Probl. Anal. Issues Anal. Vol. 12 (30), No 3, 2023, pp. 41–49
DOI: 10.15393/j3.art.2023.14570

UDC 517.18

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ON APPLICATIONS OF THE DIHEDRAL GROUP TO INTERPOLATION PROBLEMS FOR ENTIRE FUNCTIONS

Abstract. We consider a particular case of the dihedral group of rotations and study linear poly-element functional equations associated with that group. We search for a solution in the class of functions that are holomorphic in the plane with a cut along "half" of the boundary of its fundamental region and vanish at infinity. We suggest a method for the regularization of such equations based on the theory of the Carleman boundary-value problem. The inverse involutive shift is induced by the generating transformations of the group. The solution is searched in the form of a Cauchy-type integral with an unknown density. The solution is a lower function that is Borel-associated with a certain entire function of exponential type (upper function).

Key words: *properly discontinuous groups, regularization method, entire functions*

2020 Mathematical Subject Classification: *11F03, 30D20*

1. Introduction and problem statement. Let D be a simply connected domain. The involutive shift $\alpha(t)\colon \partial^+ D \to \partial^- D$ is called the Carleman shift. Applications of the boundary-value problem with the Carleman shift to various branches of analysis are given in the review article [7]. Of particular interest is the case when D is the fundamental domain of the proper discontinuous group of linear-fractional transformations.

Let D be the interior of the fundamental polygon (see [8, ch. VII, § 3]) of a finite properly discontinuous group Γ. This paper aims at describing some applications of the theory of such groups to interpolation problems for entire functions of exponential type (henceforth referred to as e. f. e. t.). For the sake of definiteness, we consider as a model problem the case of a circular sector D enclosed by the line segment $\ell_1\colon t \in \ell_1 \Rightarrow \arg t = -3^{-1}\pi$, the arc ℓ_2 of the circle $|t| = 1\colon t \in \ell_2 \Rightarrow |\arg t| < 3^{-1}\pi$, and the line segment $\ell_3\colon t \in \ell_3 \Rightarrow \arg t = \pi/3$. The vertices of the sector are $t_1 = 0$, $t_2 = \exp(-3^{-1}\pi i)$, and $t_3 = \exp(3^{-1}\pi i)$, enumerated as they are traversed along the positively oriented boundary ∂D. This is the fundamental polygon of the special case of the dihedral group (see [8, ch. VII, § 6]) with generating transformations $\sigma_1(z) = \beta z$, where $\beta = \exp(2\pi i/3)$, and $\sigma_2(z) = z^{-1}$. This group contains only six transformations. In addition to σ_1 and σ_2, the group contains the transformations $\sigma_0(z) = z$, $\sigma_3 = \sigma_1^{-1}$, $\sigma_4(z) = \beta^2 z^{-1}$, and $\sigma_5(z) = \beta z^{-1}$. Let us explain the choice of the group. The vertex t_1 is common to three congruent fundamental polygons meeting at this point. On the other hand, it was proven in [1] that a lacunary analogue of the Poincaré theta series that does not contain the Cauchy kernel as a term exists if and only if each vertex of the fundamental polygon is common to either an even or an infinite number of congruent fundamental polygons that meet at a point. It was this case that was considered in [4], where another dihedral group was studied, namely the one in which each vertex of the fundamental polygon is common to an even number of congruent fundamental polygons that meet at this point. Of course, we have in this case a finite sum instead of a theta series. The result obtained in [1] is, however, essential in what follows. The group considered in [6] also satisfies the restriction given in [1].

Let us define a Carleman involutive shift

$$\alpha(t) = \left\{\sigma_j(t), t \in \ell_j, j = \overline{1,3}\right\}$$

that changes the orientation of ∂D. The derivative $\alpha'(t)$ is discontinuous at the vertices. According to the method that was suggested for the first time in [5], we do not consider the whole boundary ∂D but its "half" Ω meeting the following three conditions:

1) Ω is a piecewise-smooth curve (or a finite set of such curves Ω_j satisfying the condition $\overline{\Omega}_k \cap \overline{\Omega}_m = \varnothing$, $k \neq m$) that does not contain congruent points.

2) $\Omega \cup \alpha(\Omega) = \partial D$ (excluding possibly the vertices t_k and the nodes of Ω).

3) If B_1 is the convex hull of the set Ω, then $D \setminus B_1 \neq \varnothing$.

It is obvious that a certain degree of arbitrariness is possible in the choice of Ω. In any case, the choice is restricted by the condition that the fixed points 0 and 1 of the shift as well as one of the vertices t_2 or t_3, must belong to its closure. The goal of the paper is to study the functional equation

$$(Vf)(z) = f(z) + \sum_{j=1}^{N} G_j(z) f[\sigma_j(z)] = g(z);\ z \in D, \tag{1}$$

$N = 4, 5$, under the following assumptions.

I. The coefficients $G_j(z)$ and the right-hand side term $g(z)$ are holomorphic in the closure $\overline{D}$.

II. The solution $f(z)$ is holomorphic outside Ω and $f(\infty) = 0$. Its boundary values $f^{\pm}(t)$ satisfy a Hölder condition on any compact in Ω. Moreover, at most logarithmic singularities are allowed at the nodes of Ω. We denote this class of solutions by B.

The paper consists of three sections. Firstly, we study Equation (1) in the particular case when $\forall j\ G_j(z) \equiv 1$. Then, we study Equation (1) in the general case. At last, we consider some interpolation problems for e. f. e. t. induced by Equation (1). It is obvious that $f(z) \in B$ is the lower function Borel-associated with the e. f. e. t. $F(z)$ (upper function) (see [2, § 1, 1.1]).

2. The study of the functional equation.

Assume that $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 \subset \ell_1 \cup \ell_3$ and $\Omega_2 \subset \ell_2$. We shall search for a solution to Equation (1) in the form of a Cauchy-type integral

$$f(z) = \frac{1}{2\pi i} \int_{\Omega} (\tau - z)^{-1} \varphi(\tau) d\tau \tag{2}$$

with unknown density $\varphi(\tau)$. According to (2), we can rewrite Equation (1) as

$$(E\varphi)(z) \equiv \frac{1}{2\pi i} \int_{\Omega} A(z,\tau) \varphi(\tau) d\tau = g(z),\ z \in D, \tag{3}$$

where

$$A(z,\tau) = (\tau - z)^{-1} + \sum_{j=1}^{N} [\tau - \sigma_j(z)]^{-1}. \tag{4}$$

Assume that $z \to t \in \Omega$. Then, according to the Sokhotsky formulas, we have $(E^{+}\varphi)(t) = 2^{-1}\varphi(t) + (E\varphi)(t)$. The integral $(E\varphi)(t)$ is understood in the sense of the Cauchy principal value and is obtained by formally replacing $z \in D$ with $t \in \Omega$ in formula (3).

If $z \to \alpha(t)$, then $(E^{+}\varphi)(\alpha(t)) = -2^{-1}\varphi(t) + (E\varphi)(\alpha(t))$. Therefore, $(E^{+}\varphi)(t) - (E^{+}\varphi)(\alpha(t)) = (T\varphi)(t)$, where

$$(T\varphi)(t) \equiv \varphi(t) + \frac{1}{2\pi i}\int\limits_{\Omega} K(t,\tau)\varphi(\tau)d\tau = g(t) - g\left[\alpha(t)\right], \tag{5}$$

and

$$K(t,\tau) = A(t,\tau) - A\left[\alpha(t),\tau\right]. \tag{6}$$

We shall consider two cases.

1) $N = 5$. Kernel (4) includes all the group transformations and is automorphic in the first variable, i. e., $K(t,\tau)=0$ and $\varphi(t)=g(t)-g[\alpha(t)]$. So, using the identity

$$\sigma'(\tau)\left[\sigma(\tau)-z\right]^{-1} = \left[\tau - \sigma^{-1}(z)\right]^{-1} - \left[\tau - \sigma^{-1}(\infty)\right]^{-1},$$

valid for a linear fractional function, we obtain

$$(E\varphi)(z) = g(z) - \frac{3}{\pi i}\int\limits_{\alpha(\Omega_2)} \tau^{-1} g(z)d\tau$$

for $z \in D$. Here, condition 2) was used on the set Ω, the above identity and the equality $\frac{1}{2\pi i}\int\limits_{\partial D} g(\tau)A(z,\tau)d\tau = g(z), z \in D$.

Theorem 1. *In the case we are considering, Equation (1) is solvable if and only if the solvability condition*

$$\int\limits_{\alpha(\Omega_2)} \tau^{-1} g(\tau)d\tau = 0 \tag{7}$$

holds.

In other words, the solvability picture for a fixed right-hand side term depends on the choice of the set Ω_2.

2) $N = 4$. This is possible while preserving the boundedness condition of kernel (7) since $\sigma_5\left(\overline{\Omega}\right) \cap \overline{\Omega} = \varnothing$. Here we rely on T. Carleman's idea (see [3]) based on constructing a kernel using those group transformations that map the initial polygon to congruent polygons having either a

common edge or a common vertex with the initial polygon. The transformation σ_5 is not one of those transformations. Thus,

$$K(t,\tau) = (K(t,\tau)) = (\tau - \beta/\alpha(t))^{-1} - \left(\tau - \beta t^{-1}\right). \tag{8}$$

In what follows, we assume, for the sake of definiteness, that $\Omega_1 = \ell_1$, and Ω_2 is the "half" of ℓ_2 connecting the points t_2 and 1.

If $N < 4$, then kernel (6) is no longer bounded. It will have point singularities at the vertices. The point is that for $j < 4$ $\sigma_j(\overline{\Omega}) \cap \overline{\Omega} \neq \emptyset$. Meanwhile, $\sigma_5(\overline{\Omega}) \cap \overline{\Omega} = \emptyset$.

Lemma 1. *The homogeneous equation $T\varphi = 0$ has only the trivial solution.*

Proof. We will rely on the contraction mapping principle in the Banach space $C(\Omega)$. Let us show that the operator T performs a contraction mapping. Assume that

$$M = \max|\varphi(t)|, \ t \in \Omega, \tag{9}$$

and that this equality is attained for $t \in \Omega_1 \Rrightarrow \alpha(t) = \beta t$. Thus, kernel (8) can be written as

$$K(t,\tau) = \left(\tau - t^{-1}\right)^{-1} - \left(\tau - \beta t^{-1}\right) = (1-\beta)\left[(t\tau - 1)\left(\tau - \beta t^{-1}\right)\right]^{-1},$$

and $|K| \leqslant \sqrt{3}$. The length of the curve Ω is $d = 1 + 3^{-1}\pi$. Therefore, $\sqrt{3}d < 2\pi \Rightarrow \varphi \equiv 0$.

Assume that equality (9) is attained for $t \in \Omega_2 \Rightarrow \alpha(t) = t^{-1}$. This means that kernel (8) can be written as

$$K(t,\tau) = (\tau - \beta t)^{-1} - \left(\tau - \beta t^{-1}\right) = \beta\left(t^{-1} - t\right)\left[(\tau - \beta t)\left(\tau - \beta t^{-1}\right)\right]^{-1},$$

i. e., $|K| \leqslant \sqrt{3}$ and $\sqrt{3}d < 2\pi \Rightarrow \varphi \equiv 0$. This finishes the proof of the lemma. □

Corollary. *The inhomogeneous Fredholm equation* (5) *is solvable.*

Proof. It is obvious that (5) $\Rightarrow (E\varphi)(z) = g(z) + C, \ z \in D$. Let us fix a point $z_0 \in D$. The equivalence condition for the regularization we have applied to Equation (1) is

$$(E\varphi)(z_0) = g(z_0).$$

For every function $g(z) \not\equiv \text{const}$, we can find such a constant C_g that Equation (1) with the right-hand side term $g(z) + C_g$ is solvable. □

Theorem 2. *Assume that $N = 4$. In this case, Equation (1) is solvable if and only if the condition*

$$(E\varphi)(z_0) = g(z_0), \ \ z_0 \in D, \tag{10}$$

holds (this condition ensures the equivalence of the regularization). Here we have $\varphi(t) = T^{-1}[g(t) - g(\alpha(t))]$.

3. Regularization of the equation. Let us proceed to the regularization of Equation (1) in the general case. Assume that $N = 4$ and $G(t) = \{G_3(\beta t),\, t \in \ell_1; G_2(t^{-1}), t \in \ell_2\}$. Assume also that

$$1 + G(t) \neq 0, \ \ t \in \overline{\Omega}. \tag{11}$$

In this case,

$$A(z,\tau) = (\tau - z)^{-1} + \sum_{j=1}^{4} G_j(t)\,[\tau - \sigma_j(t)]^{-1}.$$

Reasoning in the same manner as in Section 2, we infer that

$$(T\varphi)(t) \equiv 2^{-1}(1 + G(t))\,\varphi(t) + \frac{1}{2\pi i}\int_{\Omega} K(t,\tau)\varphi(\tau)d\tau = G(t)g(t) - g[\alpha(t)]. \tag{12}$$

Moreover,

$$K(t,\tau) = G(t)A(t,\tau) - A\,[\alpha(t),\tau]\,. \tag{13}$$

Let us find the conditions on the coefficients that ensure that kernel (12) is bounded. For the sake of brevity, we write only the terms of $K(t,\tau)$ that can approach infinity for a given t.

1) If $t \in \Omega_1$, then we have

$$\begin{aligned} &K(t,\tau) = \\ &= [G_3(\beta t)G_1(t) - 1]\,(\tau - \beta t)^{-1} + [G_3(t)G_3(\beta t) - G_1(\beta t)]\,(\tau - \beta^2 t)^{-1} + \\ &\qquad + [G_4(t)G_3(\beta t) - G_1(\beta t)]\left(\tau - \beta^2 t^{-1}\right) + \ldots \end{aligned}$$

Let us analyze the nature of the singularities of the kernel, assuming firstly that $\tau, t \to 0+0$. Consider the triangle with vertices $0, \tau, \beta t$ or $\beta^2 t$. It is obvious that $|\tau - \beta^k t| > |t|, k = 1, 2$. Assume now that $\tau, t \to t_2 - 0$. In this case, the relation $|\tau - \beta^2 t^{-1}| > |t - t_2|$ holds, since points τ and $\beta^2 t^{-1}$ are on the ray $\arg t = -3^{-1}\pi$ on opposite sides with respect to the point t_2. The same inequality is valid when $t \to t_2 - 0$ and $\tau \to t_2 + 0$, which follows from the obvious inequality $|\exp(i\gamma) - h| > h - 1$ as $\gamma \to 0+0$ and $h \to 1 + 0$.

2) If $t \in \Omega_2$, then

$$K(t,\tau) = \left[G_2(t)G_2(t^{-1}) - 1\right]\left(\tau - t^{-1}\right)^{-1} + \\ + \left[G_4(t)G_2(t^{-1}) - G_3(t^{-1})\right]\left(\tau - \beta^2 t^{-1}\right)^{-1} + \dots .$$

If $\tau, t \to 1 - 0$, then $|\tau - t^{-1}| > |t - 1|$, since $\tau = \exp(i\gamma)$, $t = \exp(i\mu)$, and $\gamma, \mu \to 0 - 0$. Unfortunately, the inequality $|\tau - \beta^2 t^{-1}| > |t - t_2|$ is not valid when $\tau \to t_2 - 0$ and $t \to t_2 + 0$. However, the other inequalities we have obtained make it possible to formulate the following result.

Theorem 3. *Assume that (11) holds and, moreover, $G_3(0)G_1(0) = 1$, $G_3^2(0) = G_1(0)$, $G_4(t_2)G_3(t_3) = G_1(t_3)$, $G_2^2(1) = 1$, and $G_4(t)G_2(t^{-1}) = = G_3(t^{-1})$, $t \in \Omega_2$. Then kernel (13) is bounded, which means that (12) is a Fredholm equation.*

The following is an example of a set of coefficients that satisfy all the conditions of Theorem 3: $G_1(z) = G_2(z) = G_3(z) = z^2 - z + 1$, $G_4(z) = 1$.

Remark 1. *If we replace the condition that all the coefficients in the problem and the right-hand side term are holomorphic in the closure of D (this is the class of functions $A[D]$) with a weaker condition, namely that these functions are holomorphic in D and satisfy a Hölder condition on the boundary ∂D (the class $A(D)$), then kernel (13) has weak pole singularities at the nodes of Ω.*

4. Applications to interpolation problems. Let us proceed now to the main task of this paper: the application of functional equation (1) to interpolation problems for e. f. e. t. We restrict ourselves to the simplest case of this equation, namely when $N = 5$ and $\forall j\ G_j(z) \equiv 1$. We assume, for the sake of definiteness, that $\Omega_1 = \ell_1$ and Ω_2 is a "half" of the arc ℓ_2 ($t \in \Omega_2 \Rrightarrow \operatorname{Im} t < 0$). The conjugate indicator diagram of the e. f. e. t. $F(z)$, which is Borel-associated with the Cauchy-type integral (2), is a "half" of D, namely the circular sector B_1 with vertices $0, t_2, 1$.

Assume that $z_0 \in D$ and $\operatorname{Im} z_0 > 0$. Consider the power series

$$g(z) = c_0 + \sum_{k=1}^{\infty} \frac{c_k\,(z - z_0)^k}{k!}, \tag{14}$$

whose radius of convergence $R > \max\left(|z_0|, |z_0 - t_2|\right)$. Fix the constant c_0 in (14) in such a manner that the solvability condition (7) holds.

Assume that $z \in D$ and $z \notin \overline{B_1}$. Since all six points $\sigma_j(z)$, $j = \overline{0,5}$, are outside $\overline{B_1}$, we can rewrite Equation (1) as

$$\sum_{j=0}^{5} \int\limits_{\arg t = \theta_j} F(t) \exp\left(-\sigma_j(z)t\right) dt = g(z), \;\; z \in D \setminus \overline{B_1}, \tag{15}$$

where $\theta_0 = 3\pi/2$, $\theta_1 = \theta_3 = \pi$, $\theta_2 = 0$, $\theta_4 = 5\pi/6$, and $\theta_5 = 7\pi/6$. By equating the Taylor coefficients of the first and second members in (15) at point z_0, we obtain

$$\sum_{j=0}^{5} \int\limits_{\arg t = \theta_j} F(t) \frac{\partial^k}{\partial z^k} \left[\exp\left(-\sigma_j(z)t\right)\right]\Big|_{z=z_0} dt = c_k, \;\; k = \overline{1,\infty}. \tag{16}$$

Theorem 4. *Interpolation problem (16) is solvable in the class of e. f. e. t. $F(z)$ Borel-associated with the lower function $f(t) \in B$.*

Since the solution of Equation (1) is found explicitly in this case, we have

$$F(z) = \int\limits_{\Omega} \left[g(\alpha(\tau)) - g(\tau)\right] \exp(z\tau) d\tau.$$

As we have already noted, the choice of Ω allows for a certain degree of arbitrariness. This makes it possible to find entire functions with distinct conjugate indicator diagrams and, therefore, with distinct indicators.

Let us consider an example. Assume that $\Omega = \bigcup d_k$, $k = \overline{1,4}$, where d_1 is the line segment with endpoints 0 and $2^{-1}t_2$, d_2 is the part of the arc ℓ_2 with endpoints $t_4 = \exp(-\pi i/12)$ and 1, d_3 is the part of the arc ℓ_2 with endpoints $t_5 = \exp(\pi i/12)$ and t_3, and, finally, d_4 is the line segment with endpoints t_3 and $2^{-1}t_3$. In this case, $\overline{B_1}$ is a curvilinear hexagon with sides $d_1, d_2, d_3, \ell_3, d_5$, and d_6, where d_5 is the line segment with endpoints $2^{-1}t_2$ and t_4, and d_6 is the line segment with endpoints 1 and t_5. The set $D \setminus B_1$ splits into two connected components. One of them is a curvilinear triangle with vertices $2^{-1}t_2$, t_2, and t_4; the other one is the circular segment cut off from D by the chord d_5.

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Received April 17, 2023.
In revised form, July 25, 2023.
Accepted August 04, 2023.
Published online August 30, 2023.

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