

UDC 517.53

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**SMIRNOV AND BERNSTEIN-TYPE INEQUALITIES,  
TAKING INTO ACCOUNT HIGHER-ORDER  
COEFFICIENTS AND FREE TERMS OF POLYNOMIALS**

**Abstract.** The starting point in the theory of differential inequalities for polynomials is the book "Investigation of aqueous solutions by specific gravity" by D. I. Mendeleev. In this work, he dealt not only with chemical, but also mathematical problems. The question raised in this book led to appearance of a large number of works on various types of differential inequalities for polynomials. In our paper, we obtain Smirnov and Bernstein-type inequalities that use higher-order coefficients and free terms of polynomials.

**Key words:** *polynomial, differential inequality, higher-order coefficient, free term*

**2020 Mathematical Subject Classification:** *30C10, 30A10*

**1. Introduction.** By  $\mathcal{P}_n$  denote the set of all polynomials of degree at most  $n \in \mathbb{N}$ . Let  $\mathbb{D}$  stand for the open unit disk  $= \{z \in \mathbb{C} : |z| < 1\}$ .

Consider the following problem (the Mendeleev problem):

*Let  $B \subset \mathbb{C}$  be a compact set,  $f(z) \in \mathcal{P}_n$  be a polynomial, such that  $|f(z)| \leq M$  for  $z \in B$ . Give an estimate for  $|f'(z)|$  on  $B$ .*

This problem in its original form was posed in 1887 by the famous chemist D. I. Mendeleev in [13, § 86]. Mendeleev considered only real polynomials of degree two and the compact set  $B = [a, b]$ . In [20, p. 340], the problem was presented in the general form.

In 1889, A. A. Markov solved the original Mendeleev problem.

**Theorem A.** [10], [11, p. 51–75] *Suppose that  $f \in \mathcal{P}_n$  and  $|f(x)| \leq M$  for  $x \in [a, b]$ . Then*

$$|f'(x)| \leq Mn^2.$$

*Here equality is attained only for the functions*

$$f(x) = \pm MT_n \left( \frac{2x - a - b}{b - a} \right)$$

where  $T_n(x) = \cos(n \arccos x)$  are the Chebyshev polynomials.

In [12], V. A. Markov obtained an estimation for the  $k$ -th derivative of a polynomial  $f$ ,  $1 \leq k \leq n$ .

Let us note that Mendelev and the Markov brothers dealt only with real polynomials.

From now on, in our article we shall consider complex polynomials and the compact set  $B = \partial\mathbb{D}$ . In this case, the solution of the Mendelev problem is the following theorem:

**Theorem B.** *Let  $f \in \mathcal{P}_n$  and suppose that  $|f(z)| \leq M$  on  $\partial\mathbb{D}$ . Then for  $z \in \partial\mathbb{D}$ :*

$$|f'(z)| \leq Mn.$$

*Equality holds only if  $f(z) = e^{i\gamma} M z^n$ ,  $\gamma \in \mathbb{R}$ .*

In the literature, Theorem B is known as the Bernstein inequality. However, it apparently was not Bernstein who first obtained this statement in the presented form. The history is the following. In 1912, Bernstein considered trigonometric polynomials

$$f(t) = \sum_{k=0}^n (a_k \cos kt + b_k \sin kt)$$

of degree  $n$ ,  $|f(t)| \leq M$  for  $t \in [0, 2\pi]$ , see [1]. He proved that  $|f'(t)| \leq 2M$ . In paper [6, p. 50], for such polynomials the estimate

$$|f'(t)| \leq M, \quad t \in [0, 2\pi], \quad (1)$$

was presented by Fejér. Inequality (1) can be also found in [7], Fekete attributes the proof of (1) to Fejér. However, Bernstein [2, p. 39] attributes the proof to E. Landau, who sent the proof of (1) to Bernstein in his letter. In [18, p. 357], M. Riesz presented the statement of Theorem B as a corollary of inequality (1).

Alternative proof of Theorem B was given by V. I. Smirnov [19], [20, ch. V, section 1, 2°, p. 346].

S. N. Bernstein generalized Theorem B in the following way:

**Theorem C.** [3] (see also [4, p. 497], [17, p. 510]) *Consider polynomials  $f$  and  $F$ , such that:*

- 1)  $\deg f \leq \deg F = n$ ,
- 2)  $|f(z)| \leq |F(z)|$  on  $\partial\mathbb{D}$ ,

3)  $F$  has all its zeros in  $\overline{\mathbb{D}}$ .

Then we have

$$|f'(z)| \leq |F'(z)| \quad \text{for } z \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

For  $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , equality holds only if  $f = e^{i\gamma}F$ ,  $\gamma \in \mathbb{R}$ .

Note that Theorem B follows from Theorem C with  $F(z) = Mz^n$ .

V. I. Smirnov proved a stronger version of Theorem C. He considered the operator

$$S_\alpha[f](z) = zf'(z) - n\alpha f(z), \quad f \in \mathcal{P}_n,$$

where  $\alpha$  is a complex parameter.

**Theorem D.** [20, ch. V, § 1, p. 356] *Let  $R \geq 1$ ,  $f$  and  $F$  be polynomials from Theorem C. Then*

$$|S_\alpha[f](z)| \leq |S_\alpha[F](z)|, \quad |z| = R, \quad (2)$$

for all  $\alpha \in \Omega_R$ , where  $\Omega_1 = \{\alpha \in \mathbb{C} : \operatorname{Re} \alpha \leq 1/2\}$  and for  $R > 1$  the set  $\Omega_R$  is the complement to the open disk with diameter  $\left[\frac{R}{R+1}, \frac{R}{R-1}\right]$ .

If  $\alpha \in \operatorname{int} \Omega_R$  and  $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , the equality in (2) holds only if  $f = e^{i\gamma}F$ ,  $\gamma \in \mathbb{R}$ .

Taking  $\alpha = 0$  in Theorem D, we have Theorem C.

A. V. Olesov obtained a supplementation of the Smirnov inequality (2). He enlarged the set of variation of the parameter  $\alpha$ . In Theorem D, this set ( $\Omega_R$ ) depends only on  $R$ . The Olesov set depends on  $R$  too and also on the higher-order coefficients and the free terms of the polynomials  $f(z)$ ,  $F(z)$ .

**Theorem E.** [15] *Let  $f(z) = a_n z^n + \dots + a_0$  and  $F(z) = b_n z^n + \dots + b_0$  be polynomials from Theorem C. Assume that  $R \geq 1$ ,  $\lambda = \frac{1-d}{1+d}$ , where*

$$d = \max_{|z|=1} \left| \frac{a_0 + zb_0}{a_n + zb_n} \right|. \quad \text{Then } d \leq 1 \text{ and}$$

$$|S_\alpha[f](z)| \leq |S_\alpha[F](z)|$$

for  $|z| = R$  and  $\alpha \in G_{R,\lambda}$ , where

$$G_{1,\lambda} = \left\{ \alpha \in \mathbb{C} : \operatorname{Re} \alpha \leq \frac{1}{2} + \frac{\lambda}{2n} \right\}$$

and for  $R > 1$  the set  $G_{R,\lambda}$  is the complement to the open disk with diameter  $\left[ \frac{R + \lambda/n}{R + 1}, \frac{R - \lambda/n}{R - 1} \right]$ .

The idea to use higher-order coefficients and the free terms in different inequalities for polynomials belongs to V. N. Dubinin [5].

More results concerning the development of this subject can be found, for example, in [17], [8], [14], [16].

In the previous theorems, it was required that all the zeroes of the polynomial  $F$  belong to  $\overline{\mathbb{D}}$ . In [9], this condition was waived. For a polynomial  $F$  having one zero outside  $\overline{\mathbb{D}}$ , the analogue of the Smirnov inequality was proved.

**Theorem F.** [9] *Let  $f$  and  $F$  be polynomials, such that*

- 1)  $\deg f \leq \deg F = n$ ,
- 2)  $|f(z)| \leq |F(z)|$ ,  $z \in \mathbb{C} \setminus \mathbb{D}$ ,
- 3)  $z_0$  is a unique zero of  $F$  lying in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ ,  $k$  is order of  $z_0$ ,  $1 \leq k \leq n - 1$ .

Let  $R \geq 1$ . Then

$$|S_\alpha[f](z)| \leq |S_\alpha[F](z)|, \quad |z| = R,$$

for  $\alpha \in D_R$ , where  $D_R$  is one of the following sets:

a) the half-plane

$$\left\{ \alpha \in \mathbb{C} : \operatorname{Re} \alpha \leq \left(1 - \frac{k}{n}\right) \frac{1}{2} - \frac{k}{n} \frac{1}{|z_0| - 1} \right\}$$

for  $R = 1$ ;

b) the complement to the strip

$$\left\{ \alpha \in \mathbb{C} : \left(1 - \frac{k}{n}\right) \frac{R}{R + 1} + \frac{k}{2n} < \operatorname{Re} \alpha < \left(1 - \frac{k}{n}\right) \frac{R}{R - 1} + \frac{k}{2n} \right\}$$

for  $R = |z_0| > 1$ ;

c) the complement to the open annulus bounded by the circles with diameters

$$\left[ \left(1 - \frac{k}{n}\right) \frac{R}{R + 1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} - \frac{|z_0|}{|R^2 - |z_0|^2|} \right), \right. \\ \left. \left(1 - \frac{k}{n}\right) \frac{R}{R - 1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} + \frac{|z_0|}{|R^2 - |z_0|^2|} \right) \right]$$

and

$$\left[ \left(1 - \frac{k}{n}\right) \frac{R}{R-1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} - \frac{|z_0|}{|R^2 - |z_0|^2|} \right), \right. \\ \left. \left(1 - \frac{k}{n}\right) \frac{R}{R+1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} + \frac{|z_0|}{|R^2 - |z_0|^2|} \right) \right]$$

for

c-1)  $M|z_0| \leq R < |z_0|$ , where  $M = \sqrt{\frac{(n-k)|z_0| + k}{(n-k)|z_0| + k|z_0|^2}}$ ,

c-2)  $|z_0| < R \leq L|z_0|$ , where  $|z_0| < n/k - 1$ ,  $L = \sqrt{\frac{(n-k)|z_0| - k}{(n-k)|z_0| - k|z_0|^2}}$ ,

c-3)  $|z_0| \geq n/k - 1$ ;

d) the complement to the open disk with diameter

$$\left[ \left(1 - \frac{k}{n}\right) \frac{R}{R+1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} - \frac{|z_0|}{|R^2 - |z_0|^2|} \right), \right. \\ \left. \left(1 - \frac{k}{n}\right) \frac{R}{R-1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} + \frac{|z_0|}{|R^2 - |z_0|^2|} \right) \right]$$

for

d-1)  $1 < R < M|z_0|$ ,

d-2)  $R > L|z_0|$ ,  $|z_0| < n/k - 1$ .

In this paper, combining ideas and methods from [9] and [15], we obtain new refinements of Bernstein's and Smirnov's inequalities.

**2. Supplementation of the Smirnov inequality.** In this section, we obtain a refinement of the Smirnov inequality, basing on Theorem E. As in [9], we consider the case when the polynomial  $F$  has one zero outside  $\overline{\mathbb{D}}$ .

**Theorem 1.** Suppose  $f(z) = a_n z^n + \dots + a_0$  and  $F(z) = b_n z^n + \dots + b_0$  be polynomials, such that

1)  $\deg f \leq \deg F = n$ ,

2)  $|f(z)| \leq |F(z)|$ ,  $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ ,

3)  $z_0$  is a unique zero of  $F$  lying in  $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ ,  $k$  is order of  $z_0$ ,  $1 \leq k \leq n-1$ .

Let  $R \geq 1$ ,  $d = \max_{|z|=1} \left| \frac{a_0 + z b_0}{a_n + z b_n} \right|$ ,  $\mu = \frac{|z_0|^k - d}{|z_0|^k + d}$ . Then

$$|S_\alpha[f](z)| \leq |S_\alpha[F](z)|, \quad |z| = R, \quad (3)$$

for  $\alpha \in D_{R,\mu}$ , where  $D_{R,\mu}$  is one of the following sets:

a) the half-plane

$$\left\{ \alpha \in \mathbb{C} : \operatorname{Re} \alpha \leq \left(1 - \frac{k}{n}\right) \left(\frac{1}{2} + \frac{\mu}{2n}\right) - \frac{k}{n} \frac{1}{|z_0| - 1} \right\}$$

for  $R = 1$ ;

b) the complement to the strip

$$\left\{ \alpha \in \mathbb{C} : \left(1 - \frac{k}{n}\right) \frac{R + \mu/n}{R + 1} + \frac{k}{2n} < \operatorname{Re} \alpha < \left(1 - \frac{k}{n}\right) \frac{R - \mu/n}{R - 1} + \frac{k}{2n} \right\}$$

for  $R = |z_0| > 1$ ;

c) the complement to the open annulus bounded by the circles with diameters

$$\left[ \left(1 - \frac{k}{n}\right) \frac{R + \mu/n}{R + 1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} - \frac{|z_0|}{|R^2 - |z_0|^2|} \right), \right. \\ \left. \left(1 - \frac{k}{n}\right) \frac{R - \mu/n}{R - 1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} + \frac{|z_0|}{|R^2 - |z_0|^2|} \right) \right]$$

and

$$\left[ \left(1 - \frac{k}{n}\right) \frac{R - \mu/n}{R - 1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} - \frac{|z_0|}{|R^2 - |z_0|^2|} \right), \right. \\ \left. \left(1 - \frac{k}{n}\right) \frac{R + \mu/n}{R + 1} + \frac{k}{n} R \left( \frac{R}{R^2 - |z_0|^2} + \frac{|z_0|}{|R^2 - |z_0|^2|} \right) \right]$$

for

$$c-1) m|z_0| \leq R < |z_0|, \text{ where } m = \sqrt{\frac{(n-k)(n-\mu)|z_0| + kn}{(n-k)(n-\mu)|z_0| + kn|z_0|^2}};$$

c-2)  $|z_0| < R \leq l|z_0|$ , where

$$|z_0| < \left(\frac{n}{k} - 1\right) \left(1 - \frac{\mu}{n}\right), \quad l = \sqrt{\frac{(n-k)(n-\mu)|z_0| - kn}{(n-k)(n-\mu)|z_0| - kn|z_0|^2}};$$

c-3)  $|z_0| \geq (n/k - 1)(1 - \mu/n)$ ;

d) the complement to the open disk with diameter

Therefore,  $\alpha = 0$  belongs to  $D_{R,\mu}$  if and only if the domain  $G_{R,\lambda}$  from Theorem E contains

$$\beta(0) = -\frac{k}{n-k} \frac{z}{z-z_0}$$

for all  $z$ ,  $|z| = R$ .

The image  $\Gamma_R$  of the circle  $\{z \in \mathbb{C}: |z| = R\}$  under the function  $-\frac{k}{n-k} \frac{z}{z-z_0}$  is the circle with diameter  $\left[-\frac{k}{n-k} \frac{R}{|z_0|+R}, \frac{k}{n-k} \frac{R}{|z_0|-R}\right]$ . By Theorem E, for  $R > 1$  the set  $G_{R,\lambda}$  is the complement to the open disk, bounded by the circle  $T$  with diameter  $\left[\frac{R+\lambda/n}{R+1}, \frac{R-\lambda/n}{R-1}\right]$ . Hence,  $\beta(0) \in G_{R,\lambda}$  for all  $z$ ,  $|z| = R$ , only in one of the following cases:

1)  $\Gamma_R$  lies to the left of  $T$ , i.e.,

$$\frac{k}{n-k} \frac{R}{|z_0|-R} \leq \frac{R+\lambda/n}{R+1}; \quad (16)$$

2)  $\Gamma_R$  lies to the right of  $T$ , i.e.,

$$-\frac{k}{n-k} \frac{R}{|z_0|+R} \geq \frac{R-\lambda/n}{R-1}; \quad (17)$$

3) the interior of  $\Gamma_R$  contains  $T$  or  $T$  is tangent to  $\Gamma_R$ , i.e.,

$$-\frac{k}{n-k} \frac{R}{|z_0|+R} \leq \frac{R+\lambda/n}{R+1} \quad \text{and} \quad \frac{k}{n-k} \frac{R}{|z_0|-R} \geq \frac{R-\lambda/n}{R-1}. \quad (18)$$

Consider the first case. Let us solve (16) with respect to  $R$ . Inequality (16) is equivalent to the following inequality:

$$nR^2 + \left(k - (n-k)\left(|z_0| - \frac{\lambda}{n}\right)\right)R - (n-k)\frac{\lambda}{n}|z_0| \leq 0. \quad (19)$$

Solving (19), we find that  $1 < R \leq R_1$ .

In the second case, inequality (17) is not true, because the left-hand side of (17) is negative, while the right-hand side is positive.

We finish the proof by considering the third case. The first inequality of (18) is obviously true. Let us solve the second one. This inequality can be rewritten as

$$nR^2 - \left(k + (n-k)\left(|z_0| + \frac{\lambda}{n}\right)\right)R + \frac{\lambda}{n}(n-k)|z_0| \geq 0. \quad (20)$$

Discriminant of the equation

$$nR^2 - \left( k + (n - k) \left( |z_0| + \frac{\lambda}{n} \right) \right) R + \frac{\lambda}{n} (n - k) |z_0| = 0$$

equals

$$D = \left( k + (n - k) \left( |z_0| + \frac{\lambda}{n} \right) \right)^2 - 4(n - k)\lambda|z_0|.$$

Prove that  $D > 0$ . Write  $D$  in the form

$$D = k^2 + (n - k)^2 \left( |z_0| + \frac{\lambda}{n} \right)^2 + 2(n - k) \left( |z_0| (k - 2\lambda) + \frac{k\lambda}{n} \right).$$

Clearly, if  $k - 2\lambda \geq 0$ , then  $D > 0$ .

Consider the opposite case, when  $k < 2\lambda$ . Since  $k \geq 1$  and, by Theorem E,  $\lambda \in [0, 1]$ , we have that the inequality  $k < 2\lambda$  takes place only if  $k = 1$ ,  $\lambda > 1/2$ . For  $k = 1$ ,  $D > 0$  iff

$$\left( 1 + (n - 1) \left( |z_0| + \frac{\lambda}{n} \right) \right)^2 > 4(n - 1)\lambda|z_0|. \quad (21)$$

First, note that for  $1/2 < \lambda \leq 1$

$$\begin{aligned} \left( 1 + (n - 1) \left( |z_0| + \frac{\lambda}{n} \right) \right)^2 &> (n - 1) \left( |z_0| + \frac{\lambda}{n} \right) \left( (n - 1) \left( |z_0| + \frac{\lambda}{n} \right) + 2 \right) > \\ &> (n - 1) |z_0| \left( (n - 1) |z_0| + \frac{5}{2} - \frac{1}{2n} \right). \end{aligned} \quad (22)$$

Further, note that

$$4(n - 1)\lambda|z_0| \leq 4(n - 1)|z_0|. \quad (23)$$

Taking into account (22) and (23), it is sufficient to show that

$$(n - 1) |z_0| + \frac{5}{2} - \frac{1}{2n} \geq 4. \quad (24)$$

to prove (21). Inequality (24) is equivalent to

$$|z_0| \geq \frac{3n + 1}{2n(n - 1)}. \quad (25)$$



If

$$\frac{3n+1}{2n(n-1)} < 1, \quad (26)$$

then (25) is true, because  $|z_0| > 1$ . Inequality (26) is fulfilled for  $n \geq 3$ . So, for  $n \geq 3$ ,  $k = 1$ ,  $\lambda > 1/2$  we have also proved that  $D > 0$ .

It remains to consider the case  $n = 2$ ,  $k = 1$ ,  $1/2 < \lambda \leq 1$ . In this situation,  $D > 0$  if and only if

$$1 + \left(|z_0| + \frac{\lambda}{2}\right)^2 + 2\left(|z_0| + \frac{\lambda}{2}\right) > 4\lambda|z_0|. \quad (27)$$

Rewrite (27):

$$|z_0|^2 + (2 - 3\lambda)|z_0| + \left(\frac{\lambda^2}{4} + \lambda + 1\right) > 0.$$

Since for  $1/2 < \lambda \leq 1$  discriminant of the corresponding equation is negative, it follows that (27) is true.

Consequently, in any case  $D > 0$ .

Solving (20) with respect to  $R$ , we obtain that  $R \geq R_2$ , where

$$R_2 = \frac{k + (n - k)(|z_0| + \lambda/n) + \sqrt{(k + (n - k)(|z_0| + \lambda/n))^2 - 4(n - k)\lambda|z_0|}}{2n}.$$

□

Lemma 1 will be used to prove the following theorem.

**Theorem 2.** Let  $f(z) = a_n z^n + \dots + a_0$  and  $F(z) = b_n z^n + \dots + b_0$  be polynomials, such that

- 1)  $\deg f \leq \deg F = n$ ;
- 2) all the zeros  $z_1, \dots, z_m$  of  $F$  belong to  $\overline{\mathbb{D}}$ ,  $|z_1| > |z_2| \geq |z_i|$ ,  $i = 3, \dots, m$ ,  $k$  is order of  $z_1$ ,  $1 \leq k \leq n - 1$ ;
- 3)  $|f(z)| \leq |F(z)|$ ,  $|z| \geq |z_2|$ .

Then

$$|f'(z)| \leq |F'(z)| \quad (28)$$

for all  $z$ , such that  $|z| \in [|z_2|, +\infty) \setminus (r_1, r_2)$ , where

$$r_1 = \frac{1}{2n} \left( -k|z_2| + (n - k) \left( |z_1| - \frac{\tilde{\lambda}|z_2|}{n} \right) + \right.$$

$$\begin{aligned}
& + \sqrt{\left(k|z_2| - (n-k)\left(|z_1| - \frac{\tilde{\lambda}|z_2|}{n}\right)\right)^2 + 4(n-k)\tilde{\lambda}|z_1||z_2|}, \\
r_2 &= \frac{1}{2n} \left( k|z_2| + (n-k)\left(|z_1| + \frac{\tilde{\lambda}|z_2|}{n}\right) + \right. \\
& \left. + \sqrt{\left(k|z_2| + (n-k)\left(|z_1| + \frac{\tilde{\lambda}|z_2|}{n}\right)\right)^2 - 4(n-k)\tilde{\lambda}|z_1||z_2|} \right),
\end{aligned}$$

$\tilde{\lambda} = \frac{|z_2|^n - d}{|z_2|^n + d}$ ,  $d$  is the constant from Theorem 2.

**Proof.** Firstly, suppose that  $|z_2| > 0$ . Construct new polynomials

$$\tilde{f}(w) = f(|z_2|w) = a_n|z_2|^n w^n + \dots + a_0,$$

$$\tilde{F}(w) = F(|z_2|w) = b_n|z_2|^n w^n + \dots + b_0.$$

It is easily shown that all the assumptions of Theorem theo1 hold for the polynomials  $\tilde{f}$  and  $\tilde{F}$ . The zeros  $\frac{z_2}{|z_2|}, \dots, \frac{z_m}{|z_2|}$  of the polynomial  $\tilde{F}$  lie in  $\overline{\mathbb{D}}$  and the zero  $\frac{z_1}{|z_2|}$  of order  $k$  lies outside  $\overline{\mathbb{D}}$ . Applying Theorem 2 to  $\tilde{f}$  and  $\tilde{F}$ , we get

$$|S_\alpha[\tilde{f}](w)| \leq |S_\alpha[\tilde{F}](w)|$$

for  $|w| = \tilde{R} \geq 1$  and  $\alpha \in D_{\tilde{R}, \tilde{\mu}}$ , where

$$\tilde{\mu} = \frac{\left|\frac{z_1}{z_2}\right|^k - \tilde{d}}{\left|\frac{z_1}{z_2}\right|^k + \tilde{d}} = \frac{|z_1|^k |z_2|^{n-k} - d}{|z_1|^k |z_2|^{n-k} + d},$$

$$\tilde{d} = \max_{|z|=1} \left| \frac{a_0 + zb_0}{a_n|z_2|^n + zb_n|z_2|^n} \right| = \frac{d}{|z_2|^n},$$

$d$  is the constant from Theorem 2.

By Lemma 1, if

$$1 < \tilde{R} \leq \frac{1}{2n} \left( -k + (n-k) \left( \frac{|z_1|}{|z_2|} - \frac{\tilde{\lambda}}{n} \right) + \right.$$

$$+ \sqrt{\left(k - (n - k) \left(\frac{|z_1|}{|z_2|} - \frac{\tilde{\lambda}}{n}\right)\right)^2 + 4(n - k)\tilde{\lambda}\frac{|z_1|}{|z_2|}}$$

or

$$\begin{aligned} & \frac{1}{2n} \left( k + (n - k) \left( \frac{|z_1|}{|z_2|} + \frac{\tilde{\lambda}}{n} \right) + \right. \\ & \left. + \sqrt{\left(k + (n - k) \left( \frac{|z_1|}{|z_2|} + \frac{\tilde{\lambda}}{n} \right)\right)^2 - 4(n - k)\tilde{\lambda}\frac{|z_1|}{|z_2|}} \right) \leq \tilde{R} < \frac{|z_1|}{|z_2|}, \end{aligned}$$

where

$$\tilde{\lambda} = \frac{1 - \tilde{d}}{1 + \tilde{d}} = \frac{|z_2|^n - d}{|z_2|^n + d},$$

then  $\alpha = 0 \in D_{\tilde{R}, \tilde{\mu}}$ , i.e., the inequality

$$|\tilde{f}'(w)| \leq |\tilde{F}'(w)| \tag{29}$$

takes place for  $|w| = \tilde{R}$ . Putting  $z = |z_2|w$ ,  $R = |z_2|\tilde{R}$ , from (29) we obtain the inequality

$$|f'(z)| \leq |F'(z)|,$$

which is true for  $|z| = R \in [|z_2|, |z_1|) \setminus (r_1, r_2)$ .

To prove (28) for  $|z| \geq |z_1|$ , we apply Theorem C to the polynomials  $f(|z_1|w)$ ,  $F(|z_1|w)$ .

To finish the proof, we consider the case  $|z_2| = 0$ . Then

$$f(z) = a(z - z_1)^k z^{n-k}, \quad F(z) = b(z - z_1)^k z^{n-k}, \quad |a| \leq |b|.$$

For these polynomials, (28) is true for all  $z \in \mathbb{C}$ .  $\square$

In [9, Corollary 1 from Theorem 3], the following statement was proved:

**Theorem G.** Suppose polynomials  $f$  and  $F$  satisfy the conditions of Theorem 2. If  $R \in [|z_2|, +\infty) \setminus (\rho_1, \rho_2)$ , where

$$\rho_1 = \left(1 - \frac{k}{n}\right)|z_1| - \frac{k}{n}|z_2|, \quad \rho_2 = \left(1 - \frac{k}{n}\right)|z_1| + \frac{k}{n}|z_2|,$$

then the Bernstein inequality

$$|f'(z)| \leq |F'(z)|$$

takes place for  $|z| = R$ .

Let us show that Theorem 2 is a supplementation of Theorem G. For this aim, we prove that

$$\rho_1 < r_1 \quad (30)$$

and

$$\rho_2 > r_2, \quad (31)$$

where  $\rho_1, \rho_2$  are the constants from Theorem 2,  $r_1, r_2$  are the constants from Theorem G.

Rewrite (30) in the form

$$\begin{aligned} \left(1 - \frac{k}{n}\right)|z_1| - \frac{k}{n}|z_2| &< \frac{1}{2n} \left( -k|z_2| + (n-k) \left( |z_1| - \frac{\tilde{\lambda}|z_2|}{n} \right) + \right. \\ &\left. + \sqrt{\left( k|z_2| - (n-k) \left( |z_1| - \frac{\tilde{\lambda}|z_2|}{n} \right) \right)^2 + 4(n-k)\tilde{\lambda}|z_1||z_2|} \right). \end{aligned}$$

This inequality is equivalent to

$$\begin{aligned} \sqrt{\left( k|z_2| - (n-k) \left( |z_1| - \frac{\tilde{\lambda}|z_2|}{n} \right) \right)^2 + 4(n-k)\tilde{\lambda}|z_1||z_2|} &> \\ &> (n-k) \left( |z_1| + \frac{\tilde{\lambda}|z_2|}{n} \right) - k|z_2|. \quad (32) \end{aligned}$$

If the right-hand side of (32) is negative, then (32) is always true. Further, we shall assume that the right-hand side of (32) is greater than or equal to zero. Squaring both parts of (32), we get

$$-\frac{(n-k)}{n}|z_1| + \frac{k}{n}|z_2| + 2|z_1| > \frac{(n-k)}{n}|z_1| - k|z_2|,$$

which reduces to the correct inequality

$$-|z_1| < |z_2|.$$

Therefore, (30) is true.

Now, consider inequality (31):

$$\left(1 - \frac{k}{n}\right)|z_1| + \frac{k}{n}|z_2| > \frac{1}{2n} \left( k|z_2| + (n-k) \left( |z_1| + \frac{\tilde{\lambda}|z_2|}{n} \right) + \sqrt{\left( k|z_2| + (n-k) \left( |z_1| + \frac{\tilde{\lambda}|z_2|}{n} \right) \right)^2 - 4(n-k)\tilde{\lambda}|z_1||z_2|} \right).$$

Write this as

$$\begin{aligned} \sqrt{\left( k|z_2| + (n-k) \left( |z_1| + \frac{\tilde{\lambda}|z_2|}{n} \right) \right)^2 - 4(n-k)\tilde{\lambda}|z_1||z_2|} < \\ < (n-k)|z_1| + \left( k - \frac{n-k}{n}\tilde{\lambda} \right) |z_2|. \end{aligned} \quad (33)$$

Note that the right-hand side of (33) is positive. Square both parts of (33) and obtain the inequality

$$\frac{(n-k)}{n}|z_1| + \frac{k}{n}|z_2| - 2|z_1| < -\frac{(n-k)}{n}|z_1| - \frac{k}{n}|z_2|.$$

The last inequality takes place if

$$|z_1| > |z_2|,$$

which is true. Hence, (31) is also true.

**Remark.** *Summing up, we have obtained that, generally speaking, compared to Theorem G, Theorem 2 gives us two additional annuli in  $\mathbb{D}$ , where the Bernstein inequality takes place for polynomials satisfying the conditions of Theorem G.*

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*Received July 15, 2023.*

*Accepted November 03, 2023.*

*Published online November 22, 2023.*

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