem 1. [24] *Let* $(X, M, *)$ *be a fuzzy metric space. Then* $(\mathcal{K}(X), \mathcal{H}_M, *)$ *is a fuzzy metric space. Further, if* $(X, M, *)$ *is a complete fuzzy metric space, then* $(\mathcal{K}(X), \mathcal{H}_M, *)$ *is also a complete fuzzy metric space.*

Definition 6. [2] *Let* $F\colon X \times X \to X$ *be a mapping. An element* $(x, y) \in X \times X$ *is said to be a coupled fixed point of* F *if* $F(x, y) = x$ *and* $F(y, x) = y$.

If $x = y$, then the coupled fixed point is called a strong coupled fixed point, in which case we have $F(x, x) = x$. The point $(x, x) \in X \times X$ (or simply $x \in X$) is said to be a strong coupled fixed point.

Definition 7. [19] *Let* A *and* B *be two non-empty subsets of* X. *A mapping* $F\colon X \to X$ *is said to be cyclic (with respect to* A *and* B *) if* $F(A) \subset B$ *and* $F(B) \subset A$.

The following is the definition of coupling, which is a combination of the ideas of cyclic mappings and coupled mappings. It is our main instrument in this paper.

Definition 8. [4] *Let* A *and* B *be two non-empty subsets of* X. *A mapping* $F\colon X \times X \to X$ *is said to be a coupling with respect to* A *and* B *if* $F(x, y) \in B$ *and* $F(y, x) \in A$ *whenever* $x \in A$ *and* $y \in B$.

Fractal generation by iterated system of functions is done through the construction of the Hutchinson operator [15] in a fuzzy metric space. The following is the definition of the Hutchinson operator in a fuzzy metric space.

Definition 9. *Let* $(X, M, *)$ *be a fuzzy metric space and* $\{F_i; i \in \mathbb{N}_n\}$ *be a finite collection of continuous couplings on* X, *each with respect to two non-empty closed subsets* A *and* B *of* X. *The Hutchinson operator,* *corresponding to* $\{F_i; i \in \mathbb{N}_n\}$, $\widehat{G}\colon \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$, *is defined as* $\widehat{G}(A,B) = \bigcup_{i=1}^{n} \widehat{F}_i(A,B)$, *where* $\widehat{F}_i(A,B) = \{F_i(a,b)\colon a \in A, b \in B\}$.

The above definition is valid only with respect to continuous couplings.

Borrowing the concept from [4], in the next two definitions we introduce the following notions in the context of fuzzy metric space.

Definition 10. *A Fuzzy Iterated Coupling System (FICS) consists of a complete fuzzy metric space* $(X, M, *)$, *two non-empty closed subsets* A, B *of* X, *and a finite collection of couplings* $F_i\colon X \times X \to X$ *with respect to* A, B *for all* $i \in \mathbb{N}_n$. *We denote it by* $\langle (X, M, *); A, B, F_i, i \in \mathbb{N}_n \rangle$.

Definition 11. *Let* $(X, M, *)$ *be a complete fuzzy metric space and* $\widehat{T}\colon \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ *be a mapping.* $A \in \mathcal{K}(X)$ *is said to be a strong coupled fractal of* $\widehat{T}$ *if* $\widehat{T}(A,A) = A$.

Definition 12. *Let* $(X, M, *)$ *be a fuzzy metric space and* A *and* B *be two non-empty subsets of* X. *We call a coupling* $F\colon X \times X \to X$ *with respect to* A *and* B *a fuzzy contractive coupling if there exists* $k \in (0,1)$, *such that*

$$M\left(F(x,y), F(u,v), kt\right) \geqslant \left(M(x,u,t)\right)^{\frac{1}{2}} * \left(M(y,v,t)\right)^{\frac{1}{2}}, \tag{1}$$

where $x, v \in A$ *and* $u, y \in B$.

Here the constant k is the contractivity factor of the coupling.

3. Coupled fixed-point results.

Theorem 2. *Let* A *and* B *be two non-empty closed subsets of a complete fuzzy metric space* $(X, M, *)$, *satisfying* $(FM6)$ *of Definition 2. Let* $F\colon X \times X \to X$ *be a fuzzy contractive coupling with respect to* A *and* B. *If* $*$ *is a t-norm, such that* $* \geqslant *_p$, *then* $A \cap B \neq \phi$ *and* F *has a unique strong coupled fixed point. Moreover, for arbitrary choice of* $x_0 \in A$ *and* $y_0 \in B$, *both sequences* $\{x_n\}, \{y_n\}$ *constructed as* $x_{n+1} = F(y_n, x_n)$ *and* $y_{n+1} = F(x_n, y_n)$ *converge to the strong coupled fixed point.*

Proof. From the construction of $\{x_n\}$ and $\{y_n\}$ it follows that for all $n \geqslant 0$, $x_n \in A$ and $y_n \in B$.
Now,

$$M(x_n, y_n, t) = M\left(F\left(y_{n-1}, x_{n-1}\right), F\left(x_{n-1}, y_{n-1}\right), t\right) \geqslant$$

$$\geqslant \Big(M\Big(y_{n-1}, x_{n-1}, \frac{t}{k}\Big)\Big)^{\frac{1}{2}} * \Big(M\Big(x_{n-1}, y_{n-1}, \frac{t}{k}\Big)\Big)^{\frac{1}{2}} \geqslant$$
$$\geqslant \Big(M\Big(y_{n-1}, x_{n-1}, \frac{t}{k}\Big)\Big)^{\frac{1}{2}} *_p \Big(M\Big(x_{n-1}, y_{n-1}, \frac{t}{k}\Big)\Big)^{\frac{1}{2}} =$$
$$= M\Big(x_{n-1}, y_{n-1}, \frac{t}{k}\Big) = M\Big(F(y_{n-2}, x_{n-2}), F(x_{n-2}, y_{n-2}), \frac{t}{k}\Big) \geqslant$$
$$\geqslant \Big(M\Big(y_{n-2}, x_{n-2}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}} * \Big(M\Big(x_{n-2}, y_{n-2}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}} \geqslant$$
$$\geqslant \Big(M\Big(y_{n-2}, x_{n-2}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}} *_p \Big(M\Big(x_{n-2}, y_{n-2}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}} =$$
$$= M\Big(x_{n-2}, y_{n-2}, \frac{t}{k^2}\Big) \ldots \geqslant M\Big(x_0, y_0, \frac{t}{k^n}\Big).$$

Letting $n \to \infty$ in the above inequality and using $(FM6)$, we get,

$$\lim_{n\to\infty} M(x_n, y_n, t) = 1 \text{ for all } t > 0. \tag{2}$$

Again for all $n \in \mathbb{N}$ and $t > 0$,

$$M(x_{n+1}, y_n, t) = M\left(F(y_n, x_n), F(x_{n-1}, y_{n-1}), t\right) \geqslant$$
$$\geqslant \Big(M\Big(x_{n-1}, y_n, \frac{t}{k}\Big)\Big)^{\frac{1}{2}} * \Big(M\Big(x_n, y_{n-1}, \frac{t}{k}\Big)\Big)^{\frac{1}{2}} \geqslant$$
$$\geqslant \Big(M\Big(x_{n-1}, y_n, \frac{t}{k}\Big)\Big) * \Big(M\Big(x_n, y_{n-1}, \frac{t}{k}\Big)\Big) = \text{[t-norm is monotonic]}$$
$$= M\Big(F(y_{n-2}, x_{n-2}), F(x_{n-1}, y_{n-1}), \frac{t}{k}\Big) * M\Big(F(y_{n-1}, x_{n-1}), F(x_{n-2}, y_{n-2}), \frac{t}{k}\Big) \geqslant$$
$$\geqslant \Big(\Big(M\Big(x_{n-1}, y_{n-2}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}} * \Big(M\Big(x_{n-2}, y_{n-1}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}}\Big) *$$
$$* \Big(\Big(M\Big(x_{n-2}, y_{n-1}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}} * \Big(M\Big(x_{n-1}, y_{n-2}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}}\Big) \geqslant$$
$$\geqslant \Big(\Big(M\Big(x_{n-1}, y_{n-2}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}} *_p \Big(M\Big(x_{n-2}, y_{n-1}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}}\Big) *_p$$
$$*_p \Big(\Big(M\Big(x_{n-2}, y_{n-1}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}} *_p \Big(M\Big(x_{n-1}, y_{n-2}, \frac{t}{k^2}\Big)\Big)^{\frac{1}{2}}\Big) =$$
$$= M\Big(x_{n-1}, y_{n-2}, \frac{t}{k^2}\Big) *_p M\Big(x_{n-2}, y_{n-1}, \frac{t}{k^2}\Big) =$$
$$= M\Big(F(y_{n-2}, x_{n-2}), F(x_{n-3}, y_{n-3}), \frac{t}{k^2}\Big) *_p$$
$$*_p M\Big(F(y_{n-3}, x_{n-3}), F(x_{n-2}, y_{n-2}), \frac{t}{k^2}\Big) \geqslant$$

$$\geqslant \left(\left(M\Big(x_{n-3}, y_{n-2}, \frac{t}{k^3}\Big)\right)^{\frac{1}{2}} * \left(M\Big(x_{n-2}, y_{n-3}, \frac{t}{k^3}\Big)\right)^{\frac{1}{2}}\right) *_p$$
$$*_p \left(\left(M\Big(x_{n-2}, y_{n-3}, \frac{t}{k^3}\Big)\right)^{\frac{1}{2}} * \left(M\Big(x_{n-3}, y_{n-2}, \frac{t}{k^3}\Big)\right)^{\frac{1}{2}}\right) \geqslant$$
$$\geqslant \left(\left(M\Big(x_{n-3}, y_{n-2}, \frac{t}{k^3}\Big)\right)^{\frac{1}{2}} *_p \left(M\Big(x_{n-2}, y_{n-3}, \frac{t}{k^3}\Big)\right)^{\frac{1}{2}}\right) *_p$$
$$*_p \left(\left(M\Big(x_{n-2}, y_{n-3}, \frac{t}{k^3}\Big)\right)^{\frac{1}{2}} *_p \left(M\Big(x_{n-3}, y_{n-2}, \frac{t}{k^3}\Big)\right)^{\frac{1}{2}}\right) =$$
$$= M\Big(x_{n-3}, y_{n-2}, \frac{t}{k^3}\Big) *_p M\Big(x_{n-2}, y_{n-3}, \frac{t}{k^3}\Big) = \ldots =$$
$$= M\Big(x_0, y_1, \frac{t}{k^n}\Big) *_p M\Big(x_1, y_0, \frac{t}{k^n}\Big). \quad (3)$$

Proceeding in the same way as above with the sequence $M(x_n, y_{n+1}, t)$, for all $n \in \mathbb{N}$ and $t > 0$, we get:

$$M(x_n, y_{n+1}, t) \geqslant M\Big(x_0, y_1, \frac{t}{k^n}\Big) *_p M\Big(x_1, y_0, \frac{t}{k^n}\Big). \quad (4)$$

For all $n \in \mathbb{N}$ and $t > 0$, let $\gamma_n(t) = M\Big(x_0, y_1, \frac{t}{k^n}\Big) *_p M\Big(x_1, y_0, \frac{t}{k^n}\Big)$.

Using (3), (4) and $(FM6)$, we get

$$\lim_{n\to\infty} \gamma_n(t) = 1 \text{ for all } t > 0.$$

Note that for $m > n$ and $0 < k < 1$,

$$1 > 1 - k^{m-n} = (1-k)(1 + k + k^2 + \ldots + k^{m-n-1}).$$

Therefore, for every $t > 0$,

$$t > t(1-k)(1 + k + k^2 + \cdots + k^{m-n-1}).$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in A. For $m > n$, we consider the following two cases.

Case I: $m - n$ is even.

$$M(x_n, x_m, t) \geqslant M(x_n, x_m, t(1-k)(1 + k + k^2 + \ldots + k^{m-n-1})) \geqslant$$
$$\geqslant M(x_n, y_{n+1}, t(1-k)) * M(y_{n+1}, x_{n+2}, t(1-k)k) * \cdots *$$
$$* M(x_{m-2}, y_{m-1}, t(1-k)k^{m-n-2}) * M(y_{m-1}, x_m, t(1-k)k^{m-n-1}) \geqslant$$

$$\geqslant \Big(M\Big(x_0,y_1,\frac{t(1-k)}{k^n}\Big)*_p M\Big(x_1,y_0,\frac{t(1-k)}{k^n}\Big)\Big)*\Big(M\Big(x_0,y_1,\frac{t(1-k)k}{k^{n+1}}\Big)\Big)*_p$$
$$*_p\, M\Big(x_1,y_0,\frac{t(1-k)k}{k^{n+1}}\Big) * \cdots * M\Big(x_0,y_1,\frac{t(1-k)k^{m-n-1}}{k^{m-1}}\Big)*_p$$
$$*_p\, M\Big(x_1,y_0,\frac{t(1-k)k^{m-n-1}}{k^{m-1}}\Big) = \quad (\text{using } (3),(4))$$
$$= \underbrace{\gamma_n(t(1-k)) * \gamma_n(t(1-k)) * \cdots * \gamma_n(t(1-k))}_{m-n \text{ times}}.$$

Case II: $m-n$ is odd.

$$M(x_n,x_m,t) \geqslant$$
$$\geqslant M\Big(x_n,x_m,t(1-k)\Big(1+k+k^2+\cdots+k^{m-n-2}+\frac{k^{m-n-1}}{2}+\frac{k^{m-n-1}}{2}\Big)\Big) \geqslant$$
$$\geqslant M(x_n,y_{n+1},t(1-k)) * M(y_{n+1},x_{n+2},t(1-k)k) * \ \cdots\ *$$
$$* M(x_{m-1},y_m,t(1-k)k^{m-n-2}) * M\Big(x_{m-1},y_m,t(1-k)\frac{k^{m-n-1}}{2}\Big)*$$
$$* M\Big(y_m,x_m,t(1-k)\frac{k^{m-n-1}}{2}\Big) \geqslant$$
$$\geqslant \Big(M\Big(x_0,y_1,\frac{t(1-k)}{k^n}\Big) *_p M\Big(x_1,y_0,\frac{t(1-k)}{k^n}\Big)\Big) *$$
$$* \Big(M\Big(x_0,y_1,\frac{t(1-k)k}{k^{n+1}}\Big) *_p M\Big(x_1,y_0,\frac{t(1-k)k}{k^{n+1}}\Big)\Big) *$$
$$* \ \cdots\ * \Big(M\Big(x_0,y_1,\frac{t(1-k)k^{m-n-2}}{k^{m-2}}\Big) *_p M\Big(x_1,y_0,\frac{t(1-k)k^{m-n-2}}{k^{m-2}}\Big)\Big) *$$
$$* \Big(M\Big(x_1,y_0,\frac{t(1-k)k^{m-n-1}}{2k^{m-1}}\Big) *_p M\Big(x_0,y_1,\frac{t(1-k)k^{m-n-1}}{2k^{m-1}}\Big)\Big) *$$
$$* M\Big(x_0,y_0,\frac{t(1-k)k^{m-n-1}}{2k^m}\Big) = \quad (\text{using } (2),(3) \text{ and } (4))$$
$$= \underbrace{\gamma_n(t(1-k)) * \gamma_n(t(1-k)) * \cdots * \gamma_n\Big(\frac{t(1-k)}{2}\Big)}_{m-n \text{ times}} * M(x_0,y_0,\frac{t(1-k)}{2k^{n+1}}).$$

Combining the above two cases and $(FM6)$ and $\gamma_n(t) \to 1$ as $n \to \infty$ for all $t > 0$, we see that $\{x_n\}$ is a Cauchy sequence in A. A similar approach can be taken to show that $\{y_n\}$ is also a Cauchy sequence in B.

Since A, B are closed subsets, there exists $x \in A$ and $y \in B$, such that

$$\begin{aligned} \lim_{n\to\infty} M(x_n, x, t) = 1, \; \forall\; t > 0, \\ \lim_{n\to\infty} M(y_n, y, t) = 1, \; \forall\; t > 0. \end{aligned} \tag{5}$$

Now,

$$M(x, y, t) \geqslant M\Big(x, x_n, \frac{t - kt}{2}\Big) * M(x_n, y_n, kt) * M\Big(y_n, y, \frac{t - kt}{2}\Big). \tag{6}$$

Taking limit as $n \to \infty$ in the above inequality and using (2) and (5), we get $x = y$.

Therefore, $A \cap B \neq \phi$ and $x = y \in A \cap B$.

Also,

$$\begin{aligned} M(x_n, F(x, y), t) \geqslant M(x_n, F(x, y), kt) = M(F(y_{n-1}, x_{n-1}), F(x, y), kt) \geqslant \\ \geqslant (M(y_{n-1}, x, t))^{\frac{1}{2}} * (M(x_{n-1}, y, t))^{\frac{1}{2}} = (M(y_{n-1}, y, t))^{\frac{1}{2}} * (M(x_{n-1}, x, t))^{\frac{1}{2}}. \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality and using (5), we get $x_n \to F(x, x)$. Since the topology of the fuzzy metric space is Hausdorff, we have $F(x, x) = x$. Thus (x, x) is a strong coupled fixed point of F.

To show the uniqueness of the strong coupled fixed point, let $z \neq x \in X$ be another strong coupled fixed point of F and let it be such that $F(z, z) = z$. Then

$$\begin{aligned} M(x, z, t) = M(F(x, x), F(z, z), t) \geqslant \Big(M\Big(x, z, \frac{t}{k}\Big)\Big)^{\frac{1}{2}} * \Big(M\Big(x, z, \frac{t}{k}\Big)\Big)^{\frac{1}{2}} \geqslant \\ \geqslant M\Big(x, z, \frac{t}{k}\Big). \end{aligned} \tag{7}$$

By a repeated application of (7), we have for all n:

$$M(x, z, t) \geqslant M\Big(x, z, \frac{t}{k}\Big) \geqslant M\Big(x, z, \frac{t}{k^2}\Big) \geqslant \cdots \geqslant M\left(x, z, \frac{t}{k^n}\right).$$

Taking limit as $n \to \infty$ in the above inequality, using $(FM6)$, we get $M(x, z, t) = 1$. Hence, $x = z$. Thus, F has a unique strong coupled fixed point in $A \cap B$. □

Example 1. Let $X = \mathbb{R}$ and $A = \left[0, \frac{1}{2}\right], B = \left[-\frac{1}{2}, 0\right]$. Consider the fuzzy metric space $(X, M, *)$, where $*$ is the product t-norm and $M(x, y, t) = e^{-\frac{|x-y|}{t}}$. Let $F\colon X \times X \to X$ be a mapping given by

$$F(x,y) = \begin{cases} \frac{y-x}{6} & \text{if } (x,y) \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}], \\ 2x & \text{otherwise.} \end{cases}$$

From the definition, it is immediate that F is a coupling with respect to A and B. We show that F is a fuzzy contractive coupling with respect to A and B.

Let $k = \frac{1}{3}$. For $x, v \in A$ and $y, u \in B$, we have

$$(M(x,u,t))^{\frac{1}{2}} = e^{-\frac{|x-u|}{2t}} \text{ and } (M(y,v,t))^{\frac{1}{2}} = e^{-\frac{|y-v|}{2t}}$$

and

$$M\left(F(x,y), F(u,v), kt\right) = e^{-\frac{|F(x,y)-F(u,v)|}{kt}} = e^{-\frac{|(y-x)-(v-u)|}{6kt}} = e^{-\frac{|(u-x)+(y-v)|}{2t}}.$$

Also,

$$\begin{aligned} & |(u-x)+(y-v)| \leqslant |u-x| + |y-v| \\ \text{or, } & \frac{|(u-x)+(y-v)|}{2t} \leqslant \frac{|u-x|+|y-v|}{2t} \\ \text{or, } & e^{-\frac{|(u-x)+(y-v)|}{2t}} \geqslant e^{-\frac{|u-x|+|y-v|}{2t}} \\ \text{or, } & M\left(F(x,y), F(u,v), kt\right) \geqslant (M(x,u,t))^{\frac{1}{2}} * (M(y,v,t))^{\frac{1}{2}}. \end{aligned}$$

Combining the above two cases, we conclude that F is a fuzzy contractive coupling with contractivity factor $k = \frac{1}{3}$. Thus all conditions of Theorem 2 are satisfied. Due to this Theorem, there exists a strong coupled fixed point of F, which is (0,0); that is, $F(0,0) = 0$ and also $0 \in A \cap B$, which is therefore non-null.

Corollary 1. *Let $(X, M, *)$ be a complete fuzzy metric space satisfying $(FM6)$ of Definition 2 with $*$ being stronger than the product t-norm. Let $F\colon X \times X \to X$ be a mapping satisfying the following inequality for all $x, y, u, v \in X$:*

$$M\left(F(x,y), F(u,v), kt\right) \geqslant (M(x,u,t))^{\frac{1}{2}} * (M(y,v,t))^{\frac{1}{2}}. \tag{8}$$

Then F has a strong coupled fixed point.

Proof. Take $A = B = X$ in Theorem 2. The result follows from Theorem 2. □

Remark 1. *In Example 1, F is a fuzzy contractive coupling, but the inequality (1) is not satisfied for all $x, y, u, v \in X$. For example, take $x = \frac{1}{2}, y = -\frac{1}{2}, u = \frac{3}{2}, v = -\frac{3}{2}$. Then*

$$M(F(x,y), F(u,v), kt) = M(-\frac{1}{6}, 3, kt) = e^{-\frac{19}{6kt}}.$$

$$(M(x,u,t))^{\frac{1}{2}} * (M(y,v,t))^{\frac{1}{2}} = e^{-\frac{1}{t}}.$$

If inequality (1) holds, $e^{-\frac{19}{6kt}} \geqslant e^{-\frac{1}{t}}$, which implies $k \geqslant \frac{19}{6} > 1$. This indicates that Theorem 2 properly contains its Corollary 1.

Also, note that the above result is valid for several types of t-norms, like the minimum t-norm, the $\mathcal{H}$-type t-norm, etc, which are stronger than the product t-norm.

Note 1. There exists an alternative viewpoint of coupled (and, more generally, n-tuple) fixed points, which is applicable in certain cases. A coupled fixed-point problem can be viewed as an equivalent problem in product spaces, provided certain conditions are satisfied. A good discussion on this subject is provided by Soleimani Rad et al. [27] and, also, in other subsequent works like [7], [10], [22], [25]. Even in earlier works, like [3], [5], this approach already has been adopted. One limitation of this approach is that a metric-type function compatible with that existing in the original space must be introduced in the product space. This is straightforward for the metric space. If (X, d) is a metric space, then the function D given as

$$D\left((x,y),(u,v)\right) = d(x,u) + d(y,v). \tag{9}$$

defines a metric on $X \times X$. But the above is not a universal method and, therefore, is not a unique prescription for all kinds of spaces. In the present case, if we consider the fuzzy metric space $(X, M, *_p)$ with M defined as $M(x,y,t) = \dfrac{t}{t + d(x,y)}$, where d is a prior given metric on X, then there can be the following two ways of generating fuzzy metric on the product space $X \times X$. the first is $M_1((x,y),(u,v),t) = M(x,u,t) * M(y,v,t)$. The other is $M_2((x,y),(u,v),t) = \dfrac{t}{t + d(x,u) + d(y,v)}$, which is the standard fuzzy metric corresponding to the metric defined on $X \times X$ in (9). As fuzzy metrics, they are different, as can be seen by assuming $x = 1$, $y = 2$, $u = 3$, $v = 4$. The second limitation is the difficulty of expressing the coupled inequality in the product space. Both problems may be non-trivial. In this paper, we do not consider this alternative approach to coupled fixed points.

4. Generations of fractals.

Theorem 3. *Let $(X, M, *)$ be a fuzzy metric space satisfying $(FM6)$ of Definition 2, A, B be two non-empty subsets of X, $F: X \times X \to X$ be a continuous fuzzy contractive coupling with respect to A and B with contractivity factor k. Then $\widehat{F}: \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ defined as $\widehat{F}(A, B) = \{F(a, b): a \in A, b \in B\}$ is a fuzzy contractive coupling with respect to $\mathcal{K}(A)$ and $\mathcal{K}(B)$ in the fuzzy metric space $(\mathcal{K}(X), \mathcal{H}_M, *)$ with the same contractivity factor.*

Proof. From the construction of $\widehat{F}(A, B)$ it follows that for all $C \in \mathcal{K}(A)$ and $D \in \mathcal{K}(B)$, $\widehat{F}(C, D) \in \mathcal{K}(B)$ and $\widehat{F}(D, C) \in \mathcal{K}(A)$. Let $C_1, C_2 \in \mathcal{K}(A)$ and $D_1, D_2 \in \mathcal{K}(B)$. Then

$$\begin{aligned}
&\omega\big(\widehat{F}(C_1, D_1), \widehat{F}(C_2, D_2), kt\big) = \\
&= \omega\left(\{F(c_1, d_1) : c_1 \in C_1, d_1 \in D_1\}, \{F(c_2, d_2) : c_2 \in C_2, d_2 \in D_2\}, kt\right) = \\
&= \inf_{\substack{c_1 \in C_1 \\ d_1 \in D_1}} \sup_{\substack{c_2 \in C_2 \\ d_2 \in D_2}} M(F(c_1, d_1), F(c_2, d_2), kt) \geqslant \\
&\geqslant \inf_{\substack{c_1 \in C_1 \\ d_1 \in D_1}} \sup_{\substack{c_2 \in C_2 \\ d_2 \in D_2}} (M(c_1, c_2, t))^{\frac{1}{2}} * (M(d_1, d_2, t))^{\frac{1}{2}} = \qquad \text{(by (1))} \\
&= \Big(\inf_{c_1 \in C_1} \sup_{c_2 \in C_2} M(c_1, c_2, t)\Big)^{\frac{1}{2}} * \Big(\inf_{d_1 \in D_1} \sup_{d_2 \in D_2} M(d_1, d_2, t)\Big)^{\frac{1}{2}} = \\
&= (\omega(C_1, C_2, t))^{\frac{1}{2}} * (\omega(D_1, D_2, t))^{\frac{1}{2}} \geqslant (\mathcal{H}_M(C_1, C_2, t))^{\frac{1}{2}} * (\mathcal{H}_M(D_1, D_2, t))^{\frac{1}{2}}.
\end{aligned}$$

Similarly,

$$\bar{\omega}\Big(\widehat{F}(C_1, D_1), \widehat{F}(C_2, D_2), kt\Big) \geqslant (\mathcal{H}_M(C_1, C_2, t))^{\frac{1}{2}} * (\mathcal{H}_M(D_1, D_2, t))^{\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}
&\mathcal{H}_M\Big(\widehat{F}(C_1, D_1), \widehat{F}(C_2, D_2), kt\Big) = \\
&= \min\Big\{\omega\Big(\widehat{F}(C_1, D_1), \widehat{F}(C_2, D_2), kt\Big), \bar{\omega}\Big(\widehat{F}(C_1, D_1), \widehat{F}(C_2, D_2), kt\Big)\Big\} \geqslant
\end{aligned}$$

$$\geqslant (\mathcal{H}_M(C_1, C_2, t))^{\frac{1}{2}} * (\mathcal{H}_M(D_1, D_2, t))^{\frac{1}{2}}.$$

Hence, $\widehat{F} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ is a fuzzy contractive coupling with respect to $\mathcal{K}(A)$ and $\mathcal{K}(B)$ in the Hausdorff fuzzy metric space $(\mathcal{K}(X), \mathcal{H}_M, *)$ with contractivity factor k. □

Lemma 2. *Let $(X, M, *)$ be a fuzzy metric space satisfying $(FM6)$ of Definition 2. Let $F_1, F_2, \ldots, F_n$ be a finite number of continuous fuzzy contractive couplings on $X \times X$ with respect to A and B, each with contractivity factor $k_1, k_2, \ldots, k_n$, respectively. Then the Hutchinson operator $\widehat{G} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ (Definition 9) is a fuzzy contractive coupling in the fuzzy metric space $(\mathcal{K}(X), \mathcal{H}_M, *)$ with respect to $\mathcal{K}(A)$ and $\mathcal{K}(B)$ with contractivity factor $k = \max\{k_n \,; n \in \mathbb{N}_n\}$.*

Proof. From the definition of $\widehat{G}$, its follows that for all $C \in \mathcal{K}(A)$ and $D \in \mathcal{K}(B)$, $\widehat{G}(C,D) \in \mathcal{K}(B)$ and $\widehat{G}(D, C) \in \mathcal{K}(A)$. Let $C_1, C_2 \in \mathcal{K}(A)$ and $D_1, D_2 \in \mathcal{K}(B)$;

$$\mathcal{H}_M\big(\widehat{G}(C_1, D_1), \widehat{G}(C_2, D_2), kt\big) = \mathcal{H}_M\left(\bigcup_{i=1}^{n} \widehat{F}_i(C_1, D_1), \bigcup_{i=1}^{n} \widehat{F}_i(C_2, D_2), kt\right) \geqslant$$
$$\geqslant \min_{1 \leqslant i \leqslant n} \mathcal{H}_M\left(\widehat{F}_i(C_1, D_1), \widehat{F}_i(C_2, D_2), kt\right). \qquad \text{(by Lemma 1)}$$

Again, since each $\widehat{F}_i$ is a fuzzy contractive coupling, we have, for $i = 1, 2, \ldots, n$,

$$\mathcal{H}_M\big(\widehat{F}_i(C_1, D_1), \widehat{F}_i(C_2, D_2), kt\big) \geqslant \mathcal{H}_M\big(\widehat{F}_i(C_1, D_1), \widehat{F}_i(C_2, D_2), k_i t\big) \geqslant$$
$$\geqslant (\mathcal{H}_M(C_1, C_2, t))^{\frac{1}{2}} * (\mathcal{H}_M(D_1, D_2, t))^{\frac{1}{2}}. \quad \text{(by (1))}$$

Therefore,

$$\min_{1 \leqslant i \leqslant n} \mathcal{H}_M\big(\widehat{F}_i(C_1, D_1), \widehat{F}_i(C_2, D_2), kt\big) \geqslant (\mathcal{H}_M(C_1, C_2, t))^{\frac{1}{2}} * (\mathcal{H}_M(D_1, D_2, t))^{\frac{1}{2}}.$$

Hence,

$$\mathcal{H}_M\big(\widehat{G}(C_1, D_1), \widehat{G}(C_2, D_2), kt\big) \geqslant (\mathcal{H}_M(C_1, C_2, t))^{\frac{1}{2}} * (\mathcal{H}_M(D_1, D_2, t))^{\frac{1}{2}}.$$

This completes the proof of the theorem. □

Theorem 4. *Let $(X, M, *)$ be a complete fuzzy metric space satisfying $(FM6)$ of Definition 2. Consider a Fuzzy Iterated Coupling System*

$\langle (X, M, *); A, B, F_i, i \in \mathbb{N}_n \rangle$ consisting of a finite number of continuous fuzzy contractive couplings on $X \times X$ with respect to two closed subsets A, B of X, and let $\widehat{G}\colon \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ be the corresponding Hutchinson operator (Definition 9); then there exists a unique strong coupled fractal for $\widehat{G}$, that is, there exists a $P \in \mathcal{K}(A) \cap \mathcal{K}(B)$, such that $\widehat{G}(P, P) = P$.

Further, both the iterations $\{A_n\}$ and $\{B_n\}$ constructed as $B_{n+1} = \widehat{G}(A_n, B_n)$, $A_{n+1} = \widehat{G}(B_n, A_n)$, $n \geqslant 0$, with $A_0 \in \mathcal{K}(A)$ and $B_0 \in \mathcal{K}(B)$ being arbitrarily chosen, converge to the strong coupled fractal.

Proof. By Lemma 2, $\widehat{G}$ is a fuzzy contractive coupling with contractivity factor $k = \max\{k_n : n \in \mathbb{N}_n\}$. Again, since $(X, M, *)$ is complete, so $(\mathcal{K}(X), \mathcal{H}_M, *)$ is complete. On the other hand, since A, B are closed subsets of X, $\mathcal{K}(A)$ and $\mathcal{K}(B)$ also are closed subsets of the fuzzy metric space $(\mathcal{K}(X), \mathcal{H}_M, *)$. The theorem then follows by an application of Theorem 2. □

Example 2. Let $X = \mathbb{R}$ and $A = [-2, 2]$, $B = [-1, 2]$. Consider the fuzzy metric space $(X, M, *)$, where $*$ is the minimum t-norm and $M(x, y, t) = e^{-\frac{|x-y|}{t}}$.

Let $F_1, F_2\colon X \times X \to X$ be given by $F_1(x, y) = \frac{y-x}{9}$, $F_2(x, y) = 1 + \frac{y-x}{9}$. For $x \in A = [-2, 2]$ and $y \in B = [-1, 2]$, $F_1(x, y)$, $F_2(x, y) \in B$ and $F_1(y, x)$, $F_2(y, x) \in A$. Then F_1, F_2 are couplings with respect to A, B. Then the ICS $\langle (X, M, *); A, B, F_i, i \in \mathbb{N}_2 \rangle$ generates a strong coupled fractal.

Let $A_0 = B_0 = \left[-\frac{1}{2}, \frac{3}{2}\right]$. Then the first four steps of the iteration leading to the strong coupled fractal are given in the following:

$$A_1 = B_1 = \left[-\tfrac{1}{2}, \tfrac{3}{2}\right].$$

$$A_2 = F(A_1, A_1) = \left[-\tfrac{2}{9}, \tfrac{2}{9}\right] \cup \left[\tfrac{7}{9}, \tfrac{11}{9}\right].$$

$$A_3 = F(A_2, A_2) = \left[-\tfrac{13}{81}, -\tfrac{5}{81}\right] \cup \left[-\tfrac{4}{81}, \tfrac{4}{81}\right] \cup \left[\tfrac{5}{81}, \tfrac{13}{81}\right] \cup \left[\tfrac{68}{81}, \tfrac{76}{81}\right] \cup \left[\tfrac{77}{81}, \tfrac{85}{81}\right] \cup \left[\tfrac{86}{81}, \tfrac{94}{81}\right].$$

$$A_4 = F(A_3, A_3) = \left[-\tfrac{107}{729}, -\tfrac{55}{729}\right] \cup \left[-\tfrac{26}{729}, \tfrac{26}{729}\right] \cup \left[\tfrac{55}{729}, \tfrac{107}{729}\right] \cup \left[\tfrac{622}{729}, \tfrac{674}{729}\right] \cup \left[\tfrac{703}{729}, \tfrac{755}{729}\right] \cup \left[\tfrac{784}{729}, \tfrac{836}{729}\right].$$

The first four iterations are illustrated in Figure 1.

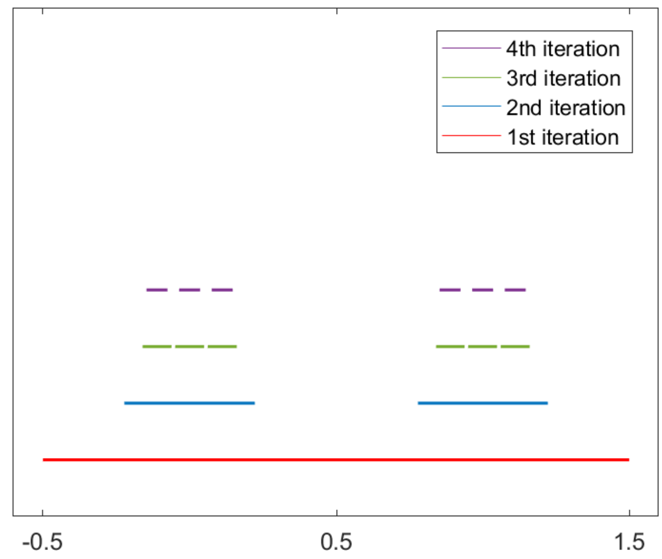

Figure 1: Iterations of the ICS in Example 2.

Example 3. Let $X = \mathbb{R}$ and $A = [-2, 2]$, $B = [-1, 2]$. Consider the fuzzy metric space $(X, M, *)$, where $*$ is the minimum t-norm and $M(x, y, t) = e^{-\frac{|x-y|}{t}}$.

Let $F_1, F_2\colon X \times X \to X$, given by $F_1(x, y) = \frac{y-x}{16}$, $F_2(x, y) = 1 + \frac{y-x}{16}$. Then the ICS $\langle (X, M, *); A, B, F_i, i \in \mathbb{N}_2 \rangle$ has an attractor.

Let $A_1 = B_1 = \left[-\frac{1}{2}, \frac{3}{2}\right]$. Then the first few iterations of the same are as follows:

$A_1 = B_1 = \left[-\frac{1}{2}, \frac{3}{2}\right]$.

$A_2 = F(A_1, A_1) = \left[-\frac{1}{8}, \frac{1}{8}\right] \cup \left[\frac{7}{8}, \frac{9}{8}\right]$.

$A_3 = F(A_2, A_2) = \left[-\frac{1}{16}, -\frac{3}{64}\right] \cup \left[-\frac{1}{64}, \frac{1}{64}\right] \cup \left[\frac{3}{64}, \frac{1}{16}\right] \cup \left[\frac{15}{16}, \frac{61}{64}\right] \cup \left[\frac{63}{64}, \frac{65}{64}\right] \cup \left[\frac{67}{64}, \frac{17}{16}\right]$.

$A_4 = F(A_3, A_3) = \left[-\frac{9}{128}, -\frac{35}{512}\right] \cup \left[-\frac{69}{1024}, -\frac{59}{1024}\right] \cup \left[-\frac{29}{512}, -\frac{7}{128}\right] \cup$
$\cup \left[-\frac{1}{128}, -\frac{3}{512}\right] \cup \left[-\frac{5}{1024}, \frac{5}{1024}\right] \cup \left[\frac{3}{512}, \frac{1}{128}\right] \cup \left[\frac{7}{128}, \frac{29}{512}\right] \cup \left[\frac{59}{1024}, \frac{69}{1024}\right] \cup$
$\cup \left[\frac{35}{512}, \frac{9}{128}\right] \cup \left[\frac{119}{128}, \frac{477}{512}\right] \cup \left[\frac{955}{1024}, \frac{965}{1024}\right] \cup \left[\frac{483}{512}, \frac{121}{128}\right] \cup \left[\frac{127}{128}, \frac{509}{512}\right] \cup \left[\frac{1019}{1024}, \frac{1029}{1024}\right] \cup$
$\cup \left[\frac{515}{512}, \frac{129}{128}\right] \cup \left[\frac{135}{128}, \frac{541}{512}\right] \cup \left[\frac{1083}{1024}, \frac{1093}{1024}\right] \cup \left[\frac{547}{512}, \frac{137}{128}\right]$.

Remark 2. *A comparison between Example 2 and Example 3 reveals an important feature in the iteration process. As an instance at the fifth stage of iteration, we have a set consisting of six intervals in the first case,* *whereas in the second case we have a set consisting of eighteen intervals. This shows that even if we are making the same initial choice, different Iterated Coupled Systems can produce very different types of iterations. This is a remarkable feature of the method of fractal generation obtained through Theorem 4.*

5. Conclusion. Fuzzy fixed point theory has its special characteristics. Due to the in-built flexibility of the fuzzy concepts, more types of contractions can be introduced in a fuzzy metric space compared to the corresponding study in an ordinary metric space. Fractal generation in Hutchinson-Barnsley's Theory utilizes contractions of various kinds. So, it is clear that this theory can be successfully utilized for the above-mentioned purpose in fuzzy metric spaces by the use of different types of contractions. In the present context, couplings are introduced, investigated for their fixed point properties, and applied for fractal generation in fuzzy metric spaces. It is possible that research in the similar ways with other types of contractive conditions might yield interesting results. This is supposed to be our future work.

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