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ON THE INVERSE PROBLEM OF THE BITSADZE–SAMARSKII TYPE FOR A FRACTIONAL PARABOLIC EQUATION

Abstract. In this paper, the inverse problem of the Bitsadze–Samarsky type is studied for a fractional order equation with a Hadamard–Caputo fractional differentiation operator. The problem is solved using the spectral method. The spectral aspects of the obtained problem are investigated, root functions are found, and their basis property is proved. The conjugate problem is investigated. The uniqueness and existence theorems for a regular solution to this problem are proved.

Key words: *Hadamard–Caputo fractional operator, Riesz basis, Le Roy function, inverse problem*

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1. Problem statement. The Hadamard integration operator of order $\alpha > 0$ is the expression [1]

$$ {}_H J_{0t}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} y(\tau) \frac{d\tau}{\tau},\ t > 0. $$

If $\alpha = 0$, we suppose that ${}_H J_{0t}^{0} y(t) = y(t)$.

For $\alpha \in (m-1, m], m = 1, 2, \ldots$, the following expression

$$ {}_{HC} D_{0t}^{\alpha} y(t) = {}_H J_{0t}^{m-\alpha} (\delta^m y(t)) \equiv \frac{1}{\Gamma(m-\alpha)} \int\limits_0^1 \left(\ln \frac{t}{\tau} \right)^{m-\alpha-1} \delta^m y(\tau) \frac{d\tau}{\tau} $$

is called the Hadamard–Caputo differentiation operator of the order $\alpha > 0$, where $\delta = t\frac{d}{dt}$ and $\delta^k = \delta \cdot \delta^{k-1}, k \geqslant 1$ [2].

Let $0 < \alpha \leqslant 1, \beta > 0$. Consider the following equation:

$$t^{-\beta}{}_{HC}D^{\alpha}_{0t}u(x,t) = u_{xx}(x,t) + g(x). \tag{1}$$

For the case $\alpha = 1, \beta = 1$, we get

$$t^{-\beta}{}_{HC}D^{\alpha}_{0t}u(t,x) = t^{-1}\Big(t\frac{\partial}{\partial t}\Big)u(t,x) = \frac{\partial u(t,x)}{\partial t}$$

and equation (1) coincides with the classical parabolic equation.

Let $\Omega = \{(x,t)\colon 0 < x < 1, 0 < t < T\}$, where T is a positive real number. For equation (1), consider the following problem in the domain Ω:

Problem *BS*. Find a pair of functions $(u(x,t),\, g(x))$ from the class

$$u(x,t),\, t^{-\beta}{}_{HC}D^{\alpha}_{0t}u(x,t),\, u_{xx}(x,t) \in C(\bar{\Omega}),\, g(x) \in C[0,1], \tag{2}$$

satisfying in the domain Ω equation (1) and the conditions

$$u(x,0) = \varphi(x),\, 0 \leqslant x \leqslant 1, \tag{3}$$

$$u(x,T) = \psi(x),\, 0 \leqslant x \leqslant 1, \tag{4}$$

$$u(0,t) = 0,\, 0 \leqslant t \leqslant T, \tag{5}$$

$$u(1,t) = u(x_0,t),\, 0 \leqslant t \leqslant T. \tag{6}$$

Here $\varphi(x), \psi(x)$ are the given functions, β, a, x_0 are the given real numbers, such that $\beta > 0,\ 0 < x_0 < 1$.

Among the first works devoted to the issues related to the solvability of nonlocal problems, we note the work of T. Carleman [3] (see also [4]), where the problem with a nonlocal condition, which consists in finding a holomorphic function in a bounded domain that connects the values of this function at different points of the boundary, is studied. This problem was reduced to a singular integral equation with deviation. We should also note the papers [5] and [6], where abstract nonlocal elliptic boundary value problems were studied.

A nonlocal boundary-value problem of a new type for an elliptic differential equation that arises in plasma theory was formulated by A. V. Bitsadze and A. A. Samarskii [7]. This problem was reduced to an integral Fredholm equation of the second kind. Using the extremum principle for elliptic equations, the uniqueness of the classical solution is proved.

Further, the results on the theory of partial differential equations and functional-differential equations made it possible to study the solvability problem for a wide class of nonlocal elliptic boundary value problems. In the monograph [8] (see also [9]), a detailed classification of nonlocal elliptic boundary value problems is given, the uniqueness and solvability of such problems in Sobolev spaces and weighted Kondratiev spaces are studied, the properties of the index, the spectral properties of the corresponding operators, the asymptotic behavior of solutions, and the smoothness of generalized solutions are considered. In addition, applications of nonlocal problems to the processes of heat distribution, diffusion, and cooling of aircraft engines are described.

Similar problems with operators of integer or fractional order with Riemann–Liouville, Caputo, and Hadamard–Caputo derivatives were studied in [10], [11], [12], [13], and for parabolic systems in [14], [15].

Note that inverse problems for a parabolic equation of fractional order with the Gerasimov–Caputo operator were also studied in [16], [17], and for degenerate equations, in [18], [19].

In this paper, an inverse problem of the Bitsadze-Samarskii type is studied for a degenerate fractional parabolic equation with the Hadamard–Caputo operator. Using the spectral method, the eigenvalues, as well as the corresponding root functions, are found, and their basis property is proved. The spectral issues of the conjugate problem are also investigated.

2. The Cauchy problem for a one-dimensional fractional differential equation.

In this subsection, we study the Cauchy problem for a one-dimensional fractional differential equation with a Hadamard–Caputo derivative.

Let $0 < \alpha \leqslant 1$, $\beta > 0$. We introduce the following operators:

$$B_{\alpha}^{\beta} y(t) = t^{-\beta}\,{}_{HC}D_{0t}^{\alpha} y(t), B_{\alpha}^{-\beta} y(t) = J_{0t}^{\alpha}[\tau^{\beta} y](t)$$

and study some properties of these operators.

From the definition of operators ${}_{H}J_{0t}^{\alpha} y(t)$ and ${}_{HC}D_{0t}^{\alpha} y(t)$, it follows that

$${}_{H}J_{0t}^{\alpha}(t^{\mu}) = \mu^{-\alpha} t^{\mu}, \alpha, \mu > 0, {}_{HC}D_{0t}^{\alpha}(t^{\mu}) = \begin{cases} 0, \ \mu = 0, \\ \mu^{\alpha} t^{\mu}, \ \mu > 0, \end{cases} \quad \alpha \in (0, 1]. \quad (7)$$

Lemma 1. *Let $\alpha \in (0, 1]$, $\beta > 0$, and $f(t) \in C[0, d]$. Then $B_{\alpha}^{-\beta} f(t) \in C[0, d]$ and the following estimate holds:*

$$\left\| B_{\alpha}^{-\beta} f(t) \right\|_{C[0,d]} \leqslant \frac{d^{\beta}}{\beta^{\alpha}} \| f(t) \|_{C[0,d]},$$

where $\|f(t)\|_{C[0,d]} = \max\limits_{0\leqslant t\leqslant d} |f(t)|$.

Proof. Let $f(t) \in C[0,d]$. Then, taking into account the definition of the operator $B_\alpha^{-\beta}$, we obtain

$$\left|B_\alpha^{-\beta} f(t)\right| = \left|\frac{1}{\Gamma(\alpha)} \int\limits_0^t \tau^\beta \left(\ln\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}\right| \leqslant$$

$$\leqslant \frac{\|f(t)\|_{C[0,d]}}{\Gamma(\alpha)} \int\limits_0^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} \tau^{\beta-1} d\tau = \frac{\|f(t)\|_{C[0,d]}}{\Gamma(\alpha)} \frac{t^\beta}{\beta^\alpha} \int\limits_0^\infty \tau^{\alpha-1} e^{-\tau} d\tau =$$

$$= \frac{\|f(t)\|_{C[0,d]}}{\beta^\alpha} t^\beta.$$

Thus, we get the following estimate:

$$\left\|B_\alpha^{-\beta} f(t)\right\|_{C[0,d]} = \max_{0\leqslant t\leqslant d} \left|B_\alpha^{-\beta} f(t)\right| \leqslant \frac{\|f(t)\|_{C[0,d]}}{\beta^\alpha} \max_{0\leqslant t\leqslant d} t^\beta = \frac{d^\beta}{\beta^\alpha} \|f(t)\|_{C[0,d]}.$$

Lemma 1 is proved. □

Lemma 2. *Let* $0 < \alpha \leqslant 1$ *and* $f(t) \in C^1[0,d]$. *Then* ${}_{HC}D_{0t}^\alpha f(t) \in C[0,d]$, ${}_{HC}D_{0t}^\alpha f(0) = 0$, *and the following estimate holds:*

$$\|{}_{HC}D_{0t}^\alpha f(t)\|_{C[0,d]} \leqslant d\|f'(t)\|_{C[0,d]}.$$

Proof. If $\alpha = 1$, then ${}_{HC}D_{0t}^1 f(t) = t\frac{d}{dt} f(t)$. Hence, for $f(t) \in C^1[0,d]$ we get ${}_{HC}D_{0t}^1 f(t) \in C[0,d]$. It is obvious that ${}_{HC}D_{0t}^1 f(t)|_{t=0} = t\frac{d}{dt} f(t)|_{t=0} = 0$.

Let $0 < \alpha < 1$. Then

$$\left|{}_{HC}D_{0t}^\alpha f(t)\right| = \left|\frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \left(\ln\frac{t}{\tau}\right)^{-\alpha} \delta f(\tau)\frac{d\tau}{\tau}\right| \leqslant$$

$$\leqslant \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \left(\ln\frac{t}{\tau}\right)^{-\alpha} |f'(\tau)|\, d\tau \leqslant$$

$$\leqslant \|f'(t)\|_{C[0,d]} \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \left(\ln\frac{t}{\tau}\right)^{-\alpha} d\tau = t\|f'(t)\|_{C[0,d]}.$$

Thus, we obtain

$$\|{}_{HC}D_{0t}^{\alpha}f(t)\|_{C[0,d]} \leqslant d\|f'(t)\|_{C[0,d]}, {}_{HC}D_{0t}^{\alpha}f(t)\,|_{t=0} = \lim_{t\to 0}\, {}_{HC}D_{0t}^{\alpha}f(t) = 0.$$

Lemma 2 is proved. □

Lemma 3. *Let $0 < \alpha \leqslant 1$, $\beta > 0$ and $f(t) \in C^1[0,d]$. Then the following equality holds:*

$$B_{\alpha}^{-\beta}\left[B_{\alpha}^{\beta}[f]\right](t) = f(t) - f(0). \tag{8}$$

Proof. As $f(t) \in C^1[0,d]$, Lemma 2 implies that the function $B_{\alpha}^{\beta}[f](t)$ belongs to the class $C[0,d]$. If $\alpha = 1$, then

$$B_1^{-\beta}\left[B_1^{\beta}[f]\right](t) = \int_0^t \tau^{\beta}B_1^{\beta}[f](\tau)\frac{d\tau}{\tau} = \int_0^t \tau^{\beta}\tau^{-\beta}\tau\frac{d}{d\tau}f(\tau)\frac{d\tau}{\tau} = f(t) - f(0).$$

For $0 < \alpha < 1$, taking into account the formula

$${}_{H}J_{0t}^{\alpha}\,{}_{H}J_{0t}^{\beta} = {}_{H}J_{0t}^{\alpha+\beta}, \quad \alpha,\beta \geqslant 0$$

from [1], we get

$$\begin{aligned}B_{\alpha}^{-\beta}\left[B_{\alpha}^{\beta}[f]\right](t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}\tau^{\beta}B_{\alpha}^{\beta}[f](\tau)\frac{d\tau}{\tau} = \\ &= {}_{H}J_{0t}^{\alpha}\left[{}_{H}J_{0t}^{1-\alpha}[\tau f']\right](t) = {}_{H}J_{0t}^{1}\left[f'\right](t) = f(t) - f(0).\end{aligned}$$

Lemma 3 is proved. □

In the domain $(0,d)$, consider the following Cauchy problem:

$$B_{\alpha}^{\beta}y(t) = \lambda y(t), \quad 0 < t < d, \tag{9}$$

$$y(0) = b, \tag{10}$$

where b is a real positive number.

A solution to this problem is a function from the class $y(t) \in [0,d]$, $B_{\alpha}^{\beta}y(t) \in C(0,d)$ satisfying equation (9) and condition (10) in the classical sense.

Let a function $y(t)$ be a solution to problem (9), (10). Applying the operator $y(t)$ to both parts of (9), we have

$$B_\alpha^{-\beta}\left[B_\alpha^\beta y\right](t) = \lambda B_\alpha^{-\beta}[y](t), \quad 0 < t < d.$$

Hence, taking into account (7) and (8), we obtain

$$y(t) = b + \lambda B_\alpha^{-\beta}[y](t), \quad 0 < t < d.$$

Thus, if $y(t)$ is a solution of problem (9), (10), then it satisfies the Volterra integral equation of the second kind of the type

$$y(t) = \frac{\lambda}{\Gamma(\alpha)}\int\limits_0^t K(t,\tau)y(\tau)d\tau + b, \tag{11}$$

where $K(t,\tau) = (\ln(t) - \ln(\tau))^{\alpha-1}\tau^{\beta-1}$.

To find a solution to the integral equation (11), we apply the method of normalized systems [20]. To do this, denote $L_1 = E$, $L_2 = \lambda B_\alpha^{-\beta}$, where E is the unit operator. Then equation (11) can be rewritten as $(L_1 - L_2)y(t) = b$. As $L_1 = E$, we get $L_1^{-1} = E$. Let $g_0 = b$. Further, we use the technique used in [20]. According to this method, consider the system

$$g_k(t) = \left(L_1^{-1}\cdot L_2\right)^k g_0, k = 1, 2, \ldots.$$

For $k = 1$, we get

$$g_1(t) = (\lambda B_\alpha^{-\beta})g_0 = \lambda B_\alpha^{-\beta}[b] = \frac{\lambda b}{\Gamma(\alpha)}\int\limits_0^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}\tau^{\beta-1}d\tau =$$

$$= \frac{\lambda b}{\Gamma(\alpha)}\int\limits_0^\infty \xi^{\alpha-1}t^{\beta-1}e^{-(\beta-1)\xi}te^{\xi}d\xi = \frac{\lambda b t^\beta}{\beta^\alpha\Gamma(\alpha)}\int\limits_0^\infty s^{\alpha-1}e^{-s}ds = \frac{\lambda b}{\beta^\alpha}t^\beta.$$

Let $k = 2$. Then we get

$$g_2(t) = (\lambda B_\alpha^{-\beta})g_1(t) = b\frac{\lambda^2}{\beta^\alpha\Gamma(\alpha)}\int\limits_0^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1}\tau^{2\beta-1}d\tau =$$

$$= b\frac{\lambda^2 t^{2\beta}}{\beta^\alpha\Gamma(\alpha)}\int\limits_0^\infty \xi^{\alpha-1}e^{-2\beta\xi}d\xi = \frac{\lambda^2 b}{2^\alpha\beta^{2\alpha}}t^{2\beta}.$$

Further, using the method of mathematical induction, for $k \geqslant 2$ the function $g_k(t)$ can be represented as

$$g_k(t) = \left(\lambda B_\alpha^{-\beta}\right)^k b = \frac{\lambda^k b}{(k!)^\alpha \beta^{k\alpha}} t^{k\beta}.$$

Lemma 4. *Let $0 < \alpha \leqslant 1$ and $\beta > 0$. Then problem (9), (10) has a unique solution, which has the form*

$$y(t) = b \sum_{k=0}^{\infty} \frac{\lambda^k}{(k!)^\alpha \beta^{k\alpha}} t^{k\beta}. \tag{12}$$

Proof. Let us first show the convergence of the series (12). To do this, we evaluate the ratio $\frac{g_{k+1}(t)}{g_k(t)}$. Then we get

$$\left|\frac{g_{k+1}(t)}{g_k(t)}\right| = \left|\frac{\lambda^{k+1} t^{\beta(k+1)}}{\beta^{\alpha(k+1)}((k+1)!)^\alpha}\right| : \left|\frac{\lambda^k t^{\beta k}}{\beta^{\alpha k}(k!)^\alpha}\right| = \frac{|\lambda| t^\beta}{\beta^\alpha k} \underset{k\to\infty}{\to} 0.$$

Therefore, by the d'Alembert criterion, the series (12) converges uniformly for $t \in [0,d]$ (in general, for $t \geqslant 0$). Since the functions $g_k(t)$, $k = 1, 2, \ldots,$ are continuous on $[0, +\infty)$, the sum of the series is also continuous in $t \in [0, d]$.

Further, applying the operator B_α^β to the functions $g_k(t)$, and taking into account (7), we have

$$B_\alpha^\beta g_0(t) = 0,$$

$$B_\alpha^\beta g_k(t) = B_\alpha^\beta \Big[\frac{\lambda^k}{(k!)^\alpha \beta^{k\alpha}} t^{k\beta}\Big] = \frac{\lambda^k}{\beta^{k\alpha-1}((k-1)!)^\alpha} t^{(k-1)\beta} = \lambda g_{k-1}(t), k \geqslant 1.$$

Then, formally applying the operator B_α^β to the series (12), we obtain

$$B_\alpha^\beta y(t) = \sum_{k=0}^{\infty} B_\alpha^\beta g_k(t) = \lambda \sum_{k=1}^{\infty} g_{k-1}(t) = \lambda \sum_{k=0}^{\infty} g_k(t) = \lambda y(t). \tag{13}$$

This implies a uniform convergence of the series $\sum\limits_{k=0}^{\infty} B_\alpha^\beta g_k(t)$ and fulfillment of the condition $B_\alpha^\beta y(t) \in C[0, d]$. It also follows from (13) that a function (12) satisfies equation (9). It is obvious that $y(0) = b$, i.e., condition (10) is also satisfied. Lemma 4 is proved. □

Remark. *Function (12) can be represented as*

$$y(t) = b \cdot L_\alpha\left(\frac{\lambda t^\beta}{\beta^\alpha}\right),$$

where $L_\alpha(z)$ is written as

$$L_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^\alpha}, \alpha > 0,\ z \in R, \tag{14}$$

and is called the Le Roy function [21].

Now, in the domain $(0,d)$, we consider an inhomogeneous equation of the form

$$B_\alpha^\beta y(t) = \lambda y(t) + a, 0 < t < d, a \neq 0, \tag{15}$$

and find its solution from the class $y(t) \in [0,d]$, $B_\alpha^\beta y(t) \in C(0,d)$, satisfying condition (10). Since the solution to equation (9) is known, then, according to the theory, it is sufficient to find a particular solution to equation (15). It is easy to see that such a solution is a constant function of the type $y(t) = -\frac{a}{\lambda}$. Then the solution to problem (15), (10) is written as

$$y(t) = \left(b + \frac{a}{\lambda}\right) L_\alpha\left(\frac{\lambda t^\beta}{\beta^\alpha}\right) - \frac{a}{\lambda}. \tag{16}$$

3. A spectral problem for a second-order differential equation.

To solve the BS problem, we apply the Fourier method, according to which we look for a non-trivial solution of a homogeneous equation $((g(x) = 0))$, corresponding to (1) in the form $u(x,t) = X(x) \cdot T(t)$. Substituting it into equation (1) and using conditions (5) and (6) to find the eigenvalues λ and eigenfunctions $X(x)$, we obtain a problem for eigenvalues in the form

$$-X''(x) = \lambda X(x), 0 < x < 1, \tag{17}$$

$$X(0) = 0,\ X(1) = X(x_0),\ 0 < x_0 < 1. \tag{18}$$

Further, we find the eigenfunctions of problem (17), (18). For $\lambda \leqslant 0$, this problem has only the trivial solution, so consider the case with $\lambda > 0$. In this case, we obtain two series of eigenvalues:

$$\lambda_{n_1} = \left(\frac{(2n-1)\pi}{1+x_0}\right)^2, \quad \lambda_{n_2} = \left(\frac{2n\pi}{1-x_0}\right)^2, n \in N, \tag{19}$$

which correspond to eigenfunctions of the form

$$X_{n_1}(x) = \sin\sqrt{\lambda_{n_1}}x, \quad X_{n_2}(x) = \sin\sqrt{\lambda_{n_2}}x,\ n \in N. \tag{20}$$

Note that among the two series of eigenvalues from (19), there are coinciding ones. Indeed, comparing λ_{s_1} and λ_{m_2} from (19), we obtain the following relation between x_0, s and m:

$$x_0 = \frac{2s - 2m - 1}{2s + 2m - 1}, \quad s, m \in N; \tag{21}$$

in this case, the corresponding values of λ_{s_1} and λ_{m_2} coincide, so that the system of eigenfunctions (20) is not complete and the problem of supplementing this system with associated functions [22] arises.

Problem (17), (18) was first studied in [22], where the completeness of the root functions of the differential operator corresponding to this problem was shown.

The spectral problem in a more general formulation than the problem of the form (17), (18), is studied in [23]. A more general basis property criterion is found, which makes it possible to study the basis property of systems of eigenfunctions and associated functions of the above-mentioned problem, as well as of its adjoint problem.

Now, since relation (21) depends on the point x_0, let us find out the nature of this point, thereby uttering the results of the aforementioned works.

Clearly, when x_0 is an irrational number from the interval $(0, 1)$, relation (21) does not hold, and, so, both eigenvalues (19) and the corresponding eigenfunctions are different. The following example shows that (21) does not hold even for some rational values of x_0. Indeed, consider the case of $x_0 = \frac{1}{2}$. Then, from (21) it follows $2s = 6m + 1$. By virtue of the fact that $m, s \in N$, $6m + 1$ is always odd, and $2s$ is even, that is, (21) does not occur under any $m, s \in N$. So, there are rational fractions in which all eigenfunctions are distinct. Now we give a criterion that shows the relationship between the values of x_0, s and m at which the equality (21) takes place, and also gives an algorithm for finding the corresponding values of $m, s \in N$.

Let x_0 be a rational number from the interval $(0, 1)$, such that $x_0 = \frac{p}{q}$, $p < q$, and p, q- are coprime natural numbers. Here are some known facts about the number x_0:

1) Numbers $p + q$ and $q - p$ at the same time are both either even or odd.

2) Let the number $q-p$ (or $p+q$) be even; then either $p+q$ or $q-p$ is a multiple of 4.

Lemma 5. *Let $x_0 \in (0,1)$ be a rational number, such that $x_0 = \frac{p}{q}$, $p < q$, p and q be odd coprime natural numbers, such that $q-p$ is a multiple of 4. Then there exists countable numbers s and m, such that for two series of eigenvalues from (19) the equality $\lambda_{s_1} = \lambda_{m_2}$ is valid.*

Proof. Indeed, from the condition on x_0, the relation (21) will take the following form:

$$s = \frac{m(p+q)}{q-p} + \frac{1}{2}. \tag{22}$$

By condition, $q-p$ is a multiple of 4, then it is represented as $q-p = 4r$, where $r \in N$. Hence, $q = p + 4r$. Then (22) takes the following form

$$s = \frac{m(p+2r)}{2r} + \frac{1}{2}. \tag{23}$$

By condition, p, q are odd numbers. Then, obviously, the number $p+2r$ is also odd, and the numbers $p+2r$ and $2r$ have no common divisors. Therefore, the condition $s \in N$ can be provided only by choosing m, and it is easy to see that it is true if and only if $m = k \cdot r$, where k is any odd natural number. Thus, at these values m from (23) we have:

$$s = \frac{k(p+2r)}{2} + \frac{1}{2},$$

and, since $k(p+2r)$ is odd, it follows that $s \in N$.

Finally, to find the values s and m, at which (21) is true, we obtained the following formulae:

$$m = k \cdot r, \quad s = \frac{m(p+q)}{q-p} + \frac{1}{2}, \tag{24}$$

where $r = \frac{q-p}{4}$, $k = 1, 3, 5, \ldots$. $\square$

Taking Lemma 1 and formula (21) into account, we divide the rational numbers from the interval $(0,1)$ into two sets Q_1 and Q_2, where the set Q_2 contains all rational numbers satisfying the conditions of Lemma 1, and Q_1 contains the remaining rational numbers.

Let us consider the case $x_0 \in Q_1$. In this case, we obtain two series of eigenvalues (19), which correspond to eigenfunctions (20), and all these functions are different.

The spectral problem (17)–(18) is not self-adjoint, and, therefore, the system (20) is not orthogonal. Then, according to the spectral theory of operators, the problem arises of studying the problem adjoint to problem (17)–(18). It is easy to determine that the following problem is adjoint to it:

$$-Y''(x) = \lambda Y(x),\, x \in (0, x_0) \cup (x_0, 1), \tag{25}$$

$$Y(0) = 0, Y(1) = 0, \tag{26}$$

$$Y(x_0 + 0) = Y(x_0 - 0), Y'(1) = Y'(x_0 + 0) - Y'(x_0 - 0). \tag{27}$$

Note that the solution of equation (25) that satisfies conditions (26) and (27) is found uniquely, that is, it has no extra conditions. This problem should be considered as two boundary-value problems with gluing conditions of the form (27).

As in the case of problem (17)–(18), consider the case $x_0 \in Q_1$. Then, solving the problem (25)–(27), we find that the numbers (19) are also eigenvalues of this problem, and the corresponding eigenfunctions have the form

$$\{Y_{n_1}(x); Y_{n_2}(x)\}, n \in N, \tag{28}$$

where

$$Y_{n_1}(x) = \begin{cases} \dfrac{4}{1+x_0} \sin \sqrt{\lambda_{n_1}} x, \quad 0 \leqslant x \leqslant x_0, \\ -\dfrac{2}{(x_0+1)\cos\sqrt{\lambda_{n_1}}} \sin \sqrt{\lambda_{n_1}}\,(1-x)\,,\; x_0 \leqslant x \leqslant 1, \end{cases}$$

$$Y_{n_2}(x) = \begin{cases} 0, \quad 0 \leqslant x \leqslant x_0, \\ -\dfrac{2}{(1-x_0)\cos\sqrt{\lambda_{n_2}}} \sin \sqrt{\lambda_{n_2}}\,(1-x)\,, \quad x_0 \leqslant x \leqslant 1, \end{cases}$$

Lemma 6. *The systems of the functions (20) and (28) are biorthonormal, i.e., it takes place*

$$(X_{s_1}(x), Y_{m_j}(x))_{L_2(0,1)} = \begin{cases} 1, \quad s = m, j = 1, \\ 0, \quad s \neq m, j = 1, 2, \end{cases}$$

$$(Y_{s_2}(x), \tilde{X}_{m_j}(x))_{L_2(0,1)} = \begin{cases} 1, \quad s = m, j = 2, \\ 0, \quad s \neq m, j = 1, 2. \end{cases}$$

The proof of Lemma 6 is carried out directly by calculating the corresponding integrals. From Lemma 6 it follows that the system $\{Y_{n_1}(x); Y_{n_2}(x)\}$ is an biorthogonal adjoint to the system $\{X_{n_1}(x); X_{n_2}(x)\}$, $n \in N$.

Lemma 7. *Let $x_0 \in Q_1$. Then:*

1) The system of the root functions of the problems (17), (18) and (25)–(27) consists only of the eigenfunctions (i.e., there are no associated functions);

2) System (20) is complete and minimal;

3) Systems of eigenfunctions (20) and (28) form the Riesz basis in $L_2(0,1)$.

Note that more detailed information about the Riesz bases can be found in [24]. Before proving the lemma, we present some definitions and facts concerning Riesz bases.

Definition 1. *[24] A complete and minimal system of the functions $\{\varphi_n(x)\}$ is called a Bessel system, if for any $f \in L_2(a,b)$ the series of squared coefficients of its biorthogonal expansion $\{\varphi_n(x)\}$ converges, i.e., $f \in L_2(a,b)$ implies that $\sum\limits_{n=1}^{\infty} \left|(f,\psi_n)_{L_2(a,b)}\right|^2 < \infty$, where $\{\psi_n(x)\}$ is the system of associated functions.*

Definition 2. *[24] A complete and minimal system of functions $\{\varphi_n(x)\}$ is called a Hilbert system, if for any sequence of numbers c_n, such that $\sum\limits_{k=1}^{\infty} c_n^2 < \infty$, there exists one and only one $f \in L_2(a,b)$, for which these c_n are the coefficients of its biorthogonal expansion in $\{\varphi_n(x)\}$, i.e.,*

$$c_n = (f,\psi_n)_{L_2(a,b)}, \quad n = 1,2,\ldots.$$

Definition 3. *[24] A complete and minimal system is called a Riesz basis, if it is both Bessel and Hilbert system at the same time.*

Proof. The proof of part 1) follows from [22] and also from Lemma 1. The completeness of system (20) was proved in [22], and the minimality follows from Lemma 6. Thus, systems (20) and (28) satisfy all three conditions A from [23]. Let us now verify the fulfillment of the conditions for the basis property of the main theorem from [23]. As follows from this theorem, for this it suffices to prove that the following two conditions are satisfied:

a) $\sum\limits_{\lambda \leqslant |\lambda_{k_i}| \leqslant \lambda+1} 1 \leqslant C_1, i = 1,2,$ for any real $\lambda \geqslant 0$,

b) $\|X_{k_i}\|_0 \cdot \|Y_{k_i}\|_0 \leqslant C_2, i = 1, 2$, for all numbers k.

The fulfillment of the first condition is verified directly. To check the second condition, we calculate the corresponding norms:

$$\|X_{n_1}(x)\|_0 = \sqrt{\frac{1}{2} - \frac{\sin 2\sqrt{\lambda_{n_1}}}{2\sqrt{\lambda_{n_1}}}}, \|X_{n_2}(x)\|_0 = \sqrt{\frac{1}{2} - \frac{\sin 2\sqrt{\lambda_{n_2}}x}{2\sqrt{\lambda_{n_1}}}},$$

$$\|Y_{n_1}(x)\|_0 = \frac{2}{1+x_0}\sqrt{2x_0 + \frac{1-x_0}{2\cos^2\sqrt{\lambda_{n_1}}} + \frac{tg\sqrt{\lambda_{n_1}}}{\sqrt{\lambda_{n_1}}}},$$

$$\|Y_{n_2}(x)\|_0 = \frac{1}{|\cos\lambda_{n_2}|}\sqrt{\frac{2}{1-x_0}},$$

where $n \in N$.

Then, since x_0 is a fixed rational number, the fulfillment of condition b) follows from Lemma 5, as well as from the finiteness of the set of values $\cos\lambda_{n_1}$ and $\cos\lambda_{n_2}$. Thus, systems (20) and (28) satisfy all conditions of the main theorem from [23]. Hence, it follows that these systems form a Riesz basis. Lemma 7 is proved. □

The spectral questions of problems (17), (18) and (25)–(27) are studied in the same way for values x_0 from Q_2. Note that in this case, these problems, in addition to eigenfunctions, also have associated functions that correspond to those eigenvalues, whose serial numbers are determined by formulae (24).

4. Existence and uniqueness of a solution to Problem *BS*.

Now we turn to the study of the existence and uniqueness of the solution of problem BS. According to the theory, we will seek the solution $u(x,t)$, $g(x)$ of problem in the form of an expansion in a specially chosen basis from the system of functions $\{X_{n_i}\}$, $i = 1, 2$ from (20):

$$u(x,t) = \sum_{n=1}^{\infty} (u_{n_1}(t) \cdot X_{n_1}(x) + u_{n_2}(t) \cdot X_{n_2}(x)), \tag{29}$$

$$g(x) = \sum_{n=1}^{\infty} (g_{n_1} \cdot X_{n_1}(x) + g_{n_2} \cdot X_{n_2}(x)), \tag{30}$$

where $u_{n_1}(t), u_{n_2}(t)$ are unknown functions, g_{n_1}, g_{n_2} are unknown constants, $n \in N$.

Substituting (29) and (30) into equation (1), we obtain the following equations for finding the functions $u_{n_1}(t), u_{n_2}(t)$ and constants g_{n_1}, g_{n_2} :

$$t^{-\beta}{}_{HC}D^{\alpha}_{0t}u_{n_i}(t) + \lambda_{n_i}u_{n_i}(t) = g_{n_i}, i = 1, 2. \tag{31}$$

From representation (29), taking into account conditions (3) and the completeness of system (20), we obtain that the unknown functions $u_{n_1}(t)$, $u_{n_2}(t)$ satisfy the conditions

$$u_{n_i}(0) = \varphi_{n_i}, \quad i = 1, 2, \tag{32}$$

where $\varphi_{n_1}, \varphi_{n_2}$ are expansion coefficients of the function $\varphi(x)$ in terms of the system of functions (20), which are found by the formulas

$$\varphi_{n_i} = \int\limits_0^1 \varphi(x)Y_{n_i}(x)dx, \tag{33}$$

and $Y_{n_i}(x), i = 1, 2$, are functions defined by formulas (28).

Applying the operator $B_{\alpha}^{-\beta}$ to both parts of equation (31), taking into account formulas (15) and (16), we find that the solution to equation (31) that satisfies the condition (32) has the form

$$u_{n_i}(t) = \varphi_{n_i} \cdot L_{\alpha}\Big(-\frac{\lambda_{n_i}t^{\beta}}{\beta^{\alpha}}\Big) + \frac{g_{n_i}}{\lambda_{n_i}}\Big[1 - L_{\alpha}\Big(-\frac{\lambda_{n_i}t^{\beta}}{\beta^{\alpha}}\Big)\Big], \; i = 1, 2, \tag{34}$$

where $L_{\alpha}(z)$ is the Le-Roy function, which has the form (14).

Now we find the unknown constants g_{n_1},g_{n_2}. To do this, we expand the function $\psi(x)$ into a series in terms of the system of functions (20):

$$\psi(x) = \sum_{n=1}^{\infty} (\psi_{n_1} \cdot X_{n_1}(x) + \psi_{n_2} \cdot X_{n_2}(x)),$$

where ψ_{n_1}, ψ_{n_2} are expansion coefficients, i.e.,

$$\psi_{n_i} = \int\limits_0^1 \psi(x)Y_{n_i}(x)dx, \; i = 1, 2. \tag{35}$$

Further, from representation (29), taking into account condition (4), we obtain that the unknown functions $u_{n_i}(t)$, $i = 1, 2$, also satisfy the conditions

$$u_{n_i}(T) = \psi_{n_i}, \; i = 1, 2. \tag{36}$$

Then, taking into account the conditions (36) and using (34) to find the constants g_{n_1},g_{n_2}, we obtain the following equations:

$$\varphi_{n_i} \cdot L_\alpha\Big(-\frac{\lambda_{n_i}T^\beta}{\beta^\alpha}\Big) + \frac{g_{n_i}}{\lambda_{n_i}}\Big[1 - L_\alpha\Big(-\frac{\lambda_{n_i}T^\beta}{\beta^\alpha}\Big)\Big] = \psi_{n_i},\ i = 1, 2.$$

From here we find

$$g_{n_i} = \frac{\lambda_{n_i}\Big(\psi_{n_i} - \varphi_{n_i} L_\alpha\Big(-\frac{\lambda_{n_i}T^\beta}{\beta^\alpha}\Big)\Big)}{1 - L_\alpha\Big(-\frac{\lambda_{n_i}T^\beta}{\beta^\alpha}\Big)},\ i = 1, 2. \tag{37}$$

Substituting all this into the expression for functions $u_{n1}(t), u_{n2}(t)$, we obtain

$$u_{n_i}(t) = \frac{L_\alpha\Big(-\frac{\lambda_{n_i}t^\beta}{\beta^\alpha}\Big) - L_\alpha\Big(-\frac{\lambda_{n_i}T^\beta}{\beta^\alpha}\Big)}{1 - L_\alpha\Big(-\frac{\lambda_{n_i}T^\beta}{\beta^\alpha}\Big)}\varphi_{n_i} + \frac{1 - L_\alpha\Big(-\frac{\lambda_{n_i}t^\beta}{\beta^\alpha}\Big)}{1 - L_\alpha\Big(-\frac{\lambda_{n_i}T^\beta}{\beta^\alpha}\Big)}\psi_{n_i},\ i = 1, 2. \tag{38}$$

Thus, a formal solution of the problem is found in the form of series (29) and (30), where the coefficients g_{n_1},g_{n_2} and functions $u_{n_1}(t)$,$u_{n_2}(t)$ are determined, respectively, by formulas (37) and (38).

Now we will consider the proof of theorems on the existence and uniqueness of a solution to the BS problem.

5. The uniqueness of a solution to Problem *BS*. In proving the uniqueness and existence of a solution to problem BS, we use the following estimate for the Le-Roy function $L_\alpha(z)$ obtained in [21].

Lemma 8. *Let $0 < \alpha < \beta < 1$. Then for any $z \geqslant 0$ the estimate is valid:*

$$e^{-z} \leqslant L_\beta(-z) < L_\alpha(-z) \leqslant \frac{1}{1+z}.$$

From here it is easy to obtain the following estimate:

Corollary 1. *For any $z > \varepsilon > 0$ it holds that*

$$\frac{1}{1 - L_\alpha(-z)} \leqslant \frac{z+1}{z} \leqslant C < \infty.$$

Theorem 1. *If a solution to the BS problem exists, it is unique.*

Proof. Let us assume the opposite. Let there be two solutions $\{u_1(x,t), g_1(x)\}$ and $\{u_2(x,t)$,$g_2(x)\}$ to the BS problem.

Introduce the notation $\tilde{u}(x,t) = u_1(x,t) - u_2(x,t)$ and $\tilde{g}(x) = g_1(x) - g_2(x)$. Then the functions $\tilde{u}(x,t)$ and $\tilde{g}(x)$ satisfy the equation

$$t^{-\beta}{}_{HC}D_{0t}^{\alpha}\tilde{u} = \tilde{u}_{xx} + \tilde{g}(x) \tag{39}$$

and conditions

$$\tilde{u}(x,0) = 0,\ \tilde{u}(x,T) = 0,\ 0 \leqslant x \leqslant 1, \tag{40}$$

$$\tilde{u}(0,t) = 0,\ \tilde{u}(1,t) = \tilde{u}(x_0,t),\ \ 0 \leqslant t \leqslant T. \tag{41}$$

Consider the functions

$$\tilde{u}_{n_i}(t) = \int\limits_0^1 \tilde{u}(x,t)Y_{n_i}(x)dx,\ i = 1,2, \tag{42}$$

where functions $Y_{n_i}(x), i = 1,2$ are determined by formulas (28).

Applying operator $B_{\alpha}^{-\beta}$ to both parts of equality (42) and taking into account equation (39), as well as conditions (40), (41), we conclude that the function $\tilde{u}_{n_i}(t)$ and constant $\tilde{g}_{n_i}$ satisfy the following equation and conditions:

$$t^{-\beta}{}_{HC}D_{0t}^{\alpha}\tilde{u}_{n_i}(t) + \lambda_{n_i}\tilde{u}_{n_i}(t) = \tilde{g}_{n_i},\ \tilde{u}_{n_i}(0) = 0,\ \ \tilde{u}_{n_i}(T) = 0, \tag{43}$$

where

$$\tilde{g}_{n_i} = (\tilde{g}(x), Y_{n_i}(x))_0.$$

From (31) and (34), we obtain that the solution of this equation, which satisfies the first boundary condition in (43)), has the form

$$\tilde{u}_{n_i}(t) = \frac{\tilde{g}_{n_i}}{\lambda_{n_i}}\left[1 - L_{\alpha}\left(-\frac{\lambda_{n_i}t^{\beta}}{\beta^{\alpha}}\right)\right],\ i = 1,2.$$

Hence, satisfying the second boundary condition in (43), we obtain

$$\frac{\tilde{g}_{n_i}}{\lambda_{n_i}}\left[1 - L_{\alpha}\left(-\frac{\lambda_{n_i}T^{\beta}}{\beta^{\alpha}}\right)\right] = 0, i = 1,2.$$

As $L_{\alpha}(0) = 1, \lambda_{n_i} \geqslant \lambda_{1_i} > 0$, we get $L_{\alpha}\left(-\frac{\lambda_{n_i}T^{\beta}}{\beta^{\alpha}}\right) \neq 1$. Then it follows that $\tilde{u}_{n_i}(t) = 0, \tilde{g}_{n_i} = 0$. Consequently, problem (43) has only a trivial solution.

As a result, we obtain that for any fixed $t \in [0,T]$ functions $\tilde{u}(x,t)$, $\tilde{g}(x)$ are orthogonal to system (28), which is complete in $L_2(0,1)$. Then $\tilde{u}(x,t) = 0$, $\tilde{g}(x) = 0$ almost everywhere in Ω and $[0,1]$, respectively. Since $u \in C(\bar{\Omega})$, $f(x) \in C[0,1]$, from here we get that $\tilde{u}(x,t) \equiv 0$, $\tilde{g}(x) \equiv 0$. The uniqueness of the solution of problem is proved. □

6. The existence of a solution to Problem *BS*. Let us prove the existence of a solution to the problem.

Theorem 2. *Let the functions $\varphi(x)$, $\psi(x)$ satisfy the conditions*

$$\varphi(x) \in C^4[0,1],\ \varphi(0) = \varphi''(0) = 0,\ \varphi(1) = \varphi(x_0),\ \varphi''(1) = \varphi''(x_0),$$

$$\psi(x) \in C^4[0,1],\ \psi(0) = \psi''(0) = 0,\ \psi(1) = \psi(x_0),\ \psi''(1) = \psi''(x_0).$$

Then a solution to problem BS exists.

Proof. Since the system (20) forms the Riesz basis in the space $L_2(0,1)$, the functions $u(x,t)$ and $g(x)$ can be represented in the form (29) and (30), where the coefficients g_{n_1},g_{n_2} and functions $u_{n_1}(t)$, $u_{n_2}(t)$ are determined, respectively, by formulas (37), (38).

It is easy to show by direct calculation that the functions $u(x,t)$ and $g(x)$, defined by series (29) and (30), satisfy equation (1) and conditions (3)–(6). It remains to show that the functions $u(x,t)$ and $g(x)$ are from class (2).

Let us show that $u_{xx} \in C(\overline{\Omega})$. From (29), differentiating twice with respect to the variable x, we obtain

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} (-\lambda_{n_1} \cdot u_{n_1}(t) X_{n_1}(x) - \lambda_{n_2} \cdot u_{n_2}(t) X_{n_2}(x)). \tag{44}$$

As $|X_{n_i}(x)| \leqslant 1$, $i = 1,2$, it follows that

$$|u_{xx}(x,t)| \leqslant \sum_{n=1}^{\infty} (\lambda_{n_1} |u_{n_1}(t)| + \lambda_{n_2} |u_{n_2}(t)|). \tag{45}$$

Let us estimate the functions $u_{n_1}(t)$ and $u_{n_2}(t)$. Taking Lemma 1 and Corollary 1 into account, from (38) we get

$$|u_{n_i}(t)| \leqslant (|\varphi_{n_i}| + |\psi_{n_i}|)\,,\ i = 1,2. \tag{46}$$

Here and below C is positive (in general, different) constant.

Further, taking into account (46) and the conditions of Theorem 2, integrating by parts the expressions for the coefficients φ_{n_i}, ψ_{n_i}, $i = 1,2$ in (33), (35), we obtain from (45)

$$|u_{xx}(x,t)| \leqslant \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_{n_1}} \left(\left|\varphi_{n_1}^{(4)}\right| + \left|\psi_{n_1}^{(4)}\right|\right) + \frac{1}{\lambda_{n_2}} \left(\left|\varphi_{n_2}^{(4)}\right| + \left|\psi_{n_2}^{(4)}\right|\right) \right),$$

where

$$\varphi_{n_i}^{(4)} = \frac{1}{\lambda_{n_i}^2} \int_0^1 \varphi^{IV}(x) Y_{n_i}(x) dx, \psi_{n_i}^{(4)} = \frac{1}{\lambda_{n_i}^2} \int_0^1 \psi^{IV}(x) Y_{n_i}(x) dx, \; i = 1,2.$$

Thus, the series (44) is majorized by the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\left|\varphi_{n_1}^{(4)}\right| + \left|\psi_{n_1}^{(4)}\right| + \left|\varphi_{n_2}^{(4)}\right| + \left|\psi_{n_2}^{(4)}\right|\right),$$

whose convergence follows from the Cauchy-Schwarz inequality, as well as from the convergence of the series $\sum\limits_{n=1}^{\infty} |\varphi_{n_i}^{(4)}|^2$, $\sum\limits_{n=1}^{\infty} |\psi_{n_i}^{(4)}|^2$, $i = 1,2$.

Then, according to the Weierstrass theorem [25], the series (44) converges absolutely and uniformly in the domain $\bar{\Omega}$, and its sum is a continuous function in this domain. In the same way, it is shown that $t^{-\beta}{}_{HC}D_{0t}^{\alpha} u(x,t) \in C(\overline{\Omega})$, and $g(x) \in C[0,1]$ follows from the fact that $t^{-\beta}{}_{HC}D_{0t}^{\alpha} u(x,t) \in C(\overline{\Omega})$, and from equation (1). $\square$

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