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A NEW GENERATOR OF THIRD-DEGREE LINEAR FORMS

Abstract. This paper examines linear forms of the third-degree, i.e., when the associated Stieltjes function satisfies a cubic equation with polynomial coefficients. A generator for third-degree forms is constructed. In fact, we study the stability of the third-degree character under this transformation that generalizes the rational spectral transformation. Moreover, we prove the stability of third-degree linear forms under standard algebraic operations. Several illustrative examples are shown.

Key words: *orthogonal polynomials, Stieltjes function, third-degree forms, rational spectral transformations*

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1. Introduction. The origin of third-degree linear forms (TDFs) lies exclusively with P. Maroni and I. Ben Salha [8]. These forms naturally extend from the well-known second-degree forms [17], [19], characterized by the fact that their formal Stieltjes function $S(u)(z) := -\sum_{n\geqslant 0} \langle u, x^n\rangle / z^{n+1}$ satisfies a cubic equation with polynomial coefficients:

$$A(z)S^3(u)(z) + B(z)S^2(u)(z) + C(z)S(u)(z) + D(z) = 0.$$

A linear form u is said to be a strict third-degree form (STDF) if it is a TDF and it cannot be reduced to a second-degree form, i.e., its Stieltjes function does not satisfy a quadratic equation with polynomial coefficients. Some properties of TDRFs are discussed in [3], [8]. In particular, every third-degree form belongs to the Laguerre-Hahn class [8], but the converse is not true. The challenging study of semiclassical linear forms has led to utilizing alternative tools, such as exploring second- and third-degree forms to describe and characterize certain semiclassical forms [2], [3], [4], [5], [6], [13], [14], [16].

On the other hand, a rational spectral transformation [22] of the formal Stieltjes function $S(z)$ is a new formal Stieltjes function defined by

$$\widetilde{S}(z) = \left(\frac{aS+b}{cS+d}\right)(z), \quad ad-bc \neq 0, \tag{1}$$

where a, b, c, and d are co-prime polynomials. In particular, when $c = 0$, the spectral transformation (1) is said to be linear. Notably, the Christoffel and Geronimus transformations (see [9], [21]) are fundamental examples of linear spectral transforms and serve as generators within the family of such transformations (see [22]).

In [7], the authors establish that the class of third-degree forms is preserved by rational spectral transformations. This fact has important consequences, particularly concerning the stability of the set of the third-degree forms under various transformations, including the so called associated forms of k-th kind, Christoffel and Geronimus transformations, co-recursive forms, inverse forms, among others. As a consequence, they provide a constructive approach in order to generate third-degree forms, emphasizing the algebraic analysis of the vector space of linear forms and the corresponding Stieltjes functions.

The aim of this contribution is to identify a new system of generators for the set of third-degree forms. Let u and v be linear forms, such that their corresponding formal Stieltjes functions $S(u)$ and $S(v)$ are related by

$$S(v)(z) = \frac{a(z)S(u)(\pi(z)) + b(z)}{c(z)S(u)(\pi(z)) + d(z)}, \quad ad-bc \neq 0, \tag{2}$$

where a, b, c, and d are co-prime polynomials and π is a monic polynomial of degree greater than or equal to 1.

Assuming that either u or v is a third-degree linear form, then can the same be said about the remaining one?

When $\pi(z) = z$, we recover a rational spectral transformation. The stability problem of third-degree characters has been investigated in [7]. We should point out that the problems when π is a monic polynomial of degree greater than or equal to 2, have not been addressed in the literature until the recent contributions [4], [5], [15], and [16], which are focused on specific cases.

The paper is structured as follows. Section 2 provides an overview of the basic background on algebraic aspects of the theory of linear forms and Orthogonal Polynomials (OP), which will be relevant for the subsequent sections. In Section 3, we review the definitions and the main properties of TDRFs. In Section 4, we start by giving, in the case where $\pi(z) = z^k + r, k \geqslant 1, r \in \mathbb{C}$, the functional link between two regular forms u and v assuming that the Stieltjes function of one of them is obtained by applying a transformation of type (2) to the Stieltjes function of the other. As a consequence, we state our main result. We deal with a stability problem, i.e., we show that for any choice of the polynomial π of degree greater than or equal to 1, the fact that the form u is of the third degree implies that the form v is also of the third degree. Furthermore, we give a partial proof of the converse result, when $\pi(z) = z^k + r$ with $k \geqslant 1$ and $r \in \mathbb{C}$. This leads us to present in Section some interesting applications concerning the stability of the class of third-degree forms under various transformations.

2. Notation and preliminaries. Let $\mathcal{P}$ be the linear space of polynomials with coefficients in $\mathbb{C}$ (the field of complex numbers) and let $\mathcal{P}'$ be its topological dual space, whose elements are called linear forms (or linear functionals). By $\langle \cdot, \cdot \rangle$, we denote the duality brackets between $\mathcal{P}$ and $\mathcal{P}'$. In particular, we denote by $\langle u, x^n \rangle := (u)_n, n \geqslant 0$, the moments of u.

An important tool is the formal Stieltjes function associated with a given regular linear form $u \in \mathcal{P}'$ defined by

$$S(u)(z) := -\sum_{n\geqslant 0} \frac{(u)_n}{z^{n+1}}. \tag{3}$$

The function $S(u)(z)$ is the zeta transform of the sequence of moments $(u)_n$ of u. Formally, $S(u)(z)$ admits the representation

$$S(u)(z) = \Big\langle u_x, \frac{1}{x-z} \Big\rangle.$$

Let us introduce some useful operations in $\mathcal{P}'$. For any linear form u, any polynomials f, g and any $(\alpha, \beta, \gamma) \in (\mathbb{C} - \{0\}) \times \mathbb{C}^2$, let u', gu, $h_\alpha u$, $\tau_\beta u$, and $(x-\gamma)^{-1}u$ be the linear forms defined by

$$\langle u', f \rangle := -\langle u, f' \rangle, \qquad \langle gu, f \rangle := \langle u, gf \rangle,$$
$$\langle h_\alpha u, f \rangle := \langle u, h_\alpha f \rangle = \langle u, f(\alpha x) \rangle, \qquad \langle \tau_\beta u, f \rangle := \langle u, \tau_{-\beta} f \rangle = \langle u, f(x+\beta) \rangle,$$
$$\langle (x-\gamma)^{-1} u, f \rangle := \langle u, \theta_\gamma f \rangle = \langle u, \tfrac{f(x)-f(\gamma)}{x-\gamma} \rangle.$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial [20]

$$(uf)(x) := \Big\langle u_\zeta, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta} \Big\rangle.$$

This allows us to define the Cauchy product of two forms

$$\langle vu, f\rangle := \langle v, uf\rangle, \quad u, v \in \mathcal{P}', \quad f \in \mathcal{P}.$$

If $f \in \mathcal{P}, u, v \in \mathcal{P}'$ and $(\alpha, \beta) \in (\mathbb{C} - \{0\}) \times \mathbb{C}$, then we have (see [20])

$$S(uv)(z) = -zS(u)(z)S(v)(z), \tag{4}$$
$$S(fu)(z) = f(z)S(u)(z) + (u\theta_0 f)(z), \tag{5}$$
$$S\big((h_{\alpha^{-1}} \circ \tau_{-\beta})u\big)(z) = \alpha S(u)(\alpha z + \beta). \tag{6}$$

Let us recall that a form u is said to be regular (quasi-definite) if there exists a monic polynomial sequence $\{p_n\}_{n\geqslant 0}$ with $\deg p_n = n$, such that [11].

$$\langle u, p_n p_m\rangle = r_n \delta_{n,m}, \quad n, m \geqslant 0,$$

where $\{r_n\}_{n\geqslant 0}$ is a sequence of nonzero complex numbers and $\delta_{n,m}$ is the Kronecker symbol. A sequence $\{p_n\}_{n\geqslant 0}$ is called a monic orthogonal polynomial sequence (MOPS, in short) with respect to the form u. It is characterized by the following three-term recurrence relation:

$$p_0(x) = 1, \;\; p_1(x) = x - \beta_0,$$
$$p_{n+2}(x) = (x - \beta_{n+1})p_{n+1}(x) - \gamma_{n+1}p_n(x), \; n \geqslant 0.$$

Here $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_{n+1}\}_{n\geqslant 0}$ are sequences of complex numbers, such that $\gamma_{n+1} \neq 0$ for all n. Conversely, if a sequence of polynomials satisfies a recurrence relation as above with $\gamma_{n+1} \neq 0$ for all n, then there exists a linear form u, such that the sequence of polynomials is orthogonal with respect to u. This is the so called Favard's theorem (see [11], [20], [18]).

3. Third-degree linear forms.

Definition 1. [3] *A linear form u is called a third-degree form (TDF) if there exist polynomials A (monic), B, C, and D, such that*

$$A(z)S^3(u)(z) + B(z)S^2(u)(z) + C(z)S(u)(z) + D(z) = 0, \tag{7}$$

where D depends on A, B, C, and u.

Remark. *In most cases, the form u is not regular. However, when it is, it is said to be a third-degree regular form (TDRF).*

The form u is a (TDRF) if and only if the following conditions hold:

$$\begin{aligned}
&A(x)u^3 - xB(x)u^2 + x^2C(x)u = 0,\\
&\langle u^3,\theta_0^2 A\rangle - \langle u^2,\theta_0 B\rangle + \langle u, C\rangle = 0,\\
&\langle u^3,\theta_0 A\rangle - \langle u^2, B\rangle + \langle u,xC(x)\rangle = 0.
\end{aligned}$$

As a consequence,

$$D(z) = \big(u^3\theta_0^3 A\big)(z) - \big(u^2\theta_0^2 B\big)(z) + \big(u\theta_0 C\big)(z).$$

Remark.

1) *A regular linear form u is called a second-degree form if the corresponding Stieltjes function satisfies a quadratic equation with polynomial coefficients M, N, R, such that [17]*

$$M(z)S^2(u)(z) + N(z)S(u)(z) + R(z) = 0. \tag{8}$$

Here M, N, R satisfy $M \neq 0$, $N^2 - 4MR \neq 0$, $R \neq 0$, according to the regularity of u.

2) *The polynomial R is given in terms of M, N, and u as*

$$R(z) = -(u^2\theta_0^2 M)(z) + (u\theta_0 N)(z).$$

3) *The polynomials A, B, and C (resp. M and N), given in (7) (resp. (8)), are not unique, because A, B, and C (resp. M and N) can be multiplied by an arbitrary polynomial. If the polynomials A, B, C, and D in (7) (resp. M, N, and R in (8)) are co-prime, then the triple (A, B, C) (resp. the pair (M, N)) is called a primitive triple (resp. the primitive pair). Such a triple (resp. pair) is unique [7] (resp. [17]).*

4) *When the form u is a third-degree regular form (TDRF) and not a second-degree form, we call it a strict third-degree regular form (STDRF) [3].*

It is well known that the Chebyshev form of the first kind $\mathcal{T} := \mathcal{J}\big(-\frac{1}{2}, -\frac{1}{2}\big)$ is a second-degree form. Indeed, its Stieltjes function is

$$S(\mathcal{T})(z) = -(z^2-1)^{-\frac{1}{2}}$$

and satisfies the quadratic equation

$$(z^2-1)S^2(\mathcal{T})(z) - 1 = 0.$$

Among the most well-known forms that are strict third-degree (STDRF), we can find the Jacobi form $\mathcal{V} := \mathcal{J}(-\frac{2}{3}, -\frac{1}{3})$ [3]. Let us remind that $\mathcal{V}$ satisfies the following equations:

$$(z+1)^2(z-1)S^3(\mathcal{V})(z) + 1 = 0.$$

The third-degree character is preserved by an affine transformation. Indeed,

Proposition 1. [3] *Let u be a (TDRF), such that (7) holds. Then, the shifted form $\widehat{u} = (h_{a^{-1}} \circ \tau_{-b})u, a \in \mathbb{C} - \{0\}, b \in \mathbb{C}$, fulfils*

$$\widehat{A}(z)S^3(\widehat{u})(z) + \widehat{B}(z)S^2(\widehat{u})(z) + \widehat{C}(z)S(\widehat{u})(z) + \widehat{D}(z) = 0,$$

with

$$\begin{aligned}
\widehat{A}(z) &= a^{-\deg A}A(az+b),\\
\widehat{B}(z) &= a^{1-\deg A}B(az+b),\\
\widehat{C}(z) &= a^{2-\deg A}C(az+b),\\
\widehat{D}(z) &= a^{3-\deg A}D(az+b).
\end{aligned}$$

Note that every second-degree form is semiclassical (see [2]). Classical strict third-degree (respectively second-degree) forms have been studied in [3] (respectively [2]) and are related to special choices of the parameters α, β of the Jacobi linear form $\mathcal{J}(\alpha, \beta)$ (see [18]). Indeed,

Theorem 1. [2] *Among the classical forms, only the Jacobi forms $\mathcal{J}(t-1/2, l-1/2)$ are second-degree forms, provided $t + l \geqslant 0$, $t, l \in \mathbb{Z}$.*

Theorem 2. [3] *Among the classical forms, only the Jacobi forms $\mathcal{J}(t+q/3, l-q/3)$ are (STDRFs), provided $t + l \geqslant -1$, $t, l \in \mathbb{Z}$, $q \in \{1,2\}$.*

4. Stability of third-degree linear functionals. In this section, we first introduce some operators in the linear space of polynomials and state some preliminary lemmas. For a fixed $\pi \in \mathcal{P}$, let $\sigma_\pi : \mathcal{P} \to \mathcal{P}$ be the linear operator defined by $\sigma_\pi[f] := f \circ \pi$ for every $f \in \mathcal{P}$, and define $\sigma_\pi^* : \mathcal{P}' \to \mathcal{P}'$ by duality. Indeed,

$$\sigma_\pi[f](x) := f(\pi(x)), \quad \langle \sigma_\pi^*(\omega), f \rangle := \langle \omega, \sigma_\pi[f] \rangle, \quad f \in \mathcal{P}, \omega \in \mathcal{P}'.$$

Remark. *Throughout this paper, the following notation will be used: $\pi_k(x) = x^k + r$, with $r \in \mathbb{C}$ and $k \geqslant 1$ is a fixed integer number.*

For a fixed integer number k, $k \geqslant 1$, we introduce the operator $\boldsymbol{\varrho}_k : \mathcal{P}' \to \mathcal{P}'$, such that for any linear form $\omega \in \mathcal{P}'$ we get

$$\boldsymbol{\varrho}_1(\omega) = \omega,$$

and, for all $k \geqslant 2$:

$$\big(\boldsymbol{\varrho}_k(\omega)\big)_{kn+j} = 0, \quad j = 0, \dots, k-2, \qquad n \geqslant 0, \tag{9}$$

$$\big(\boldsymbol{\varrho}_k(\omega)\big)_{kn+k-1} = (\omega)_n, \qquad n \geqslant 0. \tag{10}$$

Remark.

- *If $k = 2$, then $\boldsymbol{\varrho}_2$ is the anti-symmetrization operator $\boldsymbol{\alpha}$ (see [4]).*
- *If $k = 3$, then $\boldsymbol{\varrho}_3$ is the operator $\boldsymbol{\varrho}$ (see [13], [20]).*

Lemma 1. [10, Lemma 3.4] *Let π and ϕ be monic polynomials with $\deg \pi = l$ and let $\mathcal{B}_\pi := \{p_0, p_1, \dots, p_{l-1}\}$ be a simple set of polynomials. Then, to the triple $(\phi, \pi, \mathcal{B}_\pi)$ we may associate l polynomials $\phi_0, \phi_1, \dots, \phi_{l-1}$, with $\deg \phi_j \leqslant \lfloor (\deg \phi)/l \rfloor$ for all $j = 0, 1, \dots, l-1$, such that*

$$\phi = \sum_{j=0}^{l-1} p_j \sigma_\pi [\phi_j].$$

Lemma 2. [16, Lemma 3.2] *Let $\Sigma_1, \Sigma_2, \dots, \Sigma_N$ be formal power series, $f_1, f_2, \dots, f_N \in \mathcal{P}$ and*

$$\Omega(z) = \sum_{j=1}^{N} f_j(z^{-1}) \Sigma_j(z).$$

If $\Omega\big(1/\pi_k(z)\big) = 0$, where $r \in \mathbb{C}$ and k is a fixed integer, such that $k \geqslant 2$, then $\Omega(z) = 0$.

Lemma 3. *Let ω be a linear form in $\mathcal{P}'$ and let k be an integer, such that $k \geqslant 1$. The formal Stieltjes functions $S(\omega)$ and $S\big(\boldsymbol{\varrho}_k(\omega)\big)$ associated with the forms ω and $\boldsymbol{\varrho}_k(\omega)$, respectively, are related by*

$$S(\omega)\big(z^k\big) = S\big(\boldsymbol{\varrho}_k(\omega)\big)(z).$$

Proof. The proof is straightforward from the definition (9)–(10) of the operator $\boldsymbol{\varrho}_k$ and is omitted. □

Corollary. *Let ω be a linear form in $\mathcal{P}'$ and let k be an integer, such that $k \geqslant 1$. Then*

$$S(\omega)(\pi_k(z)) = S\Big(\boldsymbol{\varrho}_k\big(\tau_{-r}(\omega)\big)\Big)(z). \tag{11}$$

Proof. It is an immediate consequence of Lemma 3 with (6) taken into account. □

As an initial result, we discuss the relation between two linear forms whose formal Stieltjes functions are connected by (2).

Proposition 2. *Let u and v be two forms in $\mathcal{P}'$. The formal Stieltjes functions $S(u)$ and $S(v)$ associated with the forms u an v, respectively, are related by*

$$S(v)(z) = \frac{a(z)S(u)(\pi_k(z)) + b(z)}{c(z)S(u)(\pi_k(z)) + d(z)}, \quad ad - bc \neq 0, \tag{12}$$

where a, b, c, and d are co-prime polynomials, if and only if

$$c(x)\boldsymbol{\varrho}_k\big(\tau_{-r}(u)\big)v - xd(x)v + xa(x)\boldsymbol{\varrho}_k\big(\tau_{-r}(u)\big) = 0, \tag{13}$$

$$\Big(\boldsymbol{\varrho}_k\big(\tau_{-r}(u)\big)v\theta_0 c\Big)(x) - (vd)(x) + \Big(\boldsymbol{\varrho}_k\big(\tau_{-r}(u)\big)a\Big)(x) - xb(x) = 0. \tag{14}$$

Proof. The proof is straightforward once one substitutes $\boldsymbol{\varrho}_k\big(\tau_{-r}(u)\big)$ by u, in addition to taking into account [7, Proposition 5.1]. □

Remark 1. *In particular, when $c = 0$, i.e., the formal Stieltjes functions $S(u)$ and $S(v)$ associated with the forms u and v are related by*

$$S(v)(z) = \frac{a(z)S(u)(\pi_k(z)) + b(z)}{d(z)}, \quad ad \neq 0,$$

where a, b, and d are co-prime polynomials, if and only if u and v are related by

$$d(x)v = a(x)\boldsymbol{\varrho}_k\big(\tau_{-r}(u)\big),$$

$$(v\theta_0 d)(x) - \Big(\boldsymbol{\varrho}_k\big(\tau_{-r}(u)\big)\theta_0 a\Big)(x) + b(x) = 0.$$

Proposition 3. *Let u and v be two forms in $\mathcal{P}'$ whose associated formal Stieltjes functions $S(u)$ and $S(v)$ are related by*

$$S(v)(z) = \frac{a(z)S(u)(\pi(z)) + b(z)}{c(z)S(u)(\pi(z)) + d(z)}, \tag{15}$$

where $ad - bc \neq 0$, $a, b, c, d \in \mathcal{P}$, and π is a monic polynomial of degree greater than or equal to 1.

(i) If u is a (TDF), then v is also a (TDF). Moreover, if u satisfies

$$A_u(z)S^3(u)(z) + B_u(z)S^2(u)(z) + C_u(z)S(u)(z) + D_u(z) = 0, \quad (16)$$

then for the linear form v we get

$$A_v(z)S^3(v)(z) + B_v(z)S^2(v)(z) + C_v(z)S(v)(z) + D_v(z) = 0, \quad (17)$$

where

$$\begin{aligned} A_v(z) &= \sigma_\pi[A_u](z)d^3(z) - \sigma_\pi[B_u](z)c(z)d^2(z)+ \\ &+ \sigma_\pi[C_u](z)d(z)c^2(z) - \sigma_\pi[D_u](z)c^3(z), \\ B_v(z) &= -3\sigma_\pi[A_u](z)b(z)d^2(z) + \sigma_\pi[B_u](z)(a(z)d^2(z)+ \\ &+ 2b(z)d(z)c(z)) - \sigma_\pi[C_u](z)(b(z)c^2(z)+ \\ &+ 2d(z)c(z)a(z)) + 3\sigma_\pi[D_u](z)a(z)c^2(z), \\ C_v(z) &= 3\sigma_\pi[A_u](z)d(z)b^2(z) - \sigma_\pi[B_u](z)(c(z)b^2(z)+ \\ &+ 2b(z)a(z)d(z)) + \sigma_\pi[C_u](z)(d(z)a^2(z)+ \\ &+ 2b(z)a(z)c(z)) - 3\sigma_\pi[D_u](z)c(z)a^2(z), \\ D_v(z) &= -\sigma_\pi[A_u](z)b^3(z) + \sigma_\pi[B_u](z)a(z)b^2(z)- \\ &- \sigma_\pi[C_u](z)b(z)a^2(z) + \sigma_\pi[D_u](z)a^3(z). \end{aligned} \quad (18)$$

(ii) When $\pi = \pi_k$, if v is a (TDF), then u is also a (TDF).

Proof.
(i) The proof is similar to that of [7, Proposition 5.2].
(ii) Assume that v is a (TDF) satisfying (7)

$$A_v(z)S^3(v)(z) + B_v(z)S^2(v)(z) + C_v(z)S(v)(z) + D_v(z) = 0. \quad (19)$$

Replacing (15) in (19), and multiplying both sides of the resulting equation by $\big(c(z)S(u)(\pi_k(z)) + d(z)\big)^3$ after some computations, we get

$$A_v(z)\mathcal{H}_1(z) + B_v(z)\mathcal{H}_2(z) + C_v(z)\mathcal{H}_3(z) + D_v(z)\mathcal{H}_4(z) = 0,$$

with

$$\begin{aligned} \mathcal{H}_1(z) =& a^3(z)S^3(u)(\pi_k(z)) + 3b(z)a^2(z)S^2(u)(\pi_k(z))+ \\ &+ 3a(z)b^2(z)S(u)(v) + b^3(z), \\ \mathcal{H}_2(z) =& c(z)a^2(z)S^3(u)(\pi_k(z)) + (d(z)a^2(z) + 2b(z)a(z)c(z))S^2(u)(\pi_k(z))+ \end{aligned}$$

$$
\begin{aligned}
&+ (c(z)b^2(z) + 2b(z)a(z)d(z))S(u)(\pi_k(z)) + d(z)b^2(z),\\
\mathcal{H}_3(z) =& a(z)c^2(z)S^3(u)(\pi_k(z)) + (b(z)c^2(z) + 2d(z)c(z)a(z))S^2(u)(\pi_k(z))+\\
&+ (a(z)d^2(z) + 2b(z)d(z)c(z))S(u)(\pi_k(z)) + b(z)d^2(z),\\
\mathcal{H}_4(z) =& c^3(z)S^3(u)(\pi_k(z)) + 3d(z)c^2(z)S^2(u)(\pi_k(z))+\\
&+ 3c(z)d^2(z)S(u)(\pi_k(z)) + d^3(z).
\end{aligned}
$$

Therefore, $S(u)(\pi_k(z))$ satisfies

$$
A_u(z)S^3(u)(\pi_k(z)) + B_u(z)S^2(u)(\pi_k(z)) + C_u(z)S(u)(\pi_k(z)) + D_u(z) = 0, \tag{20}
$$

where the polynomials A_u, B_u, C_u, and D_u are

$$
\begin{aligned}
A_u(z) =& A_v(z)a^3(z) + B_v(z)c(z)a^2(z) + C_v(z)a(z)c(z)^2 + D_v(z)c^3(z),\\
B_u(z) =& 3A_v(z)b(z)a^2(z) + B_v(z)(a^2(z)d(z) + 2b(z)a(z)c(z))+\\
&+ C_v(z)(b(z)c^2(z) + 2d(z)c(z)a(z)) + 3D_v(z)d(z)c^2(z),\\
C_u(z) =& 3A_v(z)a(z)b^2(z) + B_v(z)(c(z)b^2(z) + 2b(z)a(z)d(z))+\\
&+ C_v(z)(a(z)d(z)^2 + 2b(z)d(z)c(z)) + 3D_v(z)c(z)d(z)^2,\\
D_u(z) =& A_v(z)b(z)^3 + B_v(z)d(z)b^2(z) + C_v(z)b(z)d^2(z) + D_v(z)d^3(z).
\end{aligned} \tag{21}
$$

Now, Lemma 1 ensures the existence of polynomials $A_{u,\nu}(z)$, $B_{u,\nu}(z)$, $C_{u,\nu}(z)$, and $D_{u,\nu}(z)$, $\nu = 0,1,\dots,k-1$, with each $A_{u,\nu}(z)$, $B_{u,\nu}(z)$, $C_{u,\nu}(z)$, and $D_{u,\nu}(z)$ not necessarily of degree ν, such that

$$
\begin{aligned}
A_u(z) &= \sum_{\nu=0}^{k-1} z^\nu \sigma_{\pi_k}[A_{u,\nu}](z), \quad & B_u(z) &= \sum_{\nu=0}^{k-1} z^\nu \sigma_{\pi_k}[B_{u,\nu}](z),\\
C_u(z) &= \sum_{\nu=0}^{k-1} z^\nu \sigma_{\pi_k}[C_{u,\nu}](z), \quad & D_u(z) &= \sum_{\nu=0}^{k-1} z^\nu \sigma_{\pi_k}[D_{u,\nu}](z).
\end{aligned}
$$

Thus, (20) can be rewritten as

$$
\sum_{\nu=0}^{k-1} z^\nu \mathcal{S}_\nu\left(\pi_k(z)\right) = 0, \tag{22}
$$

where

$$
\mathcal{S}_\nu(z) := A_{u,\nu}(z)S^3(u)(z) + B_{u,\nu}(z)S^2(u)(z) + C_{u,\nu}(z)S(u)(z) + D_{u,\nu}(z), \tag{23}
$$

$\nu = 0, 1, \dots, k-1$.

Next, let us consider the k-th roots of unity ϖ_ν, i.e., $\varpi_\nu = e^{\frac{2i\nu\pi}{k}}$ for $\nu = 0, 1, \dots, k-1$. After the substitutions $z \leftarrow \varpi_1^m z$, $m = 0, 1, \dots, k-1$, in (22), and taking into account $\varpi_1^k = 1$ and $\varpi_1^\nu = \varpi_\nu$, $\nu = 0, 1, \dots, k-1$, we get

$$\sum_{\nu=0}^{k-1} \varpi_j^m z^\nu \mathcal{S}_\nu\left(\pi_k(z)\right) = 0, \quad m = 0, 1, \dots, k-1.$$

By summing the k final equations, we obtain

$$\sum_{\nu=0}^{k-1} \Big(\sum_{m=0}^{k-1} \varpi_j^m \Big) z^\nu \mathcal{S}_\nu\left(\pi_k(z)\right) = 0. \tag{24}$$

It is easy to see that $\sum_{m=0}^{k-1} \varpi_j^m = 0$ for all $\nu = 1, 2, \dots, k-1$. Hence, (24) can be simplified as $k\mathcal{S}_0(\pi_k(z)) = 0$. In the same way, for each $\nu = 0, 1, \dots, k-1$, one has

$$\mathcal{S}_\nu(\pi_k(z)) = 0. \tag{25}$$

Replacing z by z^{-1} in (23), we have

$$\begin{aligned}\mathcal{S}_\nu(z^{-1}) = A_{u,\nu}(z^{-1})S^3(u)(z^{-1}) + B_{u,\nu}(z^{-1})S^2(u)(z^{-1}) + \\ + C_{u,\nu}(z^{-1})S(u)(z^{-1}) + D_{u,\nu}(z^{-1}), \ \ \nu = 0, 1, \dots, k-1.\end{aligned}$$

Using Lemma 2 with $\Omega(z) = \mathcal{S}_\nu(z^{-1})$ and taking into account (25), we can deduce $\mathcal{S}_\nu(z) = 0$, $\nu = 0, 1, \dots, k-1$. Since $(A_v, D_v) \neq (0,0)$ and $(a, c) \neq (0,0)$, from the first relation of (21) we have $A_u \neq 0$, so, there exists $0 \leqslant \nu_0 \leqslant k-1$, such that $A_{u,\nu_0} \neq 0$. Therefore, the desired result follows by noting that $\mathcal{S}_{\nu_0}(z) = 0$. □

As a consequence of Proposition 3 and under the assumption of Proposition 3, the following statements hold:

Corollary.

(i) *If u is a second-degree form (resp. (STDF)), then v is also a second-degree form (resp. (STDF)).*

(ii) *When $\pi = \pi_k$, if v is a second-degree form (resp. (STDF)), then u is also a second-degree form (resp. (STDF)).*

Remark. *When $k = 1$ and $r = 0$, we recover again the same result for the rational spectral transformation case [7].*

Corollary. *Let us assume that the polynomials* $\widehat{A} = \sigma_{\pi_k}[A_u]$, $\widehat{B} = \sigma_{\pi_k}[B_u]$, $\widehat{C} = \sigma_{\pi_k}[C_u](z)$ *and* $\widehat{D} = \sigma_{\pi_k}[D_u](z)$ *satisfy*

$$\begin{cases} d^3(z)\widehat{A} - c(z)d^2(z)\widehat{B} + d(z)c^2(z)\widehat{C} - c^3(z)\widehat{D} = \widetilde{A}, \\ -3b(z)d^2(z)\widehat{A} + (a(z)d^2(z) + 2b(z)d(z)c(z))\widehat{B} \\ -(b(z)c^2(z) + 2d(z)c(z)a(z))\widehat{C} + 3a(z)c^2(z)\widehat{D} = \widetilde{B}, \\ 3d(z)b^2(z)\widehat{A} - (c(z)b^2(z) + 2b(z)a(z)d(z))\widehat{B} + \\ (d(z)a^2(z) + 2b(z)a(z)c(z))\widehat{C} - 3c(z)a^2(z)\widehat{D} = \widetilde{C}, \\ -b^3(z)\widehat{A} + a(z)b^2(z)\widehat{B} - b(z)a^2(z)\widehat{C} + a^3(z)\widehat{D} = \widetilde{D}. \end{cases} \tag{26}$$

If we denote by $\mathcal{D}$ *the determinant of the above linear system, then we have*

$$\mathcal{D} = \Big(a(z)d(z) - b(z)c(z)\Big)^6. \tag{27}$$

Proof. The determinant $\mathcal{D}$ of the system (26) reads as

$$\mathcal{D} = \begin{vmatrix} d^3(z) & -c(z)d^2(z) & d(z)c^2(z) & -c^3(z) \\ -3b(z)d^2(z) & a(z)d^2(z) + 2b(z)d(z)c(z) & -(b(z)c^2(z) + 2d(z)c(z)a(z)) & 3a(z)c^2(z) \\ 3d(z)b^2(z) & -(c(z)b^2(z) + 2b(z)a(z)d(z)) & d(z)a^2(z) + 2b(z)a(z)c(z) & -3c(z)a^2(z) \\ -b^3(z) & a(z)b^2(z) & -b(z)a^2(z) & a^3(z) \end{vmatrix}.$$

Expanding $\mathcal{D}$ by the first row and their corresponding cofactors, we have

$$\mathcal{D} = d^3(z)\mathcal{D}_1 + 3b(z)d^2(z)\mathcal{D}_2 + 3d(z)b^2(z)\mathcal{D}_3 + b^3(z)\mathcal{D}_4, \tag{28}$$

where

$$\mathcal{D}_1 = \begin{vmatrix} a(z)d^2(z) + 2b(z)d(z)c(z) & -(b(z)c^2(z) + 2d(z)c(z)a(z)) & 3a(z)c^2(z) \\ -(c(z)b^2(z) + 2b(z)a(z)d(z)) & d(z)a^2(z) + 2b(z)a(z)c(z) & -3c(z)a^2(z) \\ a(z)b^2(z) & -b(z)a^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_2 = \begin{vmatrix} -c(z)d^2(z) & d(z)c^2(z) & -c^3(z) \\ -(c(z)b^2(z) + 2b(z)a(z)d(z)) & d(z)a^2(z) + 2b(z)a(z)c(z) & -3c(z)a^2(z) \\ a(z)b^2(z) & -b(z)a^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_3 = \begin{vmatrix} -c(z)d^2(z) & d(z)c^2(z) & -c^3(z) \\ a(z)d^2(z) + 2b(z)d(z)c(z) & -(b(z)c^2(z) + 2d(z)c(z)a(z)) & 3a(z)c^2(z) \\ a(z)b^2(z) & -b(z)a^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_4 = \begin{vmatrix} -c(z)d^2(z) & d(z)c^2(z) & -c^3(z) \\ a(z)d^2(z) + 2b(z)d(z)c(z) & -(b(z)c^2(z) + 2d(z)c(z)a(z)) & 3a(z)c^2(z) \\ -(c(z)b^2(z) + 2b(z)a(z)d(z)) & d(z)a^2(z) + 2b(z)a(z)c(z) & -3c(z)a^2(z) \end{vmatrix}.$$

First, we need to evaluate $\mathcal{D}_k, 1 \leqslant k \leqslant 4$. For $\mathcal{D}_1$, we have

$$\mathcal{D}_1 = (a(z)d^2(z) + 2b(z)d(z)c(z))\mathcal{D}_{1,1} + \\ + (c(z)b^2(z) + 2b(z)a(z)d(z))\mathcal{D}_{1,2} + a(z)b^2(z)\mathcal{D}_{1,3}, \quad (29)$$

where

$$\mathcal{D}_{1,1} = \begin{vmatrix} d(z)a^2(z) + 2b(z)a(z)c(z) & -3c(z)a^2(z) \\ -b(z)a^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_{1,2} = \begin{vmatrix} -(b(z)c^2(z) + 2d(z)c(z)a(z)) & 3a(z)c^2(z) \\ -b(z)a^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_{1,3} = \begin{vmatrix} -(b(z)c^2(z) + 2d(z)c(z)a(z)) & 3a(z)c^2(z) \\ d(z)a^2(z) + 2b(z)a(z)c(z) & -3c(z)a^2(z) \end{vmatrix}.$$

After some elementary computations, (29) becomes

$$\mathcal{D}_1 = (a(z)d^2(z) + 2b(z)d(z)c(z))(d(z)a^5(z) - b(z)c(z)a^4(z)) + \\ + 2(c(z)b^2(z) + 2b(z)a(z)d(z))(b(z)c^2(z)a^3(z) - d(z)c(z)a^4(z)) + \\ + 3a(z)b^2(z)(d(z)c^2(z)a^3(z) - b(z)c^3(z)a^2(z)) = \\ = -a^3(z)\big(a(z)d(z) - b(z)c(z)\big)^3. \quad (30)$$

In the same way, we have

$$\mathcal{D}_2 = -c(z)d^2(z)\mathcal{D}_{2,1} - d(z)c^2(z)\mathcal{D}_{2,2} - c^3(z)\mathcal{D}_{2,3},$$

where

$$\mathcal{D}_{2,1} = \begin{vmatrix} d(z)a^2(z) + 2b(z)a(z)c(z) & -3c(z)a^2(z) \\ -b(z)a^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_{2,2} = \begin{vmatrix} -(c(z)b^2(z) + 2b(z)a(z)d(z)) & -3c(z)a^2(z) \\ a(z)b^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_{2,3} = \begin{vmatrix} -(c(z)b^2(z) + 2b(z)a(z)d(z)) & d(z)a^2(z) + 2b(z)a(z)c(z) \\ a(z)b^2(z) & -b(z)a^2(z) \end{vmatrix}.$$

Clearly, we obtain

$$\mathcal{D}_2 = -c(z)d^2(z)(d(z)a^5(z)-c(z)b(z)a^4(z))-2d(z)c^2(z)(c(z)b^2(z)a^3(z)- \\ - d(z)b(z)a(z)^4) + c^3(z)(c(z)a^2(z)b^3(z) - d(z)b^2(z)a^3(z)) = \\ = \big(a(z)d(z) - b(z)c(z)\big)^3. \quad (31)$$

On the other hand,

$$\mathcal{D}_3 = -c(z)d^2(z)\mathcal{D} - d(z)c^2(z)\mathcal{D}_{3,2} - c^3(z)\mathcal{D}_{3,3},$$

where

$$\mathcal{D}_{3,1} = \begin{vmatrix} -(b(z)c^2(z) + 2d(z)c(z)a(z)) & 3a(z)c^2(z) \\ -b(z)a^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_{3,2} = \begin{vmatrix} a(z)d^2(z) + 2b(z)d(z)c(z) & 3a(z)c^2(z) \\ a(z)b^2(z) & a^3(z) \end{vmatrix},$$

$$\mathcal{D}_{3,3} = \begin{vmatrix} a(z)d^2(z) + 2b(z)d(z)c(z) & -(b(z)c^2(z) + 2d(z)c(z)a(z)) \\ a(z)b^2(z) & -b(z)a^2(z) \end{vmatrix}.$$

Thus we deduce

$$\mathcal{D}_3 = 2c(z)d^2(z)(d(z)c(z)a^4(z) - b(z)c^2(z)a^3(z))- \\ - d(z)c^2(z)(d^2(z)a^4(z) + 2d(z)c(z)b(z)a^3(z) - 3c^2(z)b^2(z)a^2(z))+ \\ + c^3(z)(a(z)c^2(z)b^3(z) - b(z)d^2(z)a^3(z)) = \\ = -\big(a(z)c^2(z)\big)\big(a(z)d(z) - b(z)c(z)\big)^3. \quad (32)$$

Finally,

$$\mathcal{D}_4 = -c(z)d^2(z)\mathcal{D}_{4,1} - d(z)c^2(z)\mathcal{D}_{4,2} - c^3(z)\mathcal{D}_{4,3},$$

where

$$\mathcal{D}_{4,1} = \begin{vmatrix} -(b(z)c^2(z) + 2d(z)c(z)a(z)) & 3a(z)c^2(z) \\ d(z)a^2(z) + 2b(z)c(z)a(z) & -3c(z)a^2(z) \end{vmatrix},$$

$$\mathcal{D}_{4,2} = \begin{vmatrix} a(z)d^2(z) + 2b(z)d(z)c(z) & 3a(z)c^2(z) \\ -(c(z)b^2(z) + 2b(z)a(z)d(z)) & -3c(z)a^2(z) \end{vmatrix}$$

and

$$\mathcal{D}_{4,3} = \begin{vmatrix} a(z)d^2(z) + 2b(z)d(z)c(z) & -(b(z)c^2(z) + 2d(z)c(z)a(z)) \\ -(c(z)b^2(z) + 2b(z)a(z)d(z)) & d(z)a^2(z) + 2b(z)c(z)a(z) \end{vmatrix}.$$

A simple computation yields

$$\begin{aligned} \mathcal{D}_4 = 3c(z)d^2(z)(b(z)a^2(z)c^3(z) - d(z)c^2(z)a^3(z)) + \\ + 3d(z)c^2(z)(c(z)d^2(z)a^3(z) - a(z)b^2(z)c^3(z)) + \\ + c^3(z)(b^3(z)c^3(z) - d^3(z)a^3(z)) = \\ = c^3(z)\big(a(z)d(z) - b(z)c(z)\big)^3. \end{aligned} \tag{33}$$

Therefore, from (28) and taking into account (30)–(33), we get (27). □

5. Applications. In this section, we deal with some interesting applications of the above transformations.

Lemma 4. *[10] Let* $\mathbf{u}$ *be a linear form in* $\mathcal{P}'$*. The formal Stieltjes series* $S(\mathbf{u})$ *and* $S\big(\sigma^*_{\pi_k}(\mathbf{u})\big)$ *associated with the regular moment linear forms* $\mathbf{u}$ *and* $\sigma^*_{\pi_k}(\mathbf{u})$ *(resp.) are related by*

$$S\big(\sigma^*_{\pi_k}(\mathbf{u})\big)\big(\pi_k(z)\big) = \frac{S(\mathbf{u})(z)}{\eta_{k-1}(z)}, \tag{34}$$

where η_{k-1} *is the polynomial defined as*

$$\eta_{k-1}(z) = \Delta_0(2, k-1; z). \tag{35}$$

Proposition 4. *Let* $\mathbf{u}$ *and* $\mathbf{w}$ *be two forms in* $\mathcal{P}'$*, such that*

$$M(x)\mathbf{w} = N(x)\sigma^*_{\pi_k}(\mathbf{u}), \tag{36}$$

where $M(x)$ *and* $N(x)$ *are polynomials.*

(a) If $\mathbf{w}$ *is a (TDF), then* $\mathbf{u}$ *is also a (TDF). Moreover, if* $\mathbf{w}$ *satisfies*

$$A_{\mathbf{w}}(z)S^3(\mathbf{w})(z) + B_{\mathbf{w}}(z)S^2(\mathbf{w})(z) + C_{\mathbf{w}}(z)S(\mathbf{w})(z) + D_{\mathbf{w}}(z) = 0,$$

then for the linear form $\mathbf{u}$ *we get*

$$A_{\mathbf{u}}(z)S^3(\mathbf{u})(z) + B_{\mathbf{u}}(z)S^2(\mathbf{u})(z) + C_{\mathbf{u}}(z)S(\mathbf{u})(z) + D_{\mathbf{u}}(z) = 0, \tag{37}$$

where

$$\begin{aligned} A_{\mathbf{u}}(z) =& \sigma_{\pi_k}[A_{\mathbf{w}}](z)\mathbf{D}^3(z),\\ B_{\mathbf{u}}(z) =& -3\sigma_{\pi_k}[A_{\mathbf{w}}](z)\mathbf{B}(z)\mathbf{D}^2(z) + \sigma_{\pi_k}[B_{\mathbf{w}}](z)\mathbf{A}(z)\mathbf{D}^2(z),\\ C_{\mathbf{u}}(z) =& 3\sigma_{\pi_k}[A_{\mathbf{w}}](z)\mathbf{D}(z)\mathbf{B}^2(z) - (z)\mathbf{D}(z)-\\ & -2\sigma_{\pi_k}[B_{\mathbf{w}}](z)\mathbf{B}(z)\mathbf{A} + \sigma_{\pi_k}[C_{\mathbf{w}}](z)\mathbf{D}(z)\mathbf{A}^2(z),\\ D_{\mathbf{u}}(z) =& \sigma_{\pi_k}[A_{\mathbf{w}}](z)\mathbf{B}^3(z) + \sigma_{\pi_k}[B_{\mathbf{w}}](z)\mathbf{A}(z)\mathbf{B}^2(z)-\\ & -\sigma_{\pi_k}[C_{\mathbf{w}}](z)\mathbf{B}(z)\mathbf{A}^2(z) + \sigma_{\pi_k}[D_{\mathbf{w}}](z)\mathbf{A}^3(z), \end{aligned} \tag{38}$$

with

$$\begin{aligned} \mathbf{A}(z) &= \eta_{k-1}(z)\sigma_{\pi_k}[M](z),\\ \mathbf{B}(z) &= -\eta_{k-1}(z)\sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z) + \eta_{k-1}(z)\sigma_{\pi_k}[\mathbf{w}\theta_0 M](z),\\ \mathbf{D}(z) &= \sigma_{\pi_k}[N](z), \end{aligned} \tag{39}$$

where the polynomial η_{k-1} *is defined in* (35)*.*

(b) *When* $\pi \in \mathcal{P}$*, if* $\mathbf{u}$ *is a (TDF), then* $\mathbf{w}$ *is also a (TDF).*

Proof. From (5) and replacing z by $\pi_k(z)$, (36) can be rewritten as

$$\begin{aligned} &\sigma_{\pi_k}[M](z)S(\mathbf{w})\big(\pi_k(z)\big) + \sigma_{\pi_k}[\mathbf{w}\theta_0 M](z) =\\ &\qquad = \sigma_{\pi_k}[N](z)S\big(\sigma^*_{\pi_k}(\mathbf{u})\big)\big(\pi_k(z)\big) + \sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z). \end{aligned} \tag{40}$$

Combining (34) and (40), one easily shows that

$$\begin{aligned} S(\mathbf{u})(z) = &\frac{\eta_{k-1}(z)\sigma_{\pi_k}[M](z)S(\mathbf{w})\big(\pi_k(z)\big)}{\sigma_{\pi_k}[N](z)}-\\ &-\frac{\eta_{k-1}(z)\sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z) + \eta_{k-1}(z)\sigma_{\pi_k}[\mathbf{w}\theta_0 M](z)}{\sigma_{\pi_k}[N](z)}. \end{aligned}$$

In other words,

$$S(\mathbf{u})(z) = \frac{\mathbf{A}(z)S(\mathbf{w})\big(\pi_k(z)\big) + \mathbf{B}(z)}{\mathbf{D}(z)},$$

where the polynomials $\mathbf{A}, \mathbf{B}$, and $\mathbf{D}$ are given in (39).

Applying Proposition 3 for $a(z) = \mathbf{A}(z), b(z) = \mathbf{B}(z), c(z) = 0$, and $d(z) = \mathbf{D}(z)$, we obtain (37) and (38). □

Remark. *From Remark 1, we observe that the formal Stieltjes functions $S(\mathbf{u})$ and $S(\mathbf{w})$ associated with the forms $\mathbf{u}$ an $\mathbf{w}$, respectively, are related by*

$$\begin{aligned}&\mathbf{D}(x)\mathbf{u} = \mathbf{A}(x)\boldsymbol{\varrho}_k\big(\tau_{-r}(\mathbf{w})\big),\\&(\mathbf{u}\theta_0\mathbf{D})(x) - \Big(\boldsymbol{\varrho}_k\big(\tau_{-r}(\mathbf{w})\big)\theta_0\mathbf{A}\Big)(x) + \mathbf{B}(x) = 0,\end{aligned}$$

where the polynomials $\mathbf{A}, \mathbf{B}$, and $\mathbf{D}$ are given in (39).

Corollary. *Let $\mathbf{u}$ be a third-degree form. Then the Christoffel transformation $\mathbf{v} = N(x)\sigma^*_{\pi_k}(\mathbf{u})$ of the form $\sigma^*_{\pi_k}(\mathbf{u})$, where $N \in \mathcal{P}$, is also a third-degree form and satisfies*

$$A_{\mathbf{v}}(z)S^3(\mathbf{v})(z) + B_{\mathbf{v}}(z)S^2(\mathbf{v})(z) + C_{\mathbf{v}}(z)S(\mathbf{v})(z) + D_{\mathbf{v}}(z) = 0,$$

with

$$\begin{aligned}\lambda A_{\mathbf{v}}(z) =&\sigma_{\pi_k}[A_{\mathbf{u}}](z)\Big(\sigma_{\pi_k}[N](z)\Big)^3,\\ \lambda B_{\mathbf{v}}(z) =&3\sigma_{\pi_k}[A_{\mathbf{u}}](z)\eta_{k-1}(z)\sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z)\Big(\sigma_{\pi_k}[N](z)\Big)^2 +\\ &+ \sigma_{\pi_k}[B_{\mathbf{u}}](z)\eta_{k-1}(z)\big(\sigma_{\pi_k}[N](z)\big)^2,\\ \lambda C_{\mathbf{v}}(z) =&3\sigma_{\pi_k}[A_{\mathbf{u}}](z)\sigma_{\pi_k}[N](z)\Big(\eta_{k-1}(z)\sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z)\Big)^2 +\\ &+ 2\sigma_{\pi_k}[B_{\mathbf{u}}](z)\eta^2_{k-1}(z)\sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z)\sigma_{\pi_k}[N](z)+\\ &+ \sigma_{\pi_k}[C_{\mathbf{u}}](z)\sigma_{\pi_k}[N](z)\eta^2_{k-1}(z),\\ \lambda D_{\mathbf{v}}(z) =&\sigma_{\pi_k}[A_{\mathbf{u}}](z)\Big(\eta_{k-1}(z)\sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z)\Big)^3 +\\ &+ \sigma_{\pi_k}[B_{\mathbf{u}}](z)\eta^3_{k-1}(z)\Big(\sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z)\Big)^2 +\\ &+ \sigma_{\pi_k}[C_{\mathbf{u}}](z)\eta^3_{k-1}(z)\sigma_{\pi_k}[\sigma^*_{\pi_k}(\mathbf{u})\theta_0 N](z) + \sigma_{\pi_k}[D_{\mathbf{u}}](z)\eta^3_{k-1}(z),\end{aligned}$$

where λ is a normalization constant chosen in order to be $A_{\mathbf{v}}$ monic.

Corollary. *Let $\mathbf{u}$ be a third-degree form. Then the Geronimus transformation $\mathbf{v}$ of the form $\sigma^*_{\pi_k}(\mathbf{u})$, i.e., $M(x)\mathbf{v} = \sigma^*_{\pi_k}(\mathbf{u})$, where $M \in \mathcal{P}$, is also a third-degree form and satisfies*

$$A_{\mathbf{v}}(z)S^3(\mathbf{v})(z) + B_{\mathbf{v}}(z)S^2(\mathbf{v})(z) + C_{\mathbf{v}}(z)S(\mathbf{v})(z) + D_{\mathbf{v}}(z) = 0,$$

with

$$\begin{aligned}\lambda A_{\mathbf{v}}(z) =&\sigma_{\pi_k}[A_{\mathbf{u}}](z),\\ \lambda B_{\mathbf{v}}(z) =& -3\sigma_{\pi_k}[A_{\mathbf{u}}](z)\eta_{k-1}(z)\sigma_{\pi_k}[\mathbf{v}\theta_0 M](z)+\\ &+\sigma_{\pi_k}[B_{\mathbf{u}}](z)\eta_{k-1}(z)\sigma_{\pi_k}[M](z),\\ \lambda C_{\mathbf{v}}(z) =&3\sigma_{\pi_k}[A_{\mathbf{u}}](z)\big(\eta_{k-1}(z)\sigma_{\pi_k}[\mathbf{v}\theta_0 M](z)\big)^2-\\ &-2\sigma_{\pi_k}[B_{\mathbf{u}}](z)\eta_{k-1}(z)\sigma_{\pi_k}[\mathbf{v}\theta_0 M](z)\eta_{k-1}(z)\sigma_{\pi_k}[M](z)+\\ &+\sigma_{\pi_k}[C_{\mathbf{u}}](z)\big(\eta_{k-1}(z)\sigma_{\pi_k}[M](z)\big)^2,\\ \lambda D_{\mathbf{v}}(z) =& -\sigma_{\pi_k}[A_{\mathbf{u}}](z)\big(\eta_{k-1}(z)\sigma_{\pi_k}[\mathbf{v}\theta_0 M](z)\big)^3+\\ &+\sigma_{\pi_k}[B_{\mathbf{u}}](z)\eta_{k-1}(z)\sigma_{\pi_k}[M](z)\big(\eta_{k-1}(z)\sigma_{\pi_k}[\mathbf{v}\theta_0 M](z)\big)^2-\\ &-\sigma_{\pi_k}[C_{\mathbf{u}}](z)\eta_{k-1}(z)\sigma_{\pi_k}[\mathbf{v}\theta_0 M](z)\big(\eta_{k-1}(z)\sigma_{\pi_k}[M](z)\big)^2+\\ &+\sigma_{\pi_k}[D_{\mathbf{u}}](z)\big(\eta_{k-1}(z)\sigma_{\pi_k}[M](z)\big)^3,\end{aligned}$$

where λ is a normalization constant chosen in order to be $A_{\mathbf{v}}$ monic.

Finally, we prove that if $\mathbf{u}$ is a third-degree form, then $\mathbf{v} = \sigma^*_{\pi_k}(\mathbf{u})+\alpha\delta_\beta$ and $\mathbf{w} = \sigma^*_{\pi_k}(\mathbf{u}) + \alpha\delta'_\beta$, where $\alpha, \beta \in \mathbb{C}$, are also third-degree forms.

Proposition 5. *Let $\mathbf{u}$ be a third-degree form. Then $\mathbf{v} = \sigma^*_{\pi_k}(\mathbf{u}) + \alpha\delta_\beta$, where $\alpha, \beta \in \mathbb{C}$, is also a third-degree form.*

Proof. From the definition of the form $\mathbf{v}$, we have

$$(\mathbf{v})_n = \big(\sigma^*_{\pi_k}(\mathbf{u})\big)_n + \alpha\beta^n, \;\; n \geqslant 0.$$

Thus,

$$S(\mathbf{v})(z) = S\big(\sigma^*_{\pi_k}(\mathbf{u})\big)(z) - \frac{\alpha}{z-\beta}.$$

By considering the change of variable $z \leftarrow \pi_k(z)$ in the previous equation, and combining the resulting equation with (34), we obtain

$$S(\mathbf{u})(z) = \frac{\big(\pi_k(z)-\beta\big)\eta_{k-1}(z)S(\mathbf{v})\big(\pi_k(z)\big) + \alpha\eta_{k-1}(z)}{\pi_k(z)-\beta}.$$

If $\alpha \neq 0$, then, applying Proposition 3 for $a(z) = \big(\pi_k(z)-\beta\big)\eta_{k-1}(z)$, $b(z) = \alpha\eta_{k-1}(z)$, $c(z) = 0$, and $d(z) = \pi_k(z)-\beta$, the requested result follows. □

Proposition 6. *Let $\mathbf{u}$ be a third-degree form. Then $\mathbf{w} = \sigma^*_{\pi_k}(\mathbf{u}) + \alpha\delta'_\beta$, where $\alpha, \beta \in \mathbb{C}$, is also a third-degree form.*

Proof. From the definition of the form $\mathbf{w}$, we have

$$(\mathbf{w})_n = \big(\sigma^*_{\pi_k}(\mathbf{u})\big)_n - \alpha n \beta^{n-1}, \ \ n \geqslant 0.$$

Then

$$S(\mathbf{w})(z) = S\big(\sigma^*_{\pi_k}(\mathbf{u})\big)(z) + \frac{\alpha}{(z-\beta)^2}.$$

By considering the change of variable $z \leftarrow \pi_k(z)$ in the previous equation, and combining the resulting equation with (34), we can show that the formal Stieltjes functions $S(\mathbf{u})$ and $S(\mathbf{w})$ are related by

$$S(\mathbf{u})(z) = \frac{\big(\pi_k(z)-\beta\big)^2 \eta_{k-1}(z) S\big(\mathbf{w}\big)\big(\pi_k(z)\big) - \alpha \eta_{k-1}(z)}{\big(\pi_k(z)-\beta\big)^2}.$$

If $\alpha \neq 0$, the desired result follows by applying Proposition 3 for $a(z) = (\pi_k(z)-\beta)^2 \eta_{k-1}(z)$, $b(z) = -\alpha\eta_{k-1}(z)$, $c(z) = 0$, $d(z) = (\pi_k(z)-\beta)^2$. □

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