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ON BOUNDED METRIC SPACES: COMMON FIXED POINT RESULTS WITH AN APPLICATION TO NONLINEAR INTEGRAL EQUATIONS

Abstract. In this article, we establish some common fixed point theorems in the setting of bounded metric spaces without using neither the compactness nor the uniform convexity of the space. Some examples are built to show the superiority of the obtained results compared to the existing ones in the literature. Moreover, we apply the main result to show the existence and uniqueness of a solution for a nonlinear integral system.

Key words: *common fixed point, compactness, uniform convexity, E-weakly of type T, nonlinear integral system*

2020 Mathematical Subject Classification: *47H09, 47H10*

1. Introduction. In the literature known by the author, fixed point theory has many applications in different branches of science. In metric fixed point theory, the most celebrated is the Banach contraction principle (1922) (for short, BCP), which played an important role in nonlinear analysis. There are plenty of generalizations of the famous BCP, which states that every selfmapping T defined on a complete metric space (X, d), such that

$$d(Tx, Ty) \leqslant kd(x, y), \tag{1}$$

where $k < 1$ for all $x, y \in X$, has a unique fixed point $u \in X$, that is $Tu = u$.

This result was a major tool to ensure the existence and uniqueness of a solution for the Fredholm integral equation:

$$x(t) = \int_0^t K(s, x(s))ds, \quad t \in [0, \tau], \tag{2}$$

where $K\colon [0,\tau]\times\mathbb{R}\to\mathbb{R}$ is a given mapping and the problem is typically to find the function x.

In the theory of fixed point, the problem of contractive selfmappings on a metric space (X,d) (that is, $d(Tx,Ty)<d(x,y)$, for all $x\neq y\in X$) was initiated by Nemytzki [12](1939). In 1962, Edelstein mentioned in [4] that to obtain a fixed point of such mappings, it is necessary to add the compactness assumption of the space.
In 1965, Browder [3] and Göhde [5] independently showed one of the most interesting extensions of BCP by proving that every nonexpansive mapping whose Lipschitz constant equals to 1, (that is $\|Tx-Ty\|\leqslant\| x-y \|$ for all $x,y\in X$) of a closed convex and bounded subset of the Banach space X has a fixed point, if the subset is supposed to be uniformly convex (for each $0<\varepsilon\leqslant 2$, there exists $\delta>0$, such that for all $\|x\|\leqslant 1, \|y\|\leqslant 1$ the condition $\|x-y\|\geqslant\varepsilon$ implies that $\|\frac{x+y}{2}\|\leqslant 1-\delta$).

In 1998, Jungck and Rhodes [6] introduced the concept of weakly compatible mappings (f,g), in other words, if they commute at coincidence points (i.e., if $fu=gu$ for some $u\in X$, then $fgu=gfu$). Moreover, they have proved some generalizations of BCP.

On the other hand, BCP has a great number of generalizations with different forms and in various spaces. One of the most prevalent is the partial metric spaces introduced by Matthews [11] as a generalization of the notion of the metric space, such that the separation axiom $d(x,x)=0$ of the metric's definition is replaced by the condition $\sigma(x,x)\leqslant\sigma(x,y)$ (in other words $\sigma(x,x)>0$ for some x). Different approaches in this area have been reported, including applications of mathematical techniques to computer science [10], [13], [17].

As mentioned in [25], Matthews [11] showed that for any partial metric σ on a nonempty set X, there exists an induced metric $d_\sigma\colon X\times X\to\mathbb{R}^+$ defined as follows:

$$d_\sigma(x,y)=2\sigma(x,y)-\sigma(x,x)-\sigma(y,y),$$

or, equivalently,

$$\sigma(x,y)=\alpha(x)+d(x,y)+\alpha(y), \tag{3}$$

where $\alpha\colon X\to\mathbb{R}^+$ is defined by $\alpha(x)=\frac{1}{2}\sigma(x,x)$ and $d(x,y)=\frac{1}{2}d_\sigma(x,y)$.

In 2011, as an extension of the result proven by Jungck, Karapänar and Yüksel [7] proved a common fixed point result for a pair of selfmappings (f,g) on a complete partial metric space X satisfying

$$\sigma(gx,gy)\leqslant k\sigma(fx,fy), \tag{4}$$

with $k < 1$ for all $x, y \in X$.

Similar to [25] but now in the case of a pair of mappings (f, g), it is obvious to see that the author in [7] used the condition

$$\alpha(gx) \leqslant k\alpha(fx) \quad \text{for all } x \in X. \tag{5}$$

Motivated by this fact, in this paper we extend condition (5) to

$$\alpha(gx) \leqslant \alpha(fx). \tag{6}$$

In 2019, a new category of contractive fixed point problems was addressed by the authors in [18]; they proved some fixed point results in a bounded metric space (X, d) without using the compactness for the following contractive mappings $T \colon X \to X$ satisfying:

$$\inf_{x \neq y \in X} \{d(x, y) - d(Tx, Ty)\} > 0. \tag{7}$$

Very recently, in 2021, the authors in [20] proved in the same direction of research, without using the compactness, the following result:

$$\inf_{x \neq y \in X} \{d(fx, fy) - d(gx, gy)\} > 0, \tag{8}$$

where (f, g) are weakly compatible mappings.

The reader can see [19], [21], [22], [23], [24], [26], [27] and references therein, for recent works in this direction.

In this paper, we introduce a new class of weakly compatible mappings via a new combination of (5) with (8) and prove a new common fixed point theorem for this new type of nonexpansive mappings (i.e., $d(gx, gy) \leqslant d(x, y)$) without using neither the compactness nor the uniform convexity. Furthermore, inspired by [2], [14], [15], we prove a theorem for a new class of weakly contractive mappings called E-weakly contractive mappings of type T. Our results extend and improve the proven results in [18], [20] and other results in the literature.

Finally, we show the existence and uniqueness of a common solution for the nonlinear integral equations

$$x(t) = \int_0^t K(s, \int_0^s K(\xi, x(\xi))d\xi)ds, \; t, s \in [0, \tau],$$

and

$$x(t) = \int_0^t K(s, x(s))ds, \, t \in [0, \tau],$$

as a generalization of the equation (2) under some new weak conditions.

2. Preliminaries. The aim of this section is to present some notions and results used in the paper. Throughout the article, we denote by $\mathbb{R}$ the set of all real numbers and by $\mathbb{N}$ the set of all positive integers.

Let (X, τ) be a topological space and $p\colon X \times X \to [0, +\infty)$ be a function. For any $\varepsilon > 0$ and any $x \in X$, let $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$.

Definition 1. *[1] The function p is said to be τ-distance if for each $x \in X$ and any neighborhood V of x, there exists $\varepsilon > 0$, such that $B_p(x,\varepsilon) \subset V$.*

Definition 2. *A sequence in a Hausdorff topological space X is a p-Cauchy if it satisfies the usual metric condition with respect to p.*

Definition 3. *[1, Definition 3.1] Let (X, τ) be a topological space with a τ-distance p.*

1) *X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim\limits_{n\to+\infty} p(x, x_n) = 0$.*
2) *X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim\limits_{n\to+\infty} x_n = x$ with respect to τ.*
3) *X is said to be p-bounded if $\sup\{p(x, y)/x, y \in X\} < +\infty$.*

Lemma 1. *[1] Let (X, τ) be a Hausdorff topological space with a τ-distance p; then*

1) *$p(x, y) = 0$ implies $x = y$.*
2) *Let (x_n) be a sequence in X, such that $\lim\limits_{n\to+\infty} p(x, x_n) = 0$ and $\lim\limits_{n\to+\infty} p(y, x_n) = 0$; then $x = y$.*

Definition 4. *[6] Two selfmappings f and g of a set X are said to be weakly compatible if they commute at there coincidence points; i.e., if $fu = gu$ for some $u \in X$, then $f \circ gu = g \circ fu$.*

Theorem 1. *[6] Let (X, d) be a metric space and $f, g\colon X \to X$ two weakly compatible mappings, such that $g(X) \subset f(X)$, and for all $x, y \in X$ we have*

$$d(gx, gy) \leqslant kd(fx, fy),$$

where $k \in [0,1)$. Then f and g have a unique common fixed point.

Theorem 2. *[18] Let $T\colon X \longrightarrow X$ be a mapping of a bounded complete metric space (X,d), such that $\inf\limits_{x\neq y\in X}\{d(x,y)-d(Tx,Ty)\} > 0$. Then T has a unique fixed point.*

Theorem 3. *[18] Let $T\colon X \longrightarrow X$ be a E-weakly contractive mapping of a bounded complete metric space (X,d). Then T has a unique fixed point.*

Theorem 4. *[20] Let (X,τ) be a p-bounded Hausdorff topological space with a τ-distance p. Let f and g be two weakly compatible selfmappings of X, satisfying the following conditions:*
i) $g(X) \subset f(X)$,
ii) $p(gx,gy) \leqslant kp(fx,fy)$,
for all $x,y \in X$ and $k<1$. If the range of f or g is S-complete subspace of X, then f and g have a unique common fixed point.

Theorem 5. *[20] Let (Xd) be a bounded complete metric space (X,d). Let f and g be two weakly compatible selfmapping of X satisfying the following conditions:*
i) $g(X) \subset f(X)$,
ii) $\inf\limits_{x\neq y}\{d(fx,fy)-d(gx,gy)\} > 0$.
Then f and g have a unique common fixed point.

Theorem 6. *[20] Let $f,g\colon X \longrightarrow X$ two weakly compatible mappings of a metric space (X,d), such that*

$$d(gx,gy) \leqslant d(fx,fy) - \phi(1+d(fx,fy)),$$

for all $x,y \in X$, where $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying $\phi(0)=0$ and $\inf\limits_{t>0}\phi(t) > 0$. Then f and g have a unique common fixed point.

Definition 5. *Let (X,d) be a metric space. A function $\alpha\colon X \to [0,+\infty)$ is said to be lower semicontinuous if for all $y \in X$ and $\{x_n\} \subset X$, such that $\lim\limits_{n\to+\infty} x_n = y$, we get*

$$\alpha(y) \leqslant \lim_{n\to+\infty}\inf \alpha(x_n).$$

In 2014, Rosa et al. [16] introduced the following new notions of g-η-admissible mapping:

Definition 6. *[16] Let $T, g\colon X \to X$ and $\eta\colon X \times X \to \mathbb{R}$. The mapping T is g-η-admissible if, for all $x, y \in X$, such that $\eta(gx, gy) \geqslant 1$, we have $\eta(Tx, Ty) \geqslant 1$. If g is the identity mapping, then T is called η-admissible.*

Definition 7. *[8] An η-admissible map T is said to be triangular η-admissible if for all $x, y, z \in X$*

$$\eta(x, z) \geqslant 1 \text{ and } \eta(z, y) \geqslant 1 \ \text{ imply } \ \eta(x, y) \geqslant 1.$$

3. Main results. In this section, we begin by proving the following Lemmas needed in the sequel.

Lemma 2. *Let (X, d) be a metric space and $p\colon X \times X \to \mathbb{R}^+$ be a function defined by*

$$p(x, y) = e^{\alpha(x)+d(x,y)+\alpha(y)} - 1, \tag{9}$$

where $\alpha\colon X \to \mathbb{R}^+$ is a function. Then p is a τ_d-distance on X, where τ_d is the metric topology.

Proof. Let (X, τ_d) be the topological space with the metric topology τ_d, let $x \in X$ and V be an arbitrary neighborhood of x; then there exists $\varepsilon > 0$, such that $B_d(x, \varepsilon) \subset V$, where $B_d(x, \varepsilon) = \{y \in X, d(x, y) < \varepsilon\}$ is the open ball.

It easy to see that $B_p(x, e^{\varepsilon} - 1) \subset B_d(x, \varepsilon)$. Indeed, let $y \in B_p(x, e^{\varepsilon} - 1)$; then $p(x, y) < e^{\varepsilon} - 1$, which implies $e^{d(x,y)+\alpha(x)+\alpha(y)} < e^{\varepsilon}$, and, hence, $d(x, y) < \varepsilon$. $\square$

Lemma 3. *Let (X, d) be a bounded metric space and $\alpha\colon X \to \mathbb{R}^+$ be a bounded function. Then the function p defined in Lemma 2 is a bounded τ-distance.*

Proof. Since (X, d) is a bounded metric space and $\alpha\colon X \to \mathbb{R}^+$ is a bounded function, then there exists $M > 0$, such that

$$\sup\{d(x, y)\colon x, y \in X\} < \frac{M}{3} \quad \text{and} \quad \sup\{\alpha(x)\colon x \in X\} < \frac{M}{3}.$$

Hence, $\sup\{p(x, y)\colon x, y \in X\} < e^M - 1$ with $e^M - 1 > 0$. $\square$

Lemma 4. *Let (X, d) be a complete metric space and $\alpha\colon X \to \mathbb{R}^+$ be a lower semicontinuous function. Then the function p defined in Lemma 2 is a S-complete τ-distance.*

Proof. Let (X, d) be a complete metric space and $\{x_n\} \subset X$ a p-Cauchy sequence, where p is the function defined in Lemma 2. Then $\lim_{n,m\to+\infty} p(x_n, x_m) = 0$, and, hence, $\lim_{n,m\to+\infty} d(x_n, x_m) = 0$, $\lim_{n\to+\infty} \alpha(x_n) = 0$. Since X is complete, there exists $u \in X$, such that $\lim_{n\to+\infty} d(u, x_n) = 0$ and by the lower semicontinuity of α, one has $\alpha(u) = 0$. Finally, we deduce that there exists $u \in X$, such that $\lim_{n\to+\infty} p(u, x_n) = 0$. □

Definition 8. *Let T, S be two selfmapping of a Hausdorff topological space (X, τ) with a τ-distance p. S is said to be T_p-continuous at $z \in X$ if for any $\{x_n\} \subset X$; $\lim_{n\to+\infty} p(Tz, Tx_n) = 0$ implies that $\lim_{n\to+\infty} p(Sz, Sx_n) = 0$.*

Lemma 5. *Let (X, τ) be a Hausdorff topological space with a τ-distance p, X be p-bounded. Let f, g be two selfmappings, such that $f(X)$ or $g(X)$ is S-complete and*

$$\eta(fx, fy)p(gx, gy) \leqslant kp(fx,fy), \tag{10}$$

for all $x, y \in X$, where $k \in [0, 1)$ and

i) $g(X) \subset f(X)$,

ii) g is an f-η-admissible mapping and triangular η-admissible,

iii) there exists $x_0 \in X$, such that $\eta(fx_0, gx_0) \geqslant 1$,

iv) g is f_p-continuous.

Then f and g have a coincidence point (i.e., there exists $u \in X$, such that $fu = gu$).

Moreover, if the following conditions hold:

(a) the pair (f, g) is weakly compatible,

(b) either $\eta(fu, fv) \geqslant 1$ or $\eta(fv, fu) \geqslant 1$ whenever $fu = gu$ and $fv = gv$, then f and g have a common fixed point.

Proof. Let $x_0 \in X$, such that $\eta(fx_0, gx_0) \geqslant 1$; from (i), we define a sequence $\{x_n\}$ by $gx_n = fx_{n+1}$, for all $n \in \mathbb{N}$.

We have $\eta(fx_0, fx_1) = \eta(fx_0, gx_0) \geqslant 1$. Since g is an f-η-admissible mapping, we obtain $\eta(gx_0, gx_1) = \eta(fx_1, fx_2) \geqslant 1$. Again, using (ii), we have $\eta(gx_1, gx_2) = \eta(fx_2, fx_3) \geqslant 1$. Repeating this process, we get $\eta(fx_n, fx_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$.

Let $m < n \in \mathbb{N}$; via the triangular η-admissibility, we obtain $\eta(fx_m, fx_n) \geqslant 1$. Then

$$\begin{aligned} p(fx_m, fx_n) &\leqslant \eta(fx_m, fx_n)p(fx_m, fx_n) \leqslant \\ &\leqslant kp(fx_{m-1}, fx_{n-1})) \leqslant \\ &\vdots \\ &\leqslant k^{n-m}(p(fx_0, fx_m)) \leqslant \\ &\leqslant k^{n-m}(M), \end{aligned} \tag{11}$$

where $M = \sup\{p(x, y) \colon x, y \in X\}$. As $\lim_{n\to+\infty} \psi^n(M) = 0$, so the sequence $\{fx_n\}$ is a p-Cauchy sequence. Since $f(X)$ is S-complete, there exists $u \in X$, such that $\lim_{n\to+\infty} p(fu, fx_n) = 0$. On the other hand, g is f_p-continuous; then $\lim_{n\to+\infty} p(gu, gx_n) = 0$. Using Lemma 1, we obtain $fu = gu$. Now, the assumption that f and g are weakly compatible implies

$$f \circ gu = g \circ fu = g \circ gu = f \circ fu. \tag{12}$$

Hence, gu is a coincidence point of f and g. Using (b), we have $\eta(g \circ gu, gu) \geqslant 1$ or $\eta(gu, g \circ gu) \geqslant 1$. Suppose that $\eta(g \circ gu, gu) \geqslant 1$ and $p(g \circ gu, gu) \neq 0$. From (10), it follows

$$p(g \circ gu, gu) \leqslant kp(f \circ gu, fu) < p(g \circ gu, gu); \tag{13}$$

this leads to a contradiction. Thus $g \circ gu = gu$. Also, $f \circ gu = g \circ fu = = g \circ gu = gu$, which implies that gu is a common fixed point of f and g. If we suppose $\eta(gu, g \circ gu) \geqslant 1$, similarly with (13), we get that gu is a common fixed point of f and g.
So, if the range of g is S-complete subspace of X, then $\lim\limits_{n\to+\infty} p(gv, gx_n) = 0$ for some $v \in X$. From (i), there exists $w \in X$, such that $gv = fw$ and the proof that gw is a common fixed point of f and g is the same as that given when $f(X)$ is S-complete. □

Let $f, g \colon X \to X$ two selfmappings on a set X. In the following, $\beta_{f,g}$ denote the class of all functions $\beta \colon X \times X \to \mathbb{R}$ satisfying: for all $x, y \in X$ such that $\beta(fx, fy) \leqslant 0$, we have $\beta(gx, gy) \leqslant 0$.

The function β is said to be triangular if for all $x, y, z \in X$

$$\beta(x, z) \leqslant 0 \text{ and } \beta(z, y) \leqslant 0 \ \text{ imply } \ \beta(x, y) \leqslant 0.$$

Now, we are able to prove our main results.

Theorem 7. *Let (X,d) be a bounded complete metric space (X,d) and $\alpha\colon X \to [0,\infty)$ is a bounded and lower semicontinuous function. Let f, $g\colon X \to X$ satisfying the following conditions:*

i) $g(X) \subset f(X)$,

ii) $\inf\limits_{x\neq y}\{\alpha(fx)+d(fx,fy)+\alpha(fy)-\alpha(gx)-d(gx,gy)-\alpha(gy)+\beta(fx,fy)\} > >0$, *where* $\beta \in \beta_{f,g}$ *and triangular.*

iii) there exists $x_0 \in X$ *such that* $\beta(fx_0,gx_0) \leqslant 0$,

iv) $\beta(a,b) \leqslant \inf\limits_{x\neq y\in X}\big\{\alpha(fx)+d(fx,fy)+\alpha(fy)-\alpha(gx)-d(gx,gy)-\alpha(gy)+ +\beta(fx,fy)\big\}$ *for all* $a,b \in X$,

v) g *is a* f*-continuous mapping.*

Then f and g have a coincidence point (i.e., there exists $u \in X$, such that $fu = gu$). Moreover, if the following conditions hold:

a') The pair (f,g) is weakly compatible,

b') either $\beta(fu,fv) \leqslant 0$ or $\beta(fv,fu) \leqslant 0$ whenever $fu = gu$ and $fv = gv$.

Then f and g have a common fixed point.

Proof. We put $\gamma = \inf\limits_{x\neq y}\{\alpha(fx)+d(fx,fy)+\alpha(fy)-\alpha(gx)-d(gx,gy)-\alpha(gy)+\beta(fx,fy)\}$; this implies that

$$\alpha(gx)+d(gx,gy)+\alpha(gy)-\beta(fx,fy) \leqslant \alpha(fx)+d(fx,fy)+\alpha(fy)-\gamma, \quad (14)$$

for all $x \neq y \in X$. Hence,

$$\eta(fx,fy)e^{\alpha(gx)+d(gx,gy)+\alpha(gy)} \leqslant ke^{\alpha(fx)+d(fx,fy)+\alpha(fy)}, \quad (15)$$

with $k = e^{-\gamma} < 1$ and $\eta(fx,fy) = e^{-\beta(fx,fy)}$. Then

$$\eta(fx,fy)p(gx,gy) \leqslant kp(fx,fy), \quad (16)$$

for all x,$y \in X$, with $p(x,y) = e^{\alpha(x)+d(x,y)+\alpha(y)} - 1$ as the τ-distance defined in Lemma 2. Now, we deduce from Lemmas 2, 3, 4, and Lemma 5 that f and g have a coincidence point. From b'), we obtain (b) in Lemma 5. Then, if a') and b') are satisfied, we conclude via Lemma 5 that f and g have a common fixed point. □

If we take $\beta = 0$, we obtain

Theorem 8. *Let (X,d) be a bounded complete metric space (X,d) and $\alpha\colon X \to [0,\infty)$ be a bounded and lower semicontinuous function. Let $f,g\colon X \to X$ satisfying the following conditions:*

i) $g(X) \subset f(X)$,

ii) $\inf_{x \neq y} \{\alpha(fx) + d(fx, fy) + \alpha(fy) - \alpha(gx) - d(gx, gy) - \alpha(gy)\} > 0$,

iii) *The pair* (f, g) *is weakly compatible.*

Then f *and* g *have a common fixed point.*

Corollary. *[20] Let* (X, d) *be a bounded complete metric space* (X, d)*. Let* f *and* g *be two weakly compatible selfmapping of* X *satisfying the following conditions:*

i) $g(X) \subset f(X)$,

ii) $\inf_{x \neq y} \{d(fx, fy) - d(gx, gy)\} > 0$.

Then f *and* g *have a unique common fixed point.*

If we take $f = Id$, we obtain the following result as a special case:

Corollary. *[25, Theorem 2.5] Let* g *be a selfmapping of a bounded complete metric space* (X, d)*. If*

$$\inf_{x \neq y} \{\alpha(x) + d(x, y) + \alpha(y) - \alpha(gx) - d(gx, gy) - \alpha(gy)\} > 0,$$

then g *has a unique fixed point* $u \in X$*.*

Corollary. *[18, Theorem 3] Let* $T \colon X \longrightarrow X$ *be a mapping of a bounded complete metric space* (X, d)*, such that* $\inf_{x \neq y \in X} \{d(x, y) - d(Tx, Ty)\} > 0$*. Then* T *has a unique fixed point.*

The following example illustrates Theorem 8.

Example 1. Let $X = \{0, 1, 2\}^2$ endowed with the metric

$$d\big((x_1, y_1), (x_2, y_2)\big) = \|(x_1, y_1) - (x_2, y_2)\|_1 = |x_1 - x_2| + |y_1 - y_2|.$$

It is clear that (X, d) is not an uniform convex space. Indeed, for $\varepsilon = 1$, $x = (1, 0)$ and $y = (0, 1)$ we have $\|x\|_1 = \|y\|_1 = 1$, $\|x - y\|_1 = 2 > 1 = \varepsilon$ and $\frac{1}{2}\|x + y\|_1 = 1 > 1 - \delta$ for each $\delta > 0$.

Define selfmapping f and g on X by

$$\begin{gathered} f(0,0) = (0,0), f(1,0) = f(0,1) = (1,1), \\ f(1{,}1) = f(0,2) = f(2,0) = (1,2), \\ f(1,2) = f(2,1) = f(2,2) = (2,2), \\ g(0,0) = g(1,0) = g(0,1) = g(1,1) = g(0,2) = g(2,0) = (0,0), \\ g(1,2) = g(2,1) = g(2,2) = (1,1), \end{gathered}$$

and a function $\alpha : X \to \mathbb{R}^+$

$$\alpha(0,0) = \alpha(1,0) = \alpha(0,1) = 0,$$
$$\alpha(1,1) = \alpha(1,2) = \alpha(2,1) = \alpha(0,2) = \alpha(2,0) = 1,\ \alpha(2,2) = \frac{3}{2}.$$

It is clear that $g(X) \subset f(X)$ and for all $x \neq y \in X$ we have

$$\alpha(fx) + \alpha(fy) + d(fx,fy) - \alpha(gx) - \alpha(gy) - d(gx,gy) \geqslant 1.$$

Then f and g satisfy all conditions of Theorem 8 and f and g have the unique common fixed point $(0,0)$.

Remark 1. *The above example illustrates the usability of Theorem 8 and shows that Theorem 8 is a real extension of [20, Theorem 2.12]. Indeed,*

$$d(f(1,0),f(2,2)) - d(g(1,0),g(2,2)) = 0 \quad \textit{or}$$
$$d(g(1,0),g(2,2)) \leqslant d(f(1,0),f(2,2)),$$

which are nonexpansive mappings.

As an application of Theorem 8, we get a result for a new class of weakly contractive maps defined as follows:

In the following, Φ is the class of all functions $\phi\colon [1,+\infty) \longrightarrow [0,+\infty)$ satisfying:

i) $\phi(t) = 0$ if and only if $t = 1$,

ii) $\inf_{t>1} \phi(t) > 0$,

iii) $\phi(t) \leqslant t$ for all $t \in [1,+\infty)$.

Definition 9. *Let (X,d) be a metric space and $f,g\colon X \longrightarrow X$ be two weakly compatible selfmappings of X, such that $g(X) \subset f(X)$. f and g are said to be E-weakly contractive of type T if*

i) For all $x \neq y \in X$, such that $fx = fy$, we have $fx = gx$,

ii) $M(x,y,g) \leqslant M(x,y,f) - \phi(M(x,y,f)+1)$, for all $x,y \in X$, where $M(x,y,f) = \alpha(fx) + d(fx,fy) + \alpha(fy)$, $\phi \in \Phi$ and $\alpha\colon X \longrightarrow [0,+\infty)$ is a bounded and lower semicontinuous function.

Theorem 9. *Let (X,d) be a bounded complete metric space and f,g be two E-weakly maps of type T on X. Then f and g have a unique common fixed point.*

Proof. Let $x \neq y \in X$; then we have the two following cases:
Case 1: If $fx = fy$, then Definition 9 implies that $fx = gx = fy = gy$. As f and g are weakly compatible, we have $g \circ fx = f \circ gx = f \circ fx = g \circ gx$. If $g \circ fx = fx$, the proof is finished. Otherwise, we have, from Definition 9:

$$\begin{aligned}\alpha(g \circ fx) + d(g \circ fx, fx) + \alpha(fx) = \alpha(g \circ fx) + d(g \circ fx, gx) + \alpha(gx) \leqslant \\ \leqslant \alpha(f \circ fx) + d(f \circ fx, fx) + \alpha(fx) - \\ - \phi(\alpha(f \circ fx) + d(f \circ fx, fx) + \alpha(fx) + 1) = \\ = \alpha(g \circ fx) + d(g \circ fx, fx) + \alpha(fx) - \\ - \phi(\alpha(g \circ fx) + d(g \circ fx, fx) + \alpha(fx) + 1),\end{aligned}$$

which is a contradiction; so, f and g have a unique common fixed point.
Case 2: If $fx \neq fy$, it follows from Definition 9, that

$$0 < \inf_{t>1} \phi(t) \leqslant \phi(M(x, y, f) + 1) \leqslant M(x, y, f) - M(x, y, g), \qquad (17)$$

hence, $\inf_{x \neq y}\{M(x, y, f) - M(x, y, g)\} > 0$. According to Theorem 8, we conclude that f and g have a unique common fixed point. □

Corollary. *[20] Let $f, g\colon X \longrightarrow X$ be two weakly compatible mappings of a bounded metric space (X, d), such that*

$$d(gx, gy) \leqslant d(fx, fy) - \phi(1 + d(fx, fy)),$$

for all $x, y \in X$, where $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying $\phi(0) = 0$ and $\inf_{t>0} \phi(t) > 0$. Then f and g have a unique common fixed point.

Corollary. *[18] Let $T\colon X \longrightarrow X$ be a E-weakly contractive mapping of a bounded complete metric space (X, d). Then T has a unique fixed point.*

Example 2. Let $X = \{0\} \cup [1, 2]$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $f, g\colon X \to X$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ 2, & \text{if } x \in [1, 2)\,, \\ 1, & \text{if } x = 2, \end{cases}$$

$$g(x) = \begin{cases} 0, & \text{if } x \in \{0\} \cup [1, 2), \\ 1, & \text{if } x = 2, \end{cases}$$

a function $\alpha\colon X \to \mathbb{R}^+$ defined by

$$\alpha(t) = \begin{cases} 0, & \text{if } t \in \{0,1\}, \\ \frac{t+1}{2}, & \text{if } t \in (1,2], \end{cases}$$

and a function $\phi\colon [1,\infty) \to [0,\infty)$ defined by

$$\phi(t) = \begin{cases} 0, & \text{if } t = 1, \\ 1, & \text{if } t > 1. \end{cases}$$

Hence, f and g satisfy all assumptions of Theorem 9 and $f0 = g0 = 0$. But the pair (f,g) does not satisfy Theorem 3.10 in [20], indeed:

$$d(g\frac{3}{2}, g2) = 1 > 0 = d(f\frac{3}{2}, f2) - \phi\big(1 + d(f\frac{3}{2}, f2)\big).$$

4. Application. Now, we prove the existence and uniqueness of a solution for the nonlinear integral equations

$$x(t) = \int\limits_0^t K(s, \int\limits_0^s K(\xi, x(\xi))d\xi)ds,\ t,s \in [0,\tau], \tag{18}$$

and

$$x(t) = \int\limits_0^t K(s, x(s))ds,\ t \in [0,\tau], \tag{19}$$

where $x \in \mathcal{CB}[0,\tau]$, the space of all continuous and bounded functions from $[0,\tau]$ into $\mathbb{R}$, with $\tau > 0$. $K\colon [0,\tau] \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping.

Let $X = \mathcal{CB}[0,\tau]$ endowed by the metric $d\colon X \times X \to \mathbb{R}^+$ defined as follows

$$d(x,y) = \sup_{t\in[0,\tau]} |x(t) - y(t)|,$$

it is clear that (X,d) is a complete metric space. Consider $\alpha\colon X \to [0,\infty)$ defined by

$$\alpha(x) = \| x \| = \sup_{t\in[0,\tau]} |x(t)|, \tag{20}$$

for all $x \in X$, which is a bounded and lower semicontinuous function. Define two mappings $f,g\colon X \to X$ by

$$g(x)(t) = \int\limits_0^t K(s, \int\limits_0^s K(\xi, x(\xi))d\xi)ds,\ t,s \in [0,\tau], \tag{21}$$

$$f(x)(t) = \int_0^t K(s, x(s))ds,\ t \in [0, \tau], \tag{22}$$

for all $x \in X$. So the problem to find a solution for the integral equations (18) and (19) is equivalent to finding a common fixed point of the mappings f and g.

Suppose that the above assumptions hold; then we have the following theorem:

Theorem 10. *If there exists $M > 0$ such that*

$$|K(s,x) - K(s,y)| \leqslant \frac{1}{\tau}|x-y| \ \text{ and } \ |K(s,x)| \leqslant \frac{1}{\tau}[|x| - \frac{1}{2}M] \tag{23}$$

or

$$|K(s,x) - K(s,y)| \leqslant \frac{1}{\tau}[|x-y| - M] \ \text{ and } \ |K(s,x)| \leqslant \frac{1}{\tau}|x|, \tag{24}$$

for all $s \in [0, \tau]$ and $x \neq y \in X$, then the nonlinear integral equations (21) and (22) have a unique common solution.

Proof. It is obvious to see that $g(X) \subset f(X)$ and f, g are weakly compatible. In the following, we will show that f and g satisfy (ii) in Theorem 8. Let $x \neq y \in X$ and $t \in [0,\tau]$; then we have the two following cases:
Case 1: In this case, we suppose that K satisfies (23), and, hence, (21) and (22) imply that

$$\begin{aligned}|g(x)(t)| &= \Big|\int_0^t K(s, \int_0^s K(\xi, x(\xi))d\xi)ds\Big| \leqslant \int_0^t \Big|K(s, \int_0^s K(\xi, x(\xi))d\xi)\Big|ds \leqslant \\ &\leqslant \int_0^t \frac{1}{\tau}\Big[\Big|\int_0^s K(\xi, x(\xi))d\xi\Big| - \frac{1}{2}M\Big]ds \leqslant \parallel fx \parallel - \frac{1}{2}M,\end{aligned} \tag{25}$$

then

$$\alpha(gx) \leqslant \alpha(fx) - \frac{1}{2}M. \tag{26}$$

Also, we have from (21), (22), and (23),

$$|g(x)(t) - g(y)(t)| = \Big|\int_0^t K(s, \int_0^s K(\xi, x(\xi))d\xi) - \int_0^t K(s, \int_0^s K(\xi, y(\xi))d\xi)\Big| =$$

$$= \Big|\int_0^t \Big[K(s, \int_0^s K(\xi,x(\xi))d\xi) - K(s, \int_0^s K(\xi, y(\xi))d\xi)\Big]ds\Big| \leqslant$$

$$\leqslant \int_0^t \Big|K(s, \int_0^s K(\xi, x(\xi))d\xi) - K(s, \int_0^s K(\xi, y(\xi))d\xi)\Big|ds \leqslant$$

$$\leqslant \int_0^t \frac{1}{\tau}\Big[\Big|\int_0^s K(\xi, x(\xi))d\xi) - \int_0^s K(\xi, y(\xi))d\xi\Big|\Big]ds \leqslant d(fx, fy), \tag{27}$$

and, hence,

$$d(gx, gy) \leqslant d(fx, fy), \tag{28}$$

for all $x \neq y \in X$. Using (26) and (28), we get

$$\inf_{x\neq y}\{\alpha(fx)+d(fx, fy)+\alpha(fy)-\alpha(gx)-d(gx, gy)-\alpha(gy)\} \geqslant M > 0. \tag{29}$$

Case 2: In this case, we assume that K satisfies (24); in a similar way, we get

$$\alpha(gx) \leqslant \alpha(fx), \tag{30}$$

$$d(gx, gy) \leqslant d(fx, fy) - M, \tag{31}$$

for all $x \neq y \in X$. It follows from (30) and (31)

$$\inf_{x\neq y}\{\alpha(fx)+d(fx, fy)+\alpha(fy)-\alpha(gx)-d(gx, gy)-\alpha(gy)\} \geqslant M > 0. \tag{32}$$

Then we deduce from (29), (32) and Theorem 8 that there exists a unique common solution of the integral equations (18) and (19). □

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