Probl. Anal. Issues Anal. Vol. 13 (31), No 1, 2024, pp. 37–49
DOI: 10.15393/j3.art.2024.14430

UDC 517.55

A. M. Kytmanov, O. V. Khodos

ON THE ROOTS OF SYSTEMS OF TRANSCENDENTAL EQUATIONS

Abstract. The article is devoted to investigation of simple and multiple roots of systems of transcendental equations. It is shown that the number is related to the number of real roots of the resultant of the system. Examples for systems of equations are given.

Key words: *systems of transcendental equations, resultant, simple root, multiple root*

2020 Mathematical Subject Classification: *32A05, 32A15, 32A27*

1. Introduction. Consider a system of transcendental equations of the form

$$\begin{cases} f_1(z) = 0, \\ \dots \\ f_n(z) = 0, \end{cases} \tag{1}$$

where $f_1(z), \dots, f_n(z)$ are entire functions of complex variables $z = (z_1, \dots, z_n)$ in $\mathbb{C}^n$.

Systems of transcendental equations arise in various fields of knowledge, for example, in chemical kinetics [4]. As a rule, the number of roots in such systems is infinite. In what follows, we will assume that the set of roots of the system (1) is discrete. Therefore, it is at most countable.

This article considers the following problem: suppose that the first coordinates of the roots of the system are found. How to find the remaining coordinates?

For systems of algebraic equations, the classical method of elimination reduces the system to a triangular form. The method is based on Sylvester's resultant theory. Having the first coordinates of all the roots, we substitute them into the previous equations, and again we get polynomials in a smaller number of variables. Solving them, we find the second coordinates of the roots, and so on. In fact, we are talking about finding several systems of resultants, which is quite difficult.

This can be done in a different way. The article [1] provides a method for finding other coordinates of the roots, based on the introduction of auxiliary functions and the calculation of power sums of roots based on the theory of multidimensional residues. For systems of transcendental equations of the form (1), we apply similar reasoning, which uses power sums to a negative degree and their number is infinite.

This method simplifies the whole procedure. For example, in the case when the roots of the system are simple, to determine another coordinate, it is not necessary to find the resultant from other variables. For algebraic systems, this method is described in [1].

Let $\mathcal{E}$ denote the set of roots with non-zero coordinates $w_{(\nu)} = \left(w_{1(\nu)}, \ldots, w_{n(\nu)}\right)$, $\nu = 1, 2, \ldots$, numbered in increasing order of modules: $|w_{(1)}| \leqslant |w_{(2)}| \leqslant \ldots \leqslant |w_{(\nu)}| \leqslant \ldots$.

Consider power sums S_α of roots of $\mathcal{E}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a non-negative multi-index (all components are non-negative and integer) and $\alpha_1 + \ldots + \alpha_n > 0$, of the form

$$S_\alpha = \sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}^{\alpha_1} \cdot w_{2(\nu)}^{\alpha_2} \cdots w_{n(\nu)}^{\alpha_n}}.$$

Lemma 1. *The series S_α converges absolutely for any multi-indices α if and only if the series*

$$\sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}}, \ \ldots, \ \sum_{\nu=1}^{\infty} \frac{1}{w_{n(\nu)}}$$

absolutely converges.

Proof. Indeed, the series

$$\Big| \sum_{\nu=1}^{\infty} \frac{1}{w_{1(\nu)}^{\alpha_1} \cdot w_{2(\nu)}^{\alpha_2} \cdot \ldots \cdot w_{n(\nu)}^{\alpha_n}} \Big| \leqslant \sum_{\nu=1}^{\infty} \frac{1}{|w_{1(\nu)}^{\alpha_1} \cdot w_{2(\nu)}^{\alpha_2} \cdot \ldots \cdot w_{n(\nu)}^{\alpha_n}|} \leqslant$$

$$\leqslant \Big(\sum_{\nu=1}^{\infty} \frac{1}{|w_{1(\nu)}|} \Big)^{\alpha_1} \cdot \Big(\sum_{\nu=1}^{\infty} \frac{1}{|w_{2(\nu)}|} \Big)^{\alpha_2} \cdot \ldots \cdot \Big(\sum_{\nu=1}^{\infty} \frac{1}{|w_{n(\nu)}|} \Big)^{\alpha_n}$$

converges as a product of convergent series.

Obviously, the opposite is also true. □

The concept of power sums for transcendental systems of equations was considered in [6], [8], [12], [13], [14]. The results of these articles were based on the calculation of power sums through the so-called residue integrals [17].

Hence, an entire function of zero genus is defined [16, chapter 7] as

$$R\left(z_1\right)=z_1^s \cdot \prod_{\eta=1}^{\infty}\left(1-\frac{z_1}{w_{1(\eta)}}\right), \tag{2}$$

where s is the multiplicity of the zero of the system (1) at zero, $s \geqslant 0$. In formula (2), the infinite product converges absolutely and uniformly on the complex plane $\mathbb{C}$.

The function $R(z_1)$ is called the *resultant* of the system (1) with respect to the variable z_1. The concept of a resultant for systems of transcendental equations is not generally accepted. For the first time, for the case of two equations, a similar concept of the resultant was introduced by N.G. Chebotarev [5] (pp. 18–27). In the recent years, this concept was considered in the works [8], [11], [14], [15]. The results of these articles were based on the calculation of power sums S_α through the so-called residue integrals [17]. See also monographs [3], [9].

2. Auxiliary results In what follows, we assume that our system (1) satisfies the conditions of Lemma 1.

We consider the cases when the roots can be simple or multiple.

Let $z_{1(\mu)}$ denote distinct nonzero roots of the resultant, $\mu=1,\ldots,$ and let each root $z_{1(\mu)}$ have multiplicity $r_\mu \geqslant 1$. The multiplicity of the root for holomorphic functions is always finite. Then the resultant looks like

$$R\left(z_1\right)=z_1^s \cdot \prod_{\mu=1}^{\infty}\left(1-\frac{z_1}{z_{1(\mu)}}\right)^{r_\mu}=\sum_{\alpha=s}^{\infty} b_\alpha z_1^\alpha. \tag{3}$$

We introduce the functions

$$\begin{aligned}
&P_j^{(t)}\left(z_1\right)= \\
&=-z_1^{s-1} \cdot \sum_{\mu=1}^{\infty}\left(\frac{1}{z_{j(\mu,1)}^t}+\cdots+\frac{1}{z_{j(\mu,r_\mu)}^t}\right) \times\left(1-\frac{z_1}{z_{1(\mu)}}\right)^{r_\mu-1} \cdot \prod_{\eta \neq \mu}\left(1-\frac{z_1}{z_{1(\eta)}}\right)^{r_\eta}= \\
&\qquad=\sum_{\beta=1}^{\infty} a_{j\beta}^{(t)} z_1^\beta, \quad t,\ s \text{ are integer}, \quad t \geqslant 0, \quad s \geqslant 1,
\end{aligned} \tag{4}$$

where $z_{j(\mu,1)}, \ldots, z_{j(\mu,r_\mu)}$ are the j-th coordinates of roots with first coordinates equal to $z_{1(\mu)}$.

Lemma 2. *The functions (4) are entire functions in the variable z_1.*

Proof.

Write the functions $P_j^{(t)}(z_1)$ as

$$P_j^{(t)}(z_1) =$$
$$= -z_1^{s-1} \cdot \prod_{\eta=1}^{\infty} \left(1 - \frac{z_1}{z_{1(\eta)}}\right) \times \sum_{\mu=1}^{\infty} \left(\left(\frac{1}{z^t_{j(\mu,1)}} + \ldots + \frac{1}{z^t_{j(\mu,r_\mu)}} \right) \cdot \frac{1}{1 - \frac{z_1}{z_{1(\mu)}}} \right).$$

The infinite product $\prod\limits_{\eta=1}^{\infty} \left(1 - \dfrac{z_1}{z_{1(\eta)}}\right)$ is an entire function of zero genus.

Let us prove that the series

$$\sum_{\mu=1}^{\infty} \left(\left(\frac{1}{z^t_{j(\mu,1)}} + \cdots + \frac{1}{z^t_{j(\mu,r_\mu)}} \right) \cdot \frac{1}{1 - \frac{z_1}{z_{1(\mu)}}} \right)$$

converges absolutely and uniformly on the complex plane $\mathbb{C}$.

We have

$$\left| \frac{1}{z^t_{j(\mu,1)}} + \cdots + \frac{1}{z^t_{j(\mu,r_\mu)}} \right| \leqslant \sum_{\mu=1}^{\infty} \left| \frac{1}{z^t_{j(\mu)}} \right|,$$

and this series converges by Lemma 1.

Let us prove that the series

$$\sum_{\nu=1}^{\infty} \frac{1}{z^t_{j(\nu)}} \cdot \frac{1}{1 - \dfrac{z_1}{z_{1(\nu)}}}$$

converges absolutely and uniformly on the complex plane $\mathbb{C}$.

By Lemma 1, the series $\sum\limits_{\nu=1}^{\infty} \dfrac{1}{|z_{1(\nu)}|}$ converges. This means

$$\lim_{\nu \to \infty} \frac{1}{|z_{1(\nu)}|} = 0,$$

and, therefore,

$$\lim_{\nu \to \infty} \left(1 - \frac{z_1}{z_{1(\nu)}}\right) = 1.$$

Since $1-\dfrac{z_1}{z_{1(\nu)}}$ is close in absolute value to one, we can assume that

$$\sum_{\nu=1}^{\infty}\left|\frac{1}{z^t_{j(\nu)}}\cdot\frac{1}{1-\frac{z_1}{z_{1(\nu)}}}\right|\leqslant 2\cdot\sum_{\nu=1}^{\infty}\frac{1}{\left|z^t_{j(\nu)}\right|}.$$

Whence it follows that the series $\sum\limits_{\nu=1}^{\infty}\dfrac{1}{z^t_{j(\nu)}}\cdot\dfrac{1}{1-\dfrac{z_1}{z_{1(\nu)}}}$ converges absolutely and uniformly on the complex plane $\mathbb{C}$.

This proves that the functions $P_j^{(t)}(z_1)$ are entire functions of the variable z_1. $\square$

Consider

$$\left.\frac{d^{r_\nu}R(z_1)}{dz_1^{r_\nu}}\right|_{z_1=z_{1(\nu)}}=(-1)^{r_\nu}\cdot r_\nu!\cdot z_{1(\nu)}^{s-r_\nu}\cdot\prod_{\mu\neq\nu}\left(1-\frac{z_{1(\nu)}}{z_{1(\mu)}}\right)^{r_\mu}.$$

Next, calculate

$$\left.\frac{d^{r_\nu-1}P_j^{(t)}(z_1)}{dz_1^{r_\nu-1}}\right|_{z_1=z_{1(\nu)}}=$$
$$=(-1)^{r_\nu}\cdot\left(\frac{1}{z^t_{j(\nu,1)}}+\cdots+\frac{1}{z^t_{j(\nu,r_\nu)}}\right)(r_\nu-1)!\cdot z_{1(\nu)}^{s-r_\nu}\cdot\prod_{\mu\neq\nu}\left(1-\frac{z_{1(\nu)}}{z_{1(\mu)}}\right)^{r_\mu}.$$

Thus, we have proved the following statement:

Lemma 3. *The equality*

$$\left.\frac{d^{r_\nu-1}P_j^{(t)}(z_1)}{dz_1^{r_\nu-1}}\middle/ r_\nu R(z_1)dz_1^{r_\nu}\right|_{z_1=z_{1(\nu)}}=\frac{1}{r_\nu}\left(\frac{1}{z^t_{j(\nu,1)}}+\cdots+\frac{1}{z^t_{j(\nu,r_\nu)}}\right)\tag{5}$$

is valid.

The relations (5) allow us to write out the inverse power sums of the j-th coordinates of the roots $z_{j(\nu,1)}$, …, $z_{j(\nu,r_\nu)}$ if $R(z_1)$ and $P_j^{(t)}(z_1)$ are known. Substituting $t=1,\dots,r_\nu$, we compute the power sums

$$\frac{1}{z^t_{j(\nu,1)}}+\cdots+\frac{1}{z^t_{j(\nu,r_\nu)}},\qquad t=1,\dots,r_\nu.$$

Using Newton's recurrent formulas (see, for example, [2, 7]), we find the polynomial P with roots $\dfrac{1}{z_{j(\nu,1)}},\dots,\dfrac{1}{z_{j(\nu,r_\nu)}}$. Therefore, we can find the roots $z_{j(\nu,1)},\dots,z_{j(\nu,r_\nu)}$.

3. Main result. Let us show that the coefficients of functions $P_j^{(t)}(z_1)$ can be expressed in terms of coefficients $R(z_1)$ and power sums. To do this, consider the auxiliary system of functions:

$$\varphi_j^{(t)}(\lambda) =$$
$$= -\lambda^{s-1}\cdot\sum_{\mu=1}^{\infty}\Big(\frac{1}{z^t_{j(\mu,1)}}+\dots+\frac{1}{z^t_{j(\mu,r_\mu)}}\Big)\cdot\frac{1}{1-\frac{\lambda}{z_{1(\mu)}}}\cdot\prod_{\eta=1}^{\infty}\Big(1-\frac{\lambda}{z_{1(\eta)}}\Big)^{r_\eta}, t\geqslant 0,\, s\geqslant 1.$$

After reduction:

$$\varphi_j^{(t)}(\lambda) =$$
$$= -\lambda^{s-1}\cdot\sum_{\mu=1}^{\infty}\Big(\frac{1}{z^t_{j(\mu,1)}}+\dots+\frac{1}{z^t_{j(\mu,r_\mu)}}\Big)\cdot\Big(1-\frac{\lambda}{z_{1(\mu)}}\Big)^{r_\mu-1}\cdot\prod_{\eta\neq\mu}\Big(1-\frac{\lambda}{z_{1(\eta)}}\Big)^{r_\eta} =$$
$$= -\lambda^{s-1}\cdot\sum_{m=0}^{\infty} a^{(t)}_{jm}\cdot\lambda^m, \qquad a^{(t)}_{j0}=1.$$

Using the geometric series formula for sufficiently small $|\lambda|$ and the formula (3) for resultant, we have:

$$\varphi_j^{(t)}(\lambda) = -\lambda^{s-1}\cdot\sum_{\mu=1}^{\infty}\Big(\frac{1}{z^t_{j(\mu,1)}}+\dots+\frac{1}{z^t_{j(\mu,r_\mu)}}\Big)\cdot\sum_{m=0}^{\infty}\Big(\frac{\lambda}{z_{1(\mu)}}\Big)^m\cdot\prod_{\eta=1}^{\infty}\Big(1-\frac{\lambda}{z_{1(\eta)}}\Big)^{r_\eta} =$$
$$= -\lambda^{s-1}\cdot\sum_{m=0}^{\infty}\lambda^m\cdot\Big(\sum_{\mu=1}^{\infty}\frac{1}{z^m_{1(\mu)}}\cdot\Big(\frac{1}{z^t_{j(\mu,1)}}+\dots+\frac{1}{z^t_{j(\mu,r_\mu)}}\Big)\Big)\cdot\prod_{\eta=1}^{\infty}\Big(1-\frac{\lambda}{z_{1(\eta)}}\Big)^{r_\eta} =$$
$$= -\frac{1}{\lambda}\cdot\Big(\sum_{m=0}^{\infty} S_{me_1+te_j}\cdot\lambda^m\Big)\cdot\Big(\sum_{k=s}^{\infty} b_k\cdot\lambda^k\Big) =$$
$$= -\frac{1}{\lambda}\cdot\sum_{l=s}^{\infty}\lambda^l\cdot\Big(\sum_{m+k=l} S_{me_1+te_j}\cdot b_k\Big) = \lambda^{s-1}\cdot\sum_{l=s}^{\infty}\lambda^{l-s}\cdot\Big(\sum_{m+k=l} S_{me_1+te_j}\cdot b_k\Big).$$

Denote $l-s=p$; then

$$\varphi_j^{(t)}(\lambda) = -\lambda^{s-1}\cdot\sum_{p=0}^{\infty}\lambda^p\cdot\Big(\sum_{m+k=p+s} S_{me_1+te_j}\cdot b_k\Big),$$

where $S_{me_1+te_j} = \sum\limits_{\nu=1}^{\infty} \dfrac{1}{z_{1(\nu)}^m \cdot z_{j(\nu)}^t}$ are power sums for the multi-index $me_1 + te_j = (m, 0, \ldots, 0, t, 0, \ldots, 0)$, the first component of the multi-index is equal to m, the j-th component is equal to t, and the other components are zeros.

We get the expressions for $a_{jp}^{(t)}$:

$$a_{jp}^{(t)} = \sum_{m+k=p+s} S_{me_1+te_j} \cdot b_k, \tag{6}$$

where $p = l - s$, $s \geqslant 1$, $l = s, s+1, \ldots, b_0 = 1$, $t \geqslant 0$, $m \geqslant 0$, $k \geqslant 0$.

The power sums $S_{me_1+te_j}$ are found differently for different types of systems of equations [8], [12], [13], [14]. Thus, we only need to know the coefficients of the resultant b_k, and the coefficients $a_{jp}^{(t)}$ are found using the formulas (6).

Thus, the following theorem is proved:

Theorem 1. *Assume that the first coordinates of the system roots (9) are known. Then, by formula (5), one can obtain power sums of j-th coordinates of the roots, having known the first coordinates. In this way, the problem of finding j-th coordinates is reduced to finding the roots of the polynomial in one variable, whose degree is equal to the number of roots of the system (9) with the given first coordinates. Moreover, the coefficients of the polynomials $P_j^{(t)}$ are found by formula (6), and the power sums $S_{me_1+te_j}$ are found in a way depending on the type of system (9) (see [8], [12], [13], [14]).*

4. Examples.

Example 1. Consider the system of equations

$$\begin{cases} z_1 + a_2 z_2^2 + b_2 z_1 z_2 + c_2 z_1 z_2^2 = 0, \\ z_2 + a_1 z_1 z_2 + b_1 z_1^2 + c_1 z_1^2 z_2 = 0. \end{cases} \tag{7}$$

Let us find the resultant of this system with respect to the variable z_1 using any method, for example, the Sylvester resultant [7]. Write down the system (7) with respect to the variable z_2:

$$\begin{cases} z_2 \left(1 + a_1 z_1 + c_1 z_1^2\right) + b_1 z_1^2 = 0, \\ z_2^2 \left(a_2 + c_2 z_1\right) + z_2 b_2 z_1 + z_1 = 0. \end{cases}$$

As a resultant, we take the determinant

$$R(z_1) = \begin{vmatrix} 1 + a_1 z_1 + c_1 z_1^2 & b_1 z_1^2 & 0 \\ 0 & 1 + a_1 z_1 + c_1 z_1^2 & b_1 z_1^2 \\ a_2 + c_2 z_1 & b_2 z_1 & z_1 \end{vmatrix} =$$
$$= z_1 \cdot \left[z_1^4 \left(c_1^2 + b_1^2 c_2 - b_1 b_2 c_1 \right) + z_1^3 \left(2a_1 c_1 + a_2 b_1^2 - a_1 b_1 b_2 \right) + \right.$$
$$\left. + z_1^2 \left(a_1^2 + 2c_1 - b_1 b_2 \right) + z_1 2a_1 + 1 \right].$$

So, the system (7) has 5 roots.

To determine the multiple roots of the resultant, we need to find its discriminant (see, for example, [7]).

Consider the polynomial

$$Q(z_1) = d_0 z_1^4 + d_1 z_1^3 + d_2 z_1^2 + d_3 z_1 + d_4 =$$
$$= z_1^4 \left(c_1^2 + b_1^2 c_2 - b_1 b_2 c_1 \right) + z_1^3 \left(2a_1 c_1 + a_2 b_1^2 - a_1 b_1 b_2 \right) +$$
$$+ z_1^2 \left(a_1^2 + 2c_1 - b_1 b_2 \right) + z_1 2a_1 + 1.$$

The discriminant of this polynomial $Q(z_1)$ is

$$D(Q) = d_0^6 \cdot \begin{vmatrix} 4 & S_1 & S_2 & S_3 \\ S_1 & S_2 & S_3 & S_4 \\ S_2 & S_3 & S_4 & S_5 \\ S_3 & S_4 & S_5 & S_6 \end{vmatrix},$$

where S_j are the power sums of the roots of the polynomial $Q(z_1)$.

These power sums can be found using Newton's recurrent formulas [7]:

$$S_j d_0 + S_{j-1} d_1 + S_{j-2} d_2 + \cdots + S_1 d_{j-1} + j d_j = 0, \quad \text{if } 1 \leqslant j \leqslant 4,$$
$$\text{and} \quad S_j d_0 + S_{j-1} d_1 + \cdots + S_{j-4} d_4 = 0, \quad \text{if } j > 4.$$

Thus, if $D(Q) = 0$, then the resultant has multiple roots. If $D(Q) \neq 0$, then the resultant has no multiple roots. It is not difficult to show that discriminant $D(Q)$ is not equal to 0 indentically.

Consider the case when $D(Q) \neq 0$, i.e., when the resultant has no multiple roots.

Write our system (7) in the form

$$f_1 = z_1 + Q_1 = 0, \qquad f_2 = z_2 + Q_2 = 0,$$

where $Q_1 = a_2 z_2^2 + b_2 z_1 z_2 + c_2 z_1 z_2^2$, $\quad Q_2 = a_1 z_1 z_2 + b_1 z_1^2 + c_1 z_1^2 z_2$.

Consider the residue integral

$$J_\beta = \frac{1}{(2\pi i)^2}\int\limits_{\gamma(r)} \frac{1}{z^{\beta+I}}\cdot\frac{df}{f} = \frac{1}{(2\pi i)^2}\int\limits_{\gamma(r_1,r_2)} \frac{1}{z_1^{\beta_1+1}\cdot z_2^{\beta_2+1}}\frac{df_1}{f_1}\wedge\frac{df_2}{f_2},$$

where $\gamma(r)=\{z\in\mathbb{C}^2\colon\ |z_1|=r_1, |z_2|=r_2\}$, $r_1>0$, $r_2>0$, $\beta=(\beta_1,\beta_2)$, is multi-index, $\beta_1\geqslant 0$, $\beta_2\geqslant 0$, $\beta_j\in\mathbb{Z}$, $I=(1,1)$.

According to the result from [12], the residue integral J_β is equal to

$$J_\beta = \sum_{\|\alpha\|\leqslant\|\beta\|+2} (-1)^{\|\alpha\|}\mathfrak{M}\left[\frac{\Delta\cdot Q_1^{\alpha_1}\cdot Q_2^{\alpha_2}}{z^{\beta+(\alpha_1+1)\beta^1+(\alpha_2+1)\beta^2}}\right], \tag{8}$$

where $\beta^1=(1,0)$, $\beta^2=(0,1)$, Δ is the Jacobian of the system of functions (7),

$$\begin{aligned}\Delta =& 1+a_1z_1+b_2z_2+(c_1-2b_1b_2)z_1^2+(c_2-2a_1a_2)z_2^2-4a_2b_1z_1z_2-\\ &-(b_2c_1+4b_1c_2)z_1^2z_2-(a_1c_2+4a_2c_1)z_1z_2^2-3c_1c_2z_1^2z_2^2,\end{aligned}$$

$\mathfrak{M}$ is a linear functional that associates the Laurent series (under the sign of the functional $\mathfrak{M}$) with its constant term.

Calculating by formula (8), we get

$$J_{(0,0)} = -a_1b_2-3a_2b_1.$$

According to the result from [12], this integral is equal to the power sum $S_{(1,1)}$.

Similarly, we can calculate power sums $S_{me_1+te_j}$ with any number to apply the equality (6), using, for example, computer system *Maple*. However, these power sums look very cumbersome and it makes no sense to give them here.

Example 2. Consider the model of Zel'dovich-Semenov ideal mixing reactor (see. [4, Ch. 2, Eq. (2.2.1)]. It has the form

$$\begin{cases}(1-x)e^{\frac{y}{1+\beta y}}-\dfrac{x}{Da}=\dfrac{dx}{d\tau},\\ (1-x)e^{\frac{y}{1+\beta y}}-\dfrac{y}{Se}=\gamma\dfrac{dy}{d\tau},\end{cases}$$

where β, D, S, e are positive parameters.

Denote $Da = a$, $Se = b$. The stationary states of the system satisfy the system of equations

$$\begin{cases} (1-x)e^{\frac{y}{1+\beta y}} - \dfrac{x}{a} = 0, \\ (1-x)e^{\frac{y}{1+\beta y}} - \dfrac{y}{b} = 0. \end{cases} \tag{9}$$

This system has been studied in [18]. We will look at it from the point of view of our article.

The system (9) obviously has no roots with zero coordinates.

From the system (9) we obtain $x = \dfrac{a}{b}\,y$.

To solve the system (9), we make the replacement $t = \dfrac{y}{1+\beta y}$. We get

$$\begin{cases} \Big(\dfrac{t}{b(1-\beta t)} - \dfrac{1}{a}\Big)e^t + \dfrac{t}{ab(1-\beta t)} = 0, \\ x = 1 - \dfrac{t}{b(1-\beta t)}\, e^{-t}. \end{cases}$$

Hence,

$$(at - b(1-\beta t))\, e^t + t = 0. \tag{10}$$

Let us denote

$$\psi(t) = (at - b(1-\beta t))\, e^t + t.$$

We can assume that the function $\psi(t)$ is a resultant of the variable t of the system (9). It is an entire function of first order growth of exponential type.

Let us show that it has no multiple zeros.

Let us consider the derivative

$$\psi' = (at - b(1-\beta t))\, e^t + 1 + e^t(a + b\beta) = 0,$$

The multiple roots of the function $\psi(t)$ satisfy the system

$$\begin{cases} \psi(t) = 0, \\ \psi'(t) = 0. \end{cases}$$

From the first and second equations of the system we obtain

$$-t + 1 + e^t(a + b\beta) = 0, \quad \text{i.e.,} \quad e^t = \frac{t-1}{a+b\beta}.$$

Then, from the first equation we get

$$t^2 + \frac{b}{a + \beta b}(1 - t) = 0.$$

Let us substitute the roots of this equation into the function $\psi(t)$. Simple calculations show that for almost all parameter values, the function $\psi(t)$ at these roots is not identically zero. Therefore, ψ and ψ' do not have common zeros. Consequently, the function $\psi(t)$ has simple zeros for almost all parameter values.

The number of real zeros of the function $\psi(t)$ was studied in [18]. It is shown that it has one or three real zeros. Therefore, Theorem 1 and Lemma 3 imply that under the same conditions the system (9) has one or three roots with real coordinates.

Acknowledgment. *The work was supported by the Krasnoyarsk Mathematical Center, funded by the Ministry of Education and Science of the Russian Federation as part of the activities for the creation and development of regional NSMCs (Agreement 075-02-2023-936).*

References

[1] Ajzenberg L. A., Bolotov V. A., Tsikh A. K. *On the solution of systems of nonlinear algebraic equations using the multidimensional logarithmic residue. On the solvability in radicals.* Sov. Math. Dokl., 1980, vol. 21, pp. 645-648; translation from Dokl. Akad. Nauk SSSR, 1980, vol. 252, pp. 11-14.

[2] Bourbaki N. *Algebra.* Paris: Hermann, 1961, vol. 2, 230 pp.

[3] Bykov V. I., Kytmanov A.M., Lazman M. Z., Passare M. *Elimination methods in polynomial computer algebra.* Springer Link, 1998, 244 p. DOI: https://doi.org/10.1007/978-94-011-5302-7

[4] Bykov V. I., Tsybenova S. B. *Nonlinear models of chemical kinetics.*, Moscow, KRASAND, 2011, 400 pp. (in Russian).

[5] Chebotarev N. G. *Collected works.* Moscow-Leningrad, Academy of Sciences of the USSR, 1949, vol. 2 (in Russian).

[6] Kulikov V. R., Stepanenko V.A. *On solutions and Waring formulas for systems of n algebraic equations in n unknowns.* Algebra i Analiz, 2014, vol. 26, no. 5, pp. 200 – 214 (in Russian). DOI: https://doi.org/10.1090/spmj/1361

[7] Kurosh A.G. *Course of higher algebra*, Moscow: Nauka, 1971, 430 pp. (in Russian).

[8] Kytmanov A. A., Kytmanov A. M., Myshkina E. K. *Residue Integrals and Waring's Formulas for Class of Systems of Transcendental Equations in* $\mathbb{C}^n$. Journal of Complex variables and Elliptic Equations, 2019, vol. 64, pp. 93–111. DOI: https://doi.org/10.48550/arXiv.1709.00791

[9] Kytmanov A. M. Âlgebraic and transcendental systems of equations., Krasnoyarsk, SibFU: 2019, 354 pp.(in Russian).

[10] Kytmanov A. M., Khodos O. V. *On localization of the zeros of an entire function of finite order of growth.* Journal Complex Analysis and Operator Theory, 2017, vol. 11, pp. 393–416.
DOI: https://doi.org/10.1007/s11785-016-0606-8

[11] Kytmanov A. M., Khodos O. V. *An Approach to the Determination of the Resultant of Two Entire Functions.* Russian Mathematics, 2018, vol. 62, no. 4, pp. 42 – 51. DOI: https://doi.org/10.3103/S1066369X18040059

[12] Kytmanov A. M., Myshkina E. K. *Evaluation of power sums of roots for systems of non-algebraic equations in* $\mathbb{C}^n$. Russian Mathematics, 2013, vol. 57, no. 12, pp. 31 – 43.
DOI: https://doi.org/10.3103/S1066369X13120049

[13] Kytmanov A. M., Myshkina E. K. *On the calculation of power sums of roots of one class of systems of non-algebraic equations.* Siberian Electronic Mathematical Reports, 2015, vol. 12, pp. 190 – 209.
DOI: https://doi.org/10.17377/semi.2015.12.016

[14] Kytmanov A. M., Myshkina E. K. *Residue integrals and Waring formulas for algebraic and transcendent systems of equations.* Russian Mathematics, 2019, vol. 63, no. 6, pp. 36 – 50.
DOI: https://doi.org/10.26907/0021-3446-2019-5-40-55

[15] Kytmanov A. M., Naprienko Ya. M. *One approach to finding the resultant of two entire function.* Complex variables and elliptic equations, 2017, vol. 62, pp. 269 – 286.
DOI: https://doi.org/10.1080/17476933.2016.1218855

[16] Markushevich A. I. *Theory of analytic functions.* M., Nauka, 1986, vol. 2 (in Russian).

[17] Passare M., Tsikh A. *Residue integrals and their Melin transforms.* Can. J. Math., 1995, vol. 47(5), pp. 1037 – 1050.
DOI: https://doi.org/10.4153/CJM-1995-055-4

[18] Khodos O.V. *On Some Systems of Non-algebraic Equations in* $\mathbb{C}^n$. J. Sib. Fed. Univ. Mathematics & Physics, 2014, vol. 7(4), pp. 455 – 465.

Received August 10, 2023.
In revised form, October 28, 2023.
Accepted November 08, 2023.
Published online December 02, 2023.

Siberian Federal University
79 Svobodny pr., Krasnoyarsk, 660049, Russia

A. M. Kytmanov
E-mail: akytmanov@sfu-kras.ru

O. V. Khodos
E-mail: khodos_olga@mail.ru