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SMIRNOV AND BERNSTEIN-TYPE INEQUALITIES, TAKING INTO ACCOUNT HIGHER-ORDER COEFFICIENTS AND FREE TERMS OF POLYNOMIALS

Abstract. The starting point in the theory of differential inequalities for polynomials is the book "Investigation of aqueous solutions by specific gravity" by D. I. Mendeleev. In this work, he dealt not only with chemical, but also mathematical problems. The question raised in this book led to appearance of a large number of works on various types of differential inequalities for polynomials. In our paper, we obtain Smirnov and Bernstein-type inequalities that use higher-order coefficients and free terms of polynomials.

Key words: *polynomial, differential inequality, higher-order coefficient, free term*

2020 Mathematical Subject Classification: *30C10, 30A10*

1. Introduction. By $\mathcal{P}_n$ denote the set of all polynomials of degree at most $n \in \mathbb{N}$. Let $\mathbb{D}$ stand for the open unit disk$= \{z \in \mathbb{C} \colon |z| < 1\}$.

Consider the following problem (the Mendeleev problem):

Let $B \subset \mathbb{C}$ be a compact set, $f(z) \in \mathcal{P}_n$ be a polynomial, such that $|f(z)| \leqslant M$ for $z \in B$. Give an estimate for $|f'(z)|$ on B.

This problem in its original form was posed in 1887 by the famous chemist D. I. Mendeleev in [13, § 86]. Mendeleev considered only real polynomials of degree two and the compact set $B = [a, b]$. In [20, p. 340], the problem was presented in the general form.

In 1889, A. A. Markov solved the original Mendeleev problem.

Theorem A. [10], [11, p. 51–75] *Suppose that $f \in \mathcal{P}_n$ and $|f(x)| \leqslant M$ for $x \in [a, b]$. Then*

$$|f'(x)| \leqslant Mn^2.$$

Here equality is attained only for the functions

$$f(x) = \pm M T_n \left(\frac{2x - a - b}{b - a} \right)$$

where $T_n(x) = \cos(n \arccos x)$ *are the Chebyshev polynomials.*

In [12], V. A. Markov obtained an estimation for the k-th derivative of a polynomial f, $1 \leqslant k \leqslant n$.

Let us note that Mendeleev and the Markov brothers dealt only with real polynomials.

From now on, in our article we shall consider complex polynomials and the compact set $B = \partial\mathbb{D}$. In this case, the solution of the Mendeleev problem is the following theorem:

Theorem B. *Let* $f \in \mathcal{P}_n$ *and suppose that* $|f(z)| \leqslant M$ *on* $\partial\mathbb{D}$. *Then for* $z \in \partial\mathbb{D}$:

$$|f'(z)| \leqslant Mn.$$

Equality holds only if $f(z) = e^{i\gamma} M z^n$, $\gamma \in \mathbb{R}$.

In the literature, Theorem B is known as the Bernstein inequality. However, it apparently was not Bernstein who first obtained this statement in the presented form. The history is the following. In 1912, Bernstein considered trigonometric polynomials

$$f(t) = \sum_{k=0}^{n} (a_k \cos kt + b_k \sin kt)$$

of degree n, $|f(t)| \leqslant M$ for $t \in [0, 2\pi]$, see [1]. He proved that $|f'(t)| \leqslant 2Mn$. In paper [6, p. 50], for such polynomials the estimate

$$|f'(t)| \leqslant Mn, \quad t \in [0, 2\pi], \tag{1}$$

was presented by Fejér. Inequality (1) can be also found in [7], Fekete attributes the proof of (1) to Fejér. However, Bernstein [2, p. 39] attributes the proof to Landau, who sent the proof of (1) to Bernstein in his letter. In [18, p. 357], M. Riesz presented the statement of Theorem B as a corollary of inequality (1).

Alternative proof of Theorem B was given by V. I. Smirnov [19], [20, ch. V, section 1, 2°, p. 346].

S. N. Bernstein generalized Theorem B in the following way:

Theorem C. [3] (see also [4, p. 497], [17, p. 510]) *Consider polynomials* f *and* F, *such that:*

1) $\deg f \leqslant \deg F = n$,

2) $|f(z)| \leqslant |F(z)|$ on $\partial\mathbb{D}$,

3) F *has all its zeros in* $\overline{\mathbb{D}}$.

Then we have

$$|f'(z)| \leqslant |F'(z)| \quad for\ z \in \mathbb{C}\backslash\mathbb{D}.$$

For $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$, *equality holds only if* $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

Note that Theorem B follows from Theorem C with $F(z) = Mz^n$.

V. I. Smirnov proved a stronger version of Theorem C. He considered the operator

$$S_\alpha[f](z) = zf'(z) - n\alpha f(z), \quad f \in \mathcal{P}_n,$$

where α is a complex parameter.

Theorem D. [20, ch. V, § 1, p. 356] *Let* $R \geqslant 1$, f *and* F *be polynomials from Theorem C. Then*

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F](z)|, \quad |z| = R, \tag{2}$$

for all $\alpha \in \Omega_R$, *where* $\Omega_1 = \{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha \leqslant 1/2\}$ *and for* $R > 1$ *the set* Ω_R *is the complement to the open disk with diameter* $\left[\dfrac{R}{R+1}, \dfrac{R}{R-1}\right]$.

If $\alpha \in \operatorname{int}\Omega_R$ *and* $z \in \mathbb{C}\backslash\overline{\mathbb{D}}$, *the equality in* (2) *holds only if* $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

Taking $\alpha = 0$ in Theorem D, we have Theorem C.

A. V. Olesov obtained a supplementation of the Smirnov inequality (2). He enlarged the set of variation of the parameter α. In Theorem D, this set (Ω_R) depends only on R. The Olesov set depends on R too and also on the higher-order coefficients and the free terms of the polynomials $f(z)$, $F(z)$.

Theorem E. [15] *Let* $f(z) = a_n z^n + \ldots + a_0$ *and* $F(z) = b_n z^n + \ldots + b_0$ *be polynomials from Theorem C. Assume that* $R \geqslant 1$, $\lambda = \dfrac{1-d}{1+d}$, *where* $d = \max\limits_{|z|=1}\left|\dfrac{a_0 + zb_0}{a_n + zb_n}\right|$. *Then* $d \leqslant 1$ *and*

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F](z)|$$

for $|z| = R$ *and* $\alpha \in G_{R,\lambda}$, *where*

$$G_{1,\lambda} = \left\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha \leqslant \frac{1}{2} + \frac{\lambda}{2n}\right\}$$

and for $R > 1$ the set $G_{R,\lambda}$ is the complement to the open disk with diameter $\left[\dfrac{R+\lambda/n}{R+1}, \dfrac{R-\lambda/n}{R-1}\right]$.

The idea to use higher-order coefficients and the free terms in different inequalities for polynomials belongs to V. N. Dubinin [5].

More results concerning the development of this subject can be found, for example, in [17], [8], [14], [16].

In the previous theorems, it was required that all the zeroes of the polynomial F belong to $\overline{\mathbb{D}}$. In [9], this condition was waived. For a polynomial F having one zero outside $\overline{\mathbb{D}}$, the analogue of the Smirnov inequality was proved.

Theorem F. [9] *Let f and F be polynomials, such that*

1) $\deg f \leqslant \deg F = n$,

2) $|f(z)| \leqslant |F(z)|, \quad z \in \mathbb{C}\backslash\mathbb{D}$,

3) *z_0 is a unique zero of F lying in $\mathbb{C}\backslash\overline{\mathbb{D}}$, k is order of z_0,* $1 \leqslant k \leqslant n-1$.

Let $R \geqslant 1$. Then

$$|S_\alpha[f](z)| \leqslant |S_\alpha[F](z)|, \quad |z| = R,$$

for $\alpha \in D_R$, where D_R is one of the following sets:

a) the half-plane

$$\left\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha \leqslant \left(1-\frac{k}{n}\right)\frac{1}{2} - \frac{k}{n}\frac{1}{|z_0|-1}\right\}$$

for $R = 1$;

b) the complement to the strip

$$\left\{\alpha \in \mathbb{C}\colon \left(1-\frac{k}{n}\right)\frac{R}{R+1} + \frac{k}{2n} < \operatorname{Re}\alpha < \left(1-\frac{k}{n}\right)\frac{R}{R-1} + \frac{k}{2n}\right\}$$

for $R = |z_0| > 1$;

c) the complement to the open annulus bounded by the circles with diameters

$$\left[\left(1-\frac{k}{n}\right)\frac{R}{R+1} + \frac{k}{n}R\left(\frac{R}{R^2-|z_0|^2} - \frac{|z_0|}{\left|R^2-|z_0|^2\right|}\right),\right.$$
$$\left.\left(1-\frac{k}{n}\right)\frac{R}{R-1} + \frac{k}{n}R\left(\frac{R}{R^2-|z_0|^2} + \frac{|z_0|}{\left|R^2-|z_0|^2\right|}\right)\right]$$

and

$$\Big[\Big(1-\frac{k}{n}\Big)\frac{R}{R-1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}-\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big),$$
$$\Big(1-\frac{k}{n}\Big)\frac{R}{R+1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}+\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big)\Big]$$

for

c-1) $M|z_0|\leqslant R<|z_0|$, *where* $M=\sqrt{\dfrac{(n-k)|z_0|+k}{(n-k)|z_0|+k|z_0|^2}}$,

c-2) $|z_0|<R\leqslant L|z_0|$, *where* $|z_0|<n/k-1$, $L=\sqrt{\dfrac{(n-k)|z_0|-k}{(n-k)|z_0|-k|z_0|^2}}$,

c-3) $|z_0|\geqslant n/k-1$;

d) the complement to the open disk with diameter

$$\Big[\Big(1-\frac{k}{n}\Big)\frac{R}{R+1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}-\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big),$$
$$\Big(1-\frac{k}{n}\Big)\frac{R}{R-1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}+\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big)\Big]$$

for

d-1) $1<R<M|z_0|$,
d-2) $R>L|z_0|$, $|z_0|<n/k-1$.

In this paper, combining ideas and methods from [9] and [15], we obtain new refinements of Bernstein's and Smirnov's inequalities.

2. Supplementation of the Smirnov inequality. In this section, we obtain a refinement of the Smirnov inequality, basing on Theorem E. As in [9], we consider the case when the polynomial F has one zero outside $\overline{\mathbb{D}}$.

Theorem 1. *Suppose* $f(z)=a_nz^n+\ldots+a_0$ *and* $F(z)=b_nz^n+\ldots+b_0$ *be polynomials, such that*

1) $\deg f\leqslant \deg F=n$,
2) $|f(z)|\leqslant|F(z)|,\quad z\in\mathbb{C}\backslash\mathbb{D}$,
3) z_0 *is a unique zero of* F *lying in* $z\in\mathbb{C}\backslash\overline{\mathbb{D}}$, k *is order of* z_0, $1\leqslant k\leqslant n-1$.

Let $R\geqslant 1$, $\quad d=\max\limits_{|z|=1}\Big|\dfrac{a_0+zb_0}{a_n+zb_n}\Big|$, $\quad \mu=\dfrac{|z_0|^k-d}{|z_0|^k+d}$. *Then*

$$|S_\alpha[f](z)|\leqslant|S_\alpha[F](z)|,\quad |z|=R, \tag{3}$$

for $\alpha \in D_{R,\mu}$, *where* $D_{R,\mu}$ *is one of the following sets:*

a) the half–plane

$$\Big\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha \leqslant \Big(1-\frac{k}{n}\Big)\Big(\frac{1}{2}+\frac{\mu}{2n}\Big)-\frac{k}{n}\frac{1}{|z_0|-1}\Big\}$$

for $R = 1$;

b) the complement to the strip

$$\Big\{\alpha \in \mathbb{C}\colon \Big(1-\frac{k}{n}\Big)\frac{R+\mu/n}{R+1}+\frac{k}{2n} < \operatorname{Re}\alpha < \Big(1-\frac{k}{n}\Big)\frac{R-\mu/n}{R-1}+\frac{k}{2n}\Big\}$$

for $R = |z_0| > 1$;

c) the complement to the open annulus bounded by the circles with diameters

$$\Big[\Big(1-\frac{k}{n}\Big)\frac{R+\mu/n}{R+1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}-\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big),\\ \Big(1-\frac{k}{n}\Big)\frac{R-\mu/n}{R-1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}+\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big)\Big]$$

and

$$\Big[\Big(1-\frac{k}{n}\Big)\frac{R-\mu/n}{R-1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}-\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big),\\ \Big(1-\frac{k}{n}\Big)\frac{R+\mu/n}{R+1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}+\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big)\Big]$$

for

c-1) $m|z_0| \leqslant R < |z_0|$, *where* $m = \sqrt{\dfrac{(n-k)(n-\mu)|z_0|+kn}{(n-k)(n-\mu)|z_0|+kn|z_0|^2}}$;

c-2) $|z_0| < R \leqslant l|z_0|$, *where*

$$|z_0| < \Big(\frac{n}{k}-1\Big)\Big(1-\frac{\mu}{n}\Big), \quad l = \sqrt{\frac{(n-k)(n-\mu)|z_0|-kn}{(n-k)(n-\mu)|z_0|-kn|z_0|^2}};$$

c-3) $|z_0| \geqslant (n/k-1)(1-\mu/n)$;

d) the complement to the open disk with diameter

$$\Bigg[\Big(1-\frac{k}{n}\Big)\frac{R+\mu/n}{R+1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}-\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big),$$
$$\Big(1-\frac{k}{n}\Big)\frac{R-\mu/n}{R-1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}+\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big)\Bigg]$$

for

d-1) $1 < R < m|z_0|$;

d-2) $R > l|z_0|,\ |z_0| < (n/k-1)(1-\mu/n)$.

Proof. To prove Theorem 1, we will use methods from the proof of Theorem 1 from [9].

Since z_0 is a zero of the polynomial F of order k, we have

$$F(z) = (z-z_0)^k F_0(z), \tag{4}$$

where the polynomial $F_0(z) = b_n z^{n-k} + \ldots + \dfrac{b_0}{(-1)^k z_0^k}$ does not vanish in $\mathbb{C}\backslash\overline{\mathbb{D}}$, $\deg F_0 = n-k$. By condition 2,

$$f(z) = (z-z_0)^k f_0(z); \tag{5}$$

here $f_0(z) = a_n z^{n-k} + \ldots + \dfrac{a_0}{(-1)^k z_0^k}$ is a polynomial of degree at most $n-k$. Also by condition 2, we have $|f_0(z)| \leqslant |F_0(z)|,\ z \in \mathbb{C}\backslash\overline{\mathbb{D}}$.

Our aim is to find values of the parameter α, such that (3) is true for fixed $R \geqslant 1$. Substitute (4) and (5) to (3) to obtain

$$\Big|z\big(k(z-z_0)^{k-1}f_0(z)+(z-z_0)^k f_0'(z)\big)-\alpha n(z-z_0)^k f_0(z)\Big| \leqslant$$
$$\leqslant \Big|z\big(k(z-z_0)^{k-1}F_0(z)+(z-z_0)^k F_0'(z)\big)-\alpha n(z-z_0)^k F_0(z)\Big|, \quad |z|=R. \tag{6}$$

Let us observe that for $z = z_0$ inequality (6) takes place for all pairs of the polynomials f_0 and F_0 and all $\alpha \in \mathbb{C}$. Therefore, we further consider the nontrivial case when $z \neq z_0$. For such z, inequality (6) is equivalent to

$$\big|S_\beta[f_0](z)\big| \leqslant \big|S_\beta[F_0](z)\big|, \quad |z| = R, \tag{7}$$

with the parameter

$$\beta = \beta(\alpha) = \frac{1}{n-k}\Big(\alpha n - k\frac{z}{z-z_0}\Big). \tag{8}$$

Since the polynomials f_0 and F_0 satisfy conditions of Theorem E (with $\deg F_0 = n-k$), inequality (7) takes place when $\beta \in G_{R,\mu}$, where

$$\mu = \frac{1-d_0}{1+d_0}, \qquad d_0 = \max_{|z|=1}\left|\frac{\frac{a_0}{(-1)^k z_0^k} + z\frac{b_0}{(-1)^k z_0^k}}{a_n + zb_n}\right|.$$

So,

$$d_0 = \frac{d}{|z_0|^k}, \qquad \mu = \frac{|z_0|^k - d}{|z_0|^k + d}.$$

Finally, inequality (3) is valid for α, such that the corresponding β (see (8)) belongs to $G_{R,\mu}$ for all z, $|z| = R$.

From (8), we have

$$\alpha = \Bigl(1 - \frac{k}{n}\Bigr)\beta + \frac{k}{n}\frac{z}{z-z_0}.$$

Hence, the desired set $D_{R,\mu}$ of values of the parameter α can be found as

$$D_{R,\mu} = \bigcap_{|z|=R} G_z,$$

where G_z is the set $\Bigl(1 - \frac{k}{n}\Bigr)G_{R,\mu}$ shifted by $\frac{k}{n}\frac{z}{z-z_0}$. Find sets G_z for a fixed $R \geqslant 1$.

a) Consider the case $R = 1$. By Theorem E, the set $G_{1,\mu}$ is the half-plane

$$\Bigl\{\beta \in \mathbb{C}\colon \operatorname{Re}\beta \leqslant \frac{1}{2} + \frac{\mu}{2n}\Bigr\}.$$

The function $\phi(z) = \dfrac{z}{z-z_0}$ maps the circle $\partial\mathbb{D}$ to the circle with diameter $\Bigl[-\dfrac{1}{|z_0|-1}, \dfrac{1}{|z_0|+1}\Bigr]$. Therefore, for $|z| = R$ the sets G_z are the half-planes

$$\Bigl\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha \leqslant \Bigl(1-\frac{k}{n}\Bigr)\Bigl(\frac{1}{2}+\frac{\mu}{2n}\Bigr) + c\Bigr\},$$

where c takes all values from the segment $\Bigl[-\dfrac{k}{n}\dfrac{1}{|z_0|-1}, \dfrac{k}{n}\dfrac{1}{|z_0|+1}\Bigr]$. The intersection $D_{R,\mu}$ of these half-planes over all c is the half-plane

$$\Bigl\{\alpha \in \mathbb{C}\colon \operatorname{Re}\alpha \leqslant \Bigl(1-\frac{k}{n}\Bigr)\Bigl(\frac{1}{2}+\frac{\mu}{2n}\Bigr) - \frac{k}{n}\frac{1}{|z_0|-1}\Bigr\}.$$

Now let $R > 1$. By Theorem E, set $G_{R,\mu}$ is the complement to the open disk with diameter $\left[\dfrac{R+\mu/n}{R+1}, \dfrac{R-\mu/n}{R-1}\right]$. Therefore, G_z is also the complement to an open disk H_z. The intersection of G_z over all z, $|z| = R$, will depend on the location of z_0 relative to the circle $|z| = R$.

b) Assume that $|z_0| = R$. Then function $\phi(z) = \dfrac{z}{z - z_0}$ maps the circle $|z| = R$ onto the straight line $\operatorname{Re} z = 1/2$. Therefore, centers of the disks H_z lie on the straight line $\operatorname{Re} z = \dfrac{k}{2n} + a\Big(1 - \dfrac{k}{n}\Big)$ (a is the center of $G_{R,\mu}$). This implies that $D_{R,\mu} = \bigcap\limits_{|z|=R} G_z$ is the complement to the strip

$$\Big\{\alpha \in \mathbb{C}\colon \Big(1 - \frac{k}{n}\Big)\frac{R+\mu/n}{R+1} + \frac{k}{2n} < \operatorname{Re}\alpha < \Big(1 - \frac{k}{n}\Big)\frac{R-\mu/n}{R-1} + \frac{k}{2n}\Big\}.$$

c), d) Now let $|z_0| \neq R$. Then the function $\phi(z) = \dfrac{z}{z - z_0}$ maps the circle $|z| = R$ to the circle of radius $\rho = \dfrac{R|z_0|}{\big|R^2 - |z_0|^2\big|}$ centred at $C = \dfrac{R^2}{R^2 - |z_0|^2}$. This yields that the centers of the disks H_z form the circle T with center

$$\frac{1}{2}\Big(1 - \frac{k}{n}\Big)\Big(\frac{R+\mu/n}{R+1} + \frac{R-\mu/n}{R-1}\Big) + \frac{k}{n}C = \Big(1 - \frac{k}{n}\Big)\frac{R^2-\mu/n}{R^2-1} + \frac{k}{n}C$$

and radius $\rho k/n$.

If radius of H_z does not exceed radius of T, i.e.,

$$\Big(1 - \frac{k}{n}\Big)\frac{R(1-\mu/n)}{R^2-1} \leqslant \frac{k}{n}\frac{R|z_0|}{\big|R^2 - |z_0|^2\big|}, \tag{9}$$

then the intersection $D_{R,\mu}$ of all G_z is the complement to the annulus

$$\Big\{\alpha \in \mathbb{C}\colon \frac{k}{n}\rho - \Big(1 - \frac{k}{n}\Big)\frac{R(1-\mu/n)}{R^2-1} <$$
$$< \Big|\alpha - \Big(1 - \frac{k}{n}\Big)\frac{R^2-\mu/n}{R^2-1} - \frac{k}{n}C\Big| < \frac{k}{n}\rho + \Big(1 - \frac{k}{n}\Big)\frac{R(1-\mu/n)}{R^2-1}\Big\}.$$

This annulus is bounded by the circles with diameters

$$\Big[\Big(1-\frac{k}{n}\Big)\frac{R+\mu/n}{R+1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}-\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big),$$
$$\Big(1-\frac{k}{n}\Big)\frac{R-\mu/n}{R-1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}+\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big)\Big]$$

and

$$\Big[\Big(1-\frac{k}{n}\Big)\frac{R-\mu/n}{R-1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}-\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big),$$
$$\Big(1-\frac{k}{n}\Big)\frac{R+\mu/n}{R+1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}+\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big)\Big].$$

In the case opposite to (9), i.e.,

$$\Big(1-\frac{k}{n}\Big)\frac{R(1-\mu/n)}{R^2-1}>\frac{k}{n}\frac{R|z_0|}{\big|R^2-|z_0|^2\big|},$$

the intersection of all G_z is the complement to the disk

$$\Big\{\alpha\in\mathbb{C}\colon\Big|\alpha-\Big(1-\frac{k}{n}\Big)\frac{R^2-\mu/n}{R^2-1}-\frac{k}{n}C\Big|<\frac{k}{n}\rho+\Big(1-\frac{k}{n}\Big)\frac{R(1-\mu/n)}{R^2-1}\Big\}.$$

Thus, $D_{R,\mu}$ is the complement to the open disk with diameter

$$\Big[\Big(1-\frac{k}{n}\Big)\frac{R+\mu/n}{R+1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}-\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big),$$
$$\Big(1-\frac{k}{n}\Big)\frac{R-\mu/n}{R-1}+\frac{k}{n}R\Big(\frac{R}{R^2-|z_0|^2}+\frac{|z_0|}{\big|R^2-|z_0|^2\big|}\Big)\Big].$$

Let us solve inequality (9) with respect to R.

If $R>|z_0|$, then (9) takes the form

$$R^2\Big(\Big(1-\frac{k}{n}\Big)\Big(1-\frac{\mu}{n}\Big)-\frac{k}{n}|z_0|\Big)\leqslant|z_0|^2\Big(\Big(1-\frac{k}{n}\Big)\Big(1-\frac{\mu}{n}\Big)-\frac{k}{n|z_0|}\Big).\quad(10)$$

If

$$\Big(1-\frac{k}{n}\Big)\Big(1-\frac{\mu}{n}\Big)-\frac{k}{n}|z_0|>0,$$

i.e.,

$$|z_0|<\frac{(n-k)(n-\mu)}{kn},\quad(11)$$

then

$$\frac{R^2}{|z_0|^2} \leqslant \frac{(n-k)(n-\mu)|z_0| - kn}{(n-k)(n-\mu)|z_0| - kn|z_0|^2}. \tag{12}$$

Since the last fraction is positive (moreover, this is more than 1), we rewrite (12) as

$$R \leqslant l|z_0|, \quad l = \sqrt{\frac{(n-k)(n-\mu)|z_0| - kn}{(n-k)(n-\mu)|z_0| - kn|z_0|^2}} > 1. \tag{13}$$

Combining (11) and (13), we get c-2). If (11) is true and $R > l|z_0|$, then we get d-2).

In the case

$$\Big(1 - \frac{k}{n}\Big)\Big(1 - \frac{\mu}{n}\Big) - \frac{k}{n}|z_0| = 0, \tag{14}$$

rewrite inequality (10) in the following way:

$$|z_0|^2 \frac{k}{n}\Big(|z_0| - \frac{1}{|z_0|}\Big) \geqslant 0.$$

The last inequality is obviously true.

For

$$\Big(1 - \frac{k}{n}\Big)\Big(1 - \frac{\mu}{n}\Big) - \frac{k}{n}|z_0| < 0, \tag{15}$$

(10) is equivalent to

$$\frac{R^2}{|z_0|^2} \geqslant \frac{\Big(1 - \frac{k}{n}\Big)\Big(1 - \frac{\mu}{n}\Big) - \frac{k}{n|z_0|}}{\Big(1 - \frac{k}{n}\Big)\Big(1 - \frac{\mu}{n}\Big) - \frac{k}{n}|z_0|}.$$

Since the left-hand side of this inequality is more than 1, the right-hand side is less than 1, so the inequality is always true. Finally, (13) and (14) give us c-3).

It remains to consider the case $1 < R < |z_0|$. For such R, inequality (9) takes the form

$$\Big(1 - \frac{k}{n}\Big)\frac{R(1-\mu/n)}{R^2 - 1} \leqslant \frac{k}{n}\frac{R|z_0|}{|z_0|^2 - R^2}.$$

Solving this inequality, we find

$$R \geqslant |z_0|m, \quad m = \sqrt{\frac{(n-k)(n-\mu)|z_0| + kn}{(n-k)(n-\mu)|z_0| + kn|z_0|^2}} < 1.$$

Hence, for $m|z_0| \leqslant R < |z_0|$ we obtain c-1). For $1 < R < m|z_0|$, we have d-1). □

Remark. *Comparing Theorem F and Theorem 1, we see that the set $D_{R,\mu}$ of variation of the parameter α in (3) from Theorem 1 is larger than the analogous set D_R from Theorem F. This is quite expected, because additional characteristics (the higher-order coefficients and the free terms of the polynomials f and F) are used for description of $D_{R,\mu}$. In case a), Theorem 1 gives us an additional strip. In case b), we have two additional strips. In c), we obtain two additional annuli, in d), an additional annulus. Also note that $m < M$, $l > L$, where M, L are the constants from Theorem F, m, l are the constants from Theorem 1. Therefore, in Theorem 1 we have larger set c) (not b)), for a larger set of R values.*

3. Refinement of the Bernstein inequality.

In this section, we obtain a refinement of the Bernstein inequality. For that, we need the following lemma.

Lemma 1. *Suppose $1 < R < |z_0|$. If $1 < R \leqslant R_1$ or $R_2 \leqslant R < |z_0|$, where*

$$R_1 = \frac{1}{2n}\left(-k + (n-k)\left(|z_0| - \frac{\lambda}{n}\right) + \right.$$
$$\left. + \sqrt{\left(k - (n-k)\left(|z_0| - \frac{\lambda}{n}\right)\right)^2 + 4(n-k)\lambda|z_0|}\right),$$

$$R_2 = \frac{1}{2n}\left(k + (n-k)\left(|z_0| + \frac{\lambda}{n}\right) + \right.$$
$$\left. + \sqrt{\left(k + (n-k)\left(|z_0| + \frac{\lambda}{n}\right)\right)^2 - 4(n-k)\lambda|z_0|}\right),$$

λ is the constant from Theorem E; then the set $D_{R,\mu}$ from Theorem 1 contains the point $\alpha = 0$.

Proof. As we have proved (see (8)), the relationship between the parameters α and β from the proof of Theorem 1 is the following;

$$\beta = \beta(\alpha) = \frac{1}{n-k}\left(\alpha n - k\frac{z}{z-z_0}\right).$$

Therefore, $\alpha = 0$ belongs to $D_{R,\mu}$ if and only if the domain $G_{R,\lambda}$ from Theorem E contains

$$\beta(0) = -\frac{k}{n-k}\frac{z}{z-z_0}$$

for all z, $|z| = R$.

The image Γ_R of the circle $\{z \in \mathbb{C}\colon |z| = R\}$ under the function $-\dfrac{k}{n-k}\dfrac{z}{z-z_0}$ is the circle with diameter $\left[-\dfrac{k}{n-k}\dfrac{R}{|z_0|+R}, \dfrac{k}{n-k}\dfrac{R}{|z_0|-R}\right]$. By Theorem E, for $R > 1$ the set $G_{R,\lambda}$ is the complement to the open disk, bounded by the circle T with diameter $\left[\dfrac{R+\lambda/n}{R+1}, \dfrac{R-\lambda/n}{R-1}\right]$. Hence, $\beta(0) \in G_{R,\lambda}$ for all z, $|z| = R$, only in one of the following cases:

1) Γ_R lies to the left of T, i.e.,

$$\frac{k}{n-k}\frac{R}{|z_0|-R} \leqslant \frac{R+\lambda/n}{R+1}; \tag{16}$$

2) Γ_R lies to the right of T, i.e.,

$$-\frac{k}{n-k}\frac{R}{|z_0|+R} \geqslant \frac{R-\lambda/n}{R-1}; \tag{17}$$

3) the interior of Γ_R contains T or T is tangent to Γ_R, i.e.,

$$-\frac{k}{n-k}\frac{R}{|z_0|+R} \leqslant \frac{R+\lambda/n}{R+1} \quad \text{and} \quad \frac{k}{n-k}\frac{R}{|z_0|-R} \geqslant \frac{R-\lambda/n}{R-1}. \tag{18}$$

Consider the first case. Let us solve (16) with respect to R. Inequality (16) is equivalent to the following inequality:

$$nR^2 + \left(k - (n-k)\left(|z_0| - \frac{\lambda}{n}\right)\right)R - (n-k)\frac{\lambda}{n}|z_0| \leqslant 0. \tag{19}$$

Solving (19), we find that $1 < R \leqslant R_1$.

In the second case, inequality (17) is not true, because the left-hand side of (17) is negative, while the right-hand side is positive.

We finish the proof by considering the third case. The first inequality of (18) is obviously true. Let us solve the second one. This inequality can be rewritten as

$$nR^2 - \left(k + (n-k)\left(|z_0| + \frac{\lambda}{n}\right)\right)R + \frac{\lambda}{n}(n-k)|z_0| \geqslant 0. \tag{20}$$

Discriminant of the equation

$$nR^2 - \Big(k + (n-k)\Big(|z_0| + \frac{\lambda}{n}\Big)\Big)R + \frac{\lambda}{n}(n-k)|z_0| = 0$$

equals

$$D = \Big(k + (n-k)\Big(|z_0| + \frac{\lambda}{n}\Big)\Big)^2 - 4(n-k)\lambda|z_0|.$$

Prove that $D > 0$. Write D in the form

$$D = k^2 + (n-k)^2\Big(|z_0| + \frac{\lambda}{n}\Big)^2 + 2(n-k)\Big(|z_0|(k-2\lambda) + \frac{k\lambda}{n}\Big).$$

Clearly, if $k - 2\lambda \geqslant 0$, then $D > 0$.

Consider the opposite case, when $k < 2\lambda$. Since $k \geqslant 1$ and, by Theorem E, $\lambda \in [0,1]$, we have that the inequality $k < 2\lambda$ takes place only if $k = 1$, $\lambda > 1/2$. For $k = 1$, $D > 0$ iff

$$\Big(1 + (n-1)\Big(|z_0| + \frac{\lambda}{n}\Big)\Big)^2 > 4(n-1)\lambda|z_0|. \tag{21}$$

First, note that for $1/2 < \lambda \leqslant 1$

$$\Big(1+(n-1)\Big(|z_0|+\frac{\lambda}{n}\Big)\Big)^2 > (n-1)\Big(|z_0|+\frac{\lambda}{n}\Big)\Big((n-1)\Big(|z_0|+\frac{\lambda}{n}\Big)+2\Big) >$$
$$> (n-1)|z_0|\Big((n-1)|z_0| + \frac{5}{2} - \frac{1}{2n}\Big). \tag{22}$$

Further, note that

$$4(n-1)\lambda|z_0| \leqslant 4(n-1)|z_0|. \tag{23}$$

Taking into account (22) and (23), it is sufficient to show that

$$(n-1)|z_0| + \frac{5}{2} - \frac{1}{2n} \geqslant 4. \tag{24}$$

to prove (21). Inequality (24) is equivalent to

$$|z_0| \geqslant \frac{3n+1}{2n(n-1)}. \tag{25}$$

If

$$\frac{3n+1}{2n(n-1)} < 1, \tag{26}$$

then (25) is true, because $|z_0| > 1$. Inequality (26) is fulfilled for $n \geqslant 3$. So, for $n \geqslant 3$, $k = 1$, $\lambda > 1/2$ we have also proved that $D > 0$.

It remains to consider the case $n = 2$, $k = 1$, $1/2 < \lambda \leqslant 1$. In this situation, $D > 0$ if and only if

$$1 + \Big(|z_0| + \frac{\lambda}{2}\Big)^2 + 2\Big(|z_0| + \frac{\lambda}{2}\Big) > 4\lambda|z_0|. \tag{27}$$

Rewrite (27):

$$|z_0|^2 + (2-3\lambda)|z_0| + \Big(\frac{\lambda^2}{4} + \lambda + 1\Big) > 0.$$

Since for $1/2 < \lambda \leqslant 1$ discriminant of the corresponding equation is negative, it follows that (27) is true.

Consequently, in any case $D > 0$.

Solving (20) with respect to R, we obtain that $R \geqslant R_2$, where

$$R_2 = \frac{k+(n-k)(|z_0|+\lambda/n)+\sqrt{\big(k+(n-k)(|z_0|+\lambda/n)\big)^2 - 4(n-k)\lambda|z_0|}}{2n}.$$

□

Lemma 1 will be used to prove the following theorem.

Theorem 2. *Let $f(z) = a_n z^n + \ldots + a_0$ and $F(z) = b_n z^n + \ldots + b_0$ be polynomials, such that*

1) $\deg f \leqslant \deg F = n$;

2) all the zeros $z_1, \ldots, z_m$ of F belong to $\overline{\mathbb{D}}$, $|z_1| > |z_2| \geqslant |z_i|$, $i = 3, \ldots, m$, k is order of z_1, $1 \leqslant k \leqslant n-1$;

3) $|f(z)| \leqslant |F(z)|$, $|z| \geqslant |z_2|$.

Then

$$|f'(z)| \leqslant |F'(z)| \tag{28}$$

for all z, such that $|z| \in [|z_2|, +\infty)\backslash(r_1, r_2)$, where

$$r_1 = \frac{1}{2n}\Bigg(-k|z_2| + (n-k)\Big(|z_1| - \frac{\widetilde{\lambda}|z_2|}{n}\Big) +$$

$$+\sqrt{(k|z_2|-(n-k)(|z_1|-\frac{\widetilde{\lambda}|z_2|}{n}))^2+4(n-k)\widetilde{\lambda}|z_1||z_2|}\Bigg),$$

$$r_2=\frac{1}{2n}\Bigg(k|z_2|+(n-k)\Big(|z_1|+\frac{\widetilde{\lambda}|z_2|}{n}\Big)+$$
$$+\sqrt{\Big(k|z_2|+(n-k)\Big(|z_1|+\frac{\widetilde{\lambda}|z_2|}{n}\Big)\Big)^2-4(n-k)\widetilde{\lambda}|z_1||z_2|}\Bigg),$$

$\widetilde{\lambda}=\dfrac{|z_2|^n-d}{|z_2|^n+d}$, *$d$ is the constant from Theorem 2.*

Proof. Firstly, suppose that $|z_2|>0$. Construct new polynomials

$$\widetilde{f}(w)=f(|z_2|w)=a_n|z_2|^n w^n+\ldots+a_0,$$

$$\widetilde{F}(w)=F(|z_2|w)=b_n|z_2|^n w^n+\ldots+b_0.$$

It is easily shown that all the assumptions of Theorem theo1 hold for the polynomials $\widetilde{f}$ and $\widetilde{F}$. The zeros $\dfrac{z_2}{|z_2|},\ldots,\dfrac{z_m}{|z_2|}$ of the polynomial $\widetilde{F}$ lie in $\overline{\mathbb{D}}$ and the zero $\dfrac{z_1}{|z_2|}$ of order k lies outside $\overline{\mathbb{D}}$. Applying Theorem 2 to $\widetilde{f}$ and $\widetilde{F}$, we get

$$|S_\alpha[\widetilde{f}](w)|\leqslant|S_\alpha[\widetilde{F}](w)|$$

for $|w|=\widetilde{R}\geqslant 1$ and $\alpha\in D_{\widetilde{R},\widetilde{\mu}}$, where

$$\widetilde{\mu}=\frac{\left|\frac{z_1}{z_2}\right|^k-\widetilde{d}}{\left|\frac{z_1}{z_2}\right|^k+\widetilde{d}}=\frac{|z_1|^k|z_2|^{n-k}-d}{|z_1|^k|z_2|^{n-k}+d},$$

$$\widetilde{d}=\max_{|z|=1}\left|\frac{a_0+zb_0}{a_n|z_2|^n+zb_n|z_2|^n}\right|=\frac{d}{|z_2|^n},$$

d is the constant from Theorem 2.

By Lemma 1, if

$$1<\widetilde{R}\leqslant\frac{1}{2n}\Bigg(-k+(n-k)\Big(\frac{|z_1|}{|z_2|}-\frac{\widetilde{\lambda}}{n}\Big)+$$

$$+\sqrt{\left(k-(n-k)\Big(\frac{|z_1|}{|z_2|}-\frac{\widetilde{\lambda}}{n}\Big)\right)^2+4(n-k)\widetilde{\lambda}\frac{|z_1|}{|z_2|}}\right)$$

or

$$\frac{1}{2n}\left(k+(n-k)\Big(\frac{|z_1|}{|z_2|}+\frac{\widetilde{\lambda}}{n}\Big)+\right.$$
$$\left.+\sqrt{\left(k+(n-k)\Big(\frac{|z_1|}{|z_2|}+\frac{\widetilde{\lambda}}{n}\Big)\right)^2-4(n-k)\widetilde{\lambda}\frac{|z_1|}{|z_2|}}\right)\leqslant \widetilde{R}<\frac{|z_1|}{|z_2|},$$

where

$$\widetilde{\lambda}=\frac{1-\widetilde{d}}{1+\widetilde{d}}=\frac{|z_2|^n-d}{|z_2|^n+d},$$

then $\alpha=0\in D_{\widetilde{R},\widetilde{\mu}}$, i.e., the inequality

$$|\widetilde{f}'(w)|\leqslant|\widetilde{F}'(w)| \tag{29}$$

takes place for $|w|=\widetilde{R}$. Putting $z=|z_2|w$, $R=|z_2|\widetilde{R}$, from (29) we obtain the inequality

$$|f'(z)|\leqslant|F'(z)|,$$

which is true for $|z|=R\in[|z_2|,|z_1|]\backslash(r_1,r_2)$.

To prove (28) for $|z|\geqslant|z_1|$, we apply Theorem C to the polynomials $f(|z_1|w)$, $F(|z_1|w)$.

To finish the proof, we consider the case $|z_2|=0$. Then

$$f(z)=a(z-z_1)^k z^{n-k},\ F(z)=b(z-z_1)^k z^{n-k},\ |a|\leqslant|b|.$$

For these polynomials, (28) is true for all $z\in\mathbb{C}$. □

In [9, Corollary 1 from Theorem 3], the following statement was proved:

Theorem G. *Suppose polynomials f and F satisfy the conditions of Theorem 2. If $R\in[|z_2|,+\infty)\backslash(\rho_1,\rho_2)$, where*

$$\rho_1=\Big(1-\frac{k}{n}\Big)|z_1|-\frac{k}{n}|z_2|,\quad \rho_2=\Big(1-\frac{k}{n}\Big)|z_1|+\frac{k}{n}|z_2|,$$

then the Bernstein inequality

$$|f'(z)|\leqslant|F'(z)|$$

takes place for $|z| = R$.

Let us show that Theorem 2 is a supplementation of Theorem G. For this aim, we prove that

$$\rho_1 < r_1 \tag{30}$$

and

$$\rho_2 > r_2, \tag{31}$$

where ρ_1, ρ_2 are the constants from Theorem 2, r_1, r_2 are the constants from Theorem G.

Rewrite (30) in the form

$$\Big(1 - \frac{k}{n}\Big)|z_1| - \frac{k}{n}|z_2| < \frac{1}{2n}\Bigg(- k|z_2| + (n-k)\Big(|z_1| - \frac{\widetilde{\lambda}|z_2|}{n}\Big) + \\ + \sqrt{\Big(k|z_2| - (n-k)\Big(|z_1| - \frac{\widetilde{\lambda}|z_2|}{n}\Big)\Big)^2 + 4(n-k)\widetilde{\lambda}|z_1||z_2|}\;\Bigg).$$

This inequality is equivalent to

$$\sqrt{\Big(k|z_2| - (n-k)\Big(|z_1| - \frac{\widetilde{\lambda}|z_2|}{n}\Big)\Big)^2 + 4(n-k)\widetilde{\lambda}|z_1||z_2|} > \\ > (n-k)\Big(|z_1| + \frac{\widetilde{\lambda}|z_2|}{n}\Big) - k|z_2|. \tag{32}$$

If the right-hand side of (32) is negative, then (32) is always true. Further, we shall assume that the right-hand side of (32) is greater than or equal to zero. Squaring both parts of (32), we get

$$-\frac{(n-k)}{n}|z_1| + \frac{k}{n}|z_2| + 2|z_1| > \frac{(n-k)}{n}|z_1| - k|z_2|,$$

which reduces to the correct inequality

$$-|z_1| < |z_2|.$$

Therefore, (30) is true.

Now, consider inequality (31):

$$\left(1-\frac{k}{n}\right)|z_1|+\frac{k}{n}|z_2|>\frac{1}{2n}\Bigg(k|z_2|+(n-k)\left(|z_1|+\frac{\widetilde{\lambda}|z_2|}{n}\right)+$$
$$+\sqrt{\left(k|z_2|+(n-k)\left(|z_1|+\frac{\widetilde{\lambda}|z_2|}{n}\right)\right)^2-4(n-k)\widetilde{\lambda}|z_1||z_2|}\Bigg).$$

Write this as

$$\sqrt{\left(k|z_2|+(n-k)\left(|z_1|+\frac{\widetilde{\lambda}|z_2|}{n}\right)\right)^2-4(n-k)\widetilde{\lambda}|z_1||z_2|}<$$
$$<(n-k)|z_1|+\left(k-\frac{n-k}{n}\widetilde{\lambda}\right)|z_2|. \quad (33)$$

Note that the right-hand side of (33) is positive. Square both parts of (33) and obtain the inequality

$$\frac{(n-k)}{n}|z_1|+\frac{k}{n}|z_2|-2|z_1|<-\frac{(n-k)}{n}|z_1|-\frac{k}{n}|z_2|.$$

The last inequality takes place if

$$|z_1|>|z_2|,$$

which is true. Hence, (31) is also true.

Remark. *Summing up, we have obtained that, generally speaking, compared to Theorem G, Theorem 2 gives us two additional annuli in* $\mathbb{D}$*, where the Bernstein inequality takes place for polynomials satisfying the conditions of Theorem G.*

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