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J. Sahoo, B. B. Jena, S. K. Paikray

ON STRONG SUMMABILITY OF THE FOURIER SERIES VIA DEFERRED RIESZ MEAN

Abstract. The strong summability technique has attracted a remarkably large number of researchers for better convergence analysis of infinite series as well as Fourier series in the study of summability theory. The of strong summability method was introduced by Fekete (Math. És Termesz Ertesitö, 34 (1916), 759–786). In this paper, we introduce the notions of strong deferred Cesàro, deferred Nörlund, and deferred Riesz summability methods. We then consider our proposed strong deferred Riesz summability mean to establish and prove a new theorem for the summability of the Fourier series of an arbitrary periodic function. Moreover, for the effectiveness of our study, we present some concluding remarks demonstrating that some earlier published results are recovered from our main non-trivial Theorem.

Key words: *strong summability, deferred Cesàro summability,* $[D\bar{N}, p_n^{(1)}, 2]$*-summability, arbitrary periodic function, Fourier series*

2020 Mathematical Subject Classification: *30D40*

1. Introduction. Let (a_n) and (b_n) be sequences of non-negative integers with

$$a_n < b_n \ (n \in \mathbb{N}) \quad \text{and} \quad \lim_{n\to\infty} b_n = \infty,$$

and let $\sum u_n$ be an infinite series with the sequence of partial sums (s_n). We define the deferred Cesàro mean of order α of (s_n) as

$$DC_n^{\alpha} = \frac{1}{E_{b_n}^{\alpha-1}} \sum_{r=a_n+1}^{b_n} s_{b_n-r} E_r^{\alpha-2},$$

where

$$E_{b_n}^{\alpha-1} = \frac{\Gamma(b_n+1)}{\Gamma(b_n+1)\Gamma(\alpha)} \quad (\alpha > 0),$$

and it is denoted by (DC, α).

The series $\sum u_n$ is said to be (DC, α)- summable to ℓ if

$$\lim_{n\to\infty} DC_n^{\alpha} = \ell.$$

Moreover, it is said to be strongly (DC, α)- summable with index q (or (DC, α, q)- summable $(\alpha > 0, q > 0)$) to ℓ if

$$\sum_{r=a_n+1}^{b_n} |DC_r^{\alpha-1} - \ell|^q = o(b_n) \quad (n \to \infty).$$

Suppose that (p_n) is a sequence of non-negative numbers with

$$P_n = \sum_{r=a_n+1}^{b_n} p_r \quad (P_{-1} = p_{-1} = 0) \quad (p_0 \neq 0). \tag{1}$$

We now consider the deferred Nörlund (DN, p_n) mean of the sequence (s_n) generated by the sequence of coefficients (p_n) as

$$t_n = \frac{1}{P_n} \sum_{r=a_n+1}^{b_n} p_{b_n-r} s_r. \tag{2}$$

The given series $\sum u_n$ is said to be strongly (DN, p_n)- summable with index q (or $[DN, p_n, q]$ $(q > 0)$-summable) to ℓ if

$$\frac{1}{|P_n|} \sum_{r=a_n+1}^{b_n} |p_r| \cdot |t_r^{'} - \ell|^q = o(1), \tag{3}$$

where

$$t_r^{'} = \frac{1}{P_n} \sum_{r=a_n+1}^{b_n} s_r \nabla p_{b_n-r} = \frac{1}{P_n} \sum_{r=a_n+1}^{b_n} u_r p_{b_n-r}, \; (\nabla p_{b_n-r} \equiv p_{b_n-r} - p_{b_n-r-1}).$$

Note that if we take

$$p_n^{\alpha} = \sum_{r=a_n+1}^{b_n} E_{b_n-r}^{\alpha-1} p_r$$

instead of p_n, then the (DN, p_n)-summability with index q can further be extended to $[DN, p_n^\alpha, q]$ $(\alpha \geqq 1, q \geqq 1)$-summability.

Also, note that if we take

$$p_{b_n - r} = E_{b_n - r}^{\alpha - 1}$$

in (2), then (t_n) mean reduces to the Cesàro mean of order α of the sequence (s_n).

Next, we define the deferred weighted $(D\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) as

$$T_n = \frac{1}{P_n} \sum_{r=a_n+1}^{b_n} s_r p_r. \tag{4}$$

The given series $\sum u_n$ is said to be strongly $(D\bar{N}, p_n)$-summable with index q (or $[D\bar{N}, p_n, q]$, $(q > 0)$-summable) to ℓ if

$$\frac{1}{|P_n|} \sum_{r=a_n+1}^{b_n} |p_r| \cdot |T_r^{'} - \ell|^q = o(1), \tag{5}$$

where

$$T_r^{'} = \frac{1}{P_n} \sum_{r=a_n+1}^{b_n} s_r \nabla p_r = \frac{1}{p_n} \sum_{r=a_n+1}^{b_n} u_r p_r, \quad (\nabla p_r \equiv p_r - p_{r-1}).$$

Again if, we take

$$p_{b_n}^\alpha = \sum_{r=a_n+1}^{b_n} E_{b_n}^{\alpha-1} p_r$$

in place of p_n, then the $(D\bar{N}, p_n)$-summability with index q can also be extended to $[D\bar{N}, p_n^\alpha, q]$ $(\alpha \geqq 1, q \geqq 1)$-summability. Subsequently, if we take

$$p_{b_n} = E_{b_n}^{\alpha-1}$$

in (2), then (T_n) mean reduces to the Cesàro mean of order α of the sequence (s_n).

2. Preliminaries and Known Results. Let $s_n(g)$ be the partial sum of the Fourier series of an arbitrary periodic function g of period $2L$, such that

$$s_n(g) = a_0 + \sum_{k=1}^{n} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right) = \sum_{k=0}^{n} u_k(x),$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} g(t) dt,$$

$$a_k = \frac{1}{L} \int_{-L}^{L} g(t) \cos \frac{k\pi t}{L} dt \qquad (k \in \mathbb{N})$$

and

$$b_k = \frac{1}{L} \int_{-L}^{L} g(t) \sin \frac{k\pi t}{L} dt \qquad (k \in \mathbb{N}).$$

Again, under the Dirichlet kernel, the partial sum takes the form

$$s_n(g) = \frac{1}{L} \int_{-L}^{L} g(\lambda + x) \mathcal{D}_n(\lambda) d\lambda,$$

where

$$\mathcal{D}_n(\lambda) = \frac{\sin \left[\frac{(n+1/2)\lambda\pi}{L} \right]}{\sin \left(\frac{\pi\lambda}{2L} \right)} \quad (\because \lambda = t - x)$$

is the *Dirichlet kernel.*

We use the following notations throughout the paper:

$$\phi = \phi_\lambda(x) = \frac{1}{2} \left[g(\lambda + x) + g(\lambda - x) - 2s_n \right]; \quad \xi_r = rp_r/p_r^{(1)} = rp_r/P_r;$$

$$\Phi(x) = \int_0^x |\phi(u)| du; \qquad \bar{N}_n(x) = \frac{1}{LP_n} \sum_{r=a_n+1}^{b_n} p_r \frac{\sin \left[\frac{(m+1/2)x\pi}{L} \right]}{\sin \left(\frac{\pi x}{2L} \right)};$$

$$\alpha(x)=\sum_{r=a_n+1}^{b_n} p_r\cos(rx);\quad \beta(x)=\sum_{r=a_n+1}^{b_n} p_r\sin(rx);$$

$$\mathcal{I}_1=\sum_{m=0}^{r}\frac{p_m}{P_r}\Big(\frac{1}{L}\int_0^{1/r}\phi(x)\frac{\sin\left[\frac{(m+1/2)x\pi}{L}\right]}{\sin\left(\frac{\pi x}{2L}\right)}\Big)dx;$$

$$\mathcal{I}_2=\sum_{m=0}^{r}\frac{p_m}{P_r}\left(\frac{1}{L}\int_{1/r}^{\delta}\phi(x)\frac{\sin\left[\frac{(m+1/2)x\pi}{L}\right]}{\sin\left(\frac{\pi x}{2L}\right)}\right)dx;$$

$$\mathcal{I}_3=\sum_{m=0}^{r}\frac{p_m}{P_r}\left(\frac{1}{L}\int_{\delta}^{L}\phi(x)\frac{\sin\left[\frac{(m+1/2)x\pi}{L}\right]}{\sin\left(\frac{\pi x}{2L}\right)}\right)dx;$$

$$l_0(t)=t+1;\quad l_1(t)=\log(t+1);$$

$$l_2(t)=\log\log(t+1),\ldots \text{ and so on, for } t>0;$$

and $\tau=[1/t]$, the largest integer less than or equal to $1/t$.

Based on a finding of Hardy and Littlewood [8] about the summability of Fourier series, Fekete [7] investigated and introduced the notion of strong summability method. Subsequently, a few researchers have imposed the idea of strong summability techniques in their research works. Additionally, matrix summability (also known as matrix transformation) is crucial for understanding summability theory because it generalizes many summability techniques, including Cesàro summability, Nörlund summability, Riesz summability, and so on. Moreover, the notion of statistical convergence via various summability means has recently attracted the wide-spread attention of many researchers due mainly to the fact that it is more powerful than the classical versions of the convergence. In this context, attention of the interested researchers are drawn towards the recent published works [1]– [6], [12]– [16], [20]– [24], [26], and [27].

Besides, in the year 1996 Mittal and Kumar [19] established certain results based on the strong Nörlund summability means and, later on, Mittal (see details below, [18]) demonstrated a result on the strong Nörlund summability of the Fourier series. Recently, Jena et al. (see details below, [10]) established a result on the strong Riesz summability of the Fourier series of a 2π periodic function.

Theorem 1. [18] *If*

$$\Phi(x) = o\left(\frac{x}{\sqrt{\prod_{m=1}^{j+1} l_m(1/x)}}\right) \quad (x \to 0), \tag{6}$$

then the series $\sum_{k=0}^{\infty} u_k(x)$ *is* $[N, p_n^{(1)}, 2]$*-summable to the sum* ℓ*, for* $x = t$*, provided that*

$$p_n = \left[\prod_{m=0}^{j} l_m(n)\right]^{-1}, \quad m = 0, 1, 2, \ldots, \tag{7}$$

holds.

Theorem 2. [10] *If*

$$\Phi(t) = o\left(\frac{t}{\sqrt{\prod_{m=1}^{k+1} l_m(1/t)}}\right) \quad (t \to 0), \tag{8}$$

then the series $\sum_{n=0}^{\infty} u_n(t)$ *is strongly* $[\bar{N}, p_n^{(1)}, 2]$*-summable to the sum* s *(the same sum), for* $t = x$*, provided that*

$$p_n = \left[\prod_{m=0}^{k} l_m(n)\right]^{-1}, \quad k = 0, 1, 2, \ldots, \tag{9}$$

holds.

From the literatures cited above, it is clear that a few works are carried out based on the strong summability of the Fourier series of 2π periodic functions. However, no such result has been developed for the strong summability of the Fourier series of arbitrary periodic functions. Motivated essentially by the above mentioned investigations and developments, we have first introduced the notion of strong deferred Cesaro, deferred Nörlund, and deferred Riesz summability methods. We then considered our proposed strong Riesz summability mean to establish and prove a new theorem for summability of the Fourier series of an arbitrary periodic function. Moreover, in the last section we present some concluding remarks in which some earlier published results are recovered from our main non-trivial Theorem.

3. Axillary Lemmas. We need to prove the following lemmas for the proof of our main Theorem 3.

Lemma 1. *Let (a_n) and (b_n) be sequences of non-negative integers and if*

$$\Phi(x) = o(x), \quad x \to 0, \tag{10}$$

then

$$\mathcal{I}_1 = o(1) \quad \text{and} \quad \mathcal{I}_3 = o(1) \quad \text{as} \quad \frac{1}{r} \to 0.$$

Proof. By using Riemann–Lebesgue lemma and the regularity condition [10], $\mathcal{I}_3 = o(1)$ as $\frac{1}{r} \to 0$.

Since (a_n) and (b_n) are sequences of non-negative integers and $D\bar{N}_r(x) = O(r)$ is uniform in $\left[0, \frac{1}{r}\right]$, so, by (10), we have

$$\mathcal{I}_1 = O\Big[\int_0^{1/r} |\phi(x)| D\bar{N}_r(x)| dx\Big] = O(r)\Phi(1/r) =$$

$$= O(r)o(1/r) = o(1) \quad \text{as} \quad \frac{1}{r} \to 0.$$

The proof is completed. □

Lemma 2. *Let $\{p_n\}$ be a non-negative and non-increasing sequence, and let (a_n) and (b_n) be sequences of non-negative integers. If condition (10) of Lemma 1 holds, then*

$$\mathcal{I}_2 = \frac{1}{LP_r} \int_{1/r}^{\delta} \phi(x)\alpha(x)dx + \frac{1}{LP_r} \int_{1/r}^{\delta} \phi(x)\beta(x)\cot(\pi x/2L)dx + o(1). \tag{11}$$

Proof. Since $p_r \geqq p_{r+1}$, $\xi_n = O(1)$, we have

$$\mathcal{I}_2 = \frac{1}{LP_r} \int_{1/r}^{\delta} \phi(x) \left[\sum_{m=0}^{\tau} + \sum_{m=\tau+1}^{r}\right] p_m \frac{\sin\left[\frac{(m+1/2)x\pi}{L}\right]}{\sin\left(\frac{\pi x}{2L}\right)} dx =$$

$$= \frac{1}{LP_r} \int\limits_{1/r}^{\delta} \frac{\phi(x)}{\sin\left(\frac{\pi x}{2L}\right)} \left[\alpha(x)\sin\left(\frac{\pi x}{2L}\right) + \beta(x)\cos\left(\frac{\pi x}{2L}\right)\right] dx +$$

$$+ \frac{1}{LP_r} \int\limits_{1/r}^{\delta} \frac{\phi(x)}{\sin\left(\frac{\pi x}{2L}\right)} \left[\sum_{m=\tau+1}^{r} p_m \sin\left(\frac{(m+1/2)x\pi}{L}\right)\right] dx =$$

$$= \mathcal{I}_{2,1} + \mathcal{I}_{2,2}. \quad (12)$$

Thus, for proving the validity of (11), it suffices to show $I_{2,2} = o(1)$ as $r \to \infty$.

For

$$\sum_{m=\tau+1}^{r} p_m \sin\left(\frac{(m+1/2)x\pi}{L}\right) = O\left(\frac{p_\tau}{r}\right)$$

and $p_m \geqq p_{m+1}$, we have

$$\mathcal{I}_{2,2} = \frac{1}{LP_r} \int\limits_{1/r}^{\delta} \frac{\phi(x)}{\sin\left(\frac{Lx}{2L}\right)} \left[\sum_{m=\tau+1}^{r} p_m \sin\left(\frac{(m+1/2)x\pi}{L}\right)\right] dx =$$

$$= O\left(\frac{1}{LP_r} \int\limits_{1/r}^{\delta} \frac{\phi(x)}{x^2} p_\tau dx\right).$$

Now, using (7) and (10) and applying integration by parts, we have

$$\mathcal{I}_{2,2} = O\left[\frac{1}{LP_r}\left(\left[\frac{\Phi(x)p(1/x)}{x^2}\right]_{1/r}^{\delta} - \int\limits_{1/r}^{\delta} \Phi(x)\left(\frac{d}{dx}\frac{1}{x\prod_{m=1}^{k} l_m(1/x)}\right)dx\right)\right] =$$

$$= O\left[\frac{1}{LP_r}\left(\left\{o\left(\frac{p(1/x)}{x}\right)\right\}_{1/r}^{\delta} + \int\limits_{1/r}^{\delta} \frac{\Phi(x)}{x^2\prod_{m=1}^{k} l_m(1/x)}\right)\right] -$$

$$- O\left[\frac{1}{LP_r}\left(\int\limits_{1/r}^{\delta} \frac{\Phi(x)}{x}\frac{d}{dx}\left(\frac{1}{\prod_{m=1}^{k} l_m(1/x)}\right)dx\right)\right].$$

Furthermore, k being fixed, we can write

$$P_r = l_{k+1}(r) = l(r).$$

Clearly, we have

$$\mathcal{I}_{2,2} = o(\xi_r) + o(1) +$$
$$+ o\left(\frac{1}{Ll(r)}\right) \cdot \left(\int\limits_{1/r}^{\delta} \frac{1}{x \prod_{m=1}^{k} l_m(1/x)} dx + \int\limits_{1/r}^{\delta} \frac{d}{dx} \left\{ \frac{1}{\prod_{m=1}^{k} l_m(1/x)} \right\} dx \right) =$$
$$= o(1) + o\left(\frac{1}{\sqrt{l(r)}}\right) \int\limits_{1/r}^{\delta} \frac{1}{\left(x \prod_{m=1}^{k} l_m(1/x)\right) \sqrt{l(1/x)}} dx +$$
$$+ o\left(\frac{1}{l(r)} \left[\frac{1}{\prod_{m=1}^{k} l_m(1/x)} \right]_{1/r}^{\delta} \right) =$$
$$= o(1) + o(\sqrt{l(r)}/\sqrt{l(r)}) + o(1/l(r)) = o(1) \quad \text{as} \quad r \to \infty. \tag{13}$$

This completes the proof of Lemma 2. □

4. Main Theorem. In this section, we wish to consider our strong deferred Riesz summability mean to prove a new theorem for summability of the Fourier series of an arbitrary periodic function, as follows:

Theorem 3. *Let (a_n) and (b_n) be a sequences of non-negative integers; if*

$$\Phi(x) = o\left(\frac{x}{\sqrt{\prod_{m=1}^{k+1} l_m(1/x)}}\right) \quad (x \to 0), \tag{14}$$

then the series $\sum\limits_{n=0}^{\infty} u_n(x)$ is strongly $[D\bar{N}, p_n^{(1)}, 2]$-summable to ℓ for $x = t$, provided that

$$p_n = \left[\prod_{m=0}^{k} l_m(n)\right]^{-1} \quad (k \in \{0\} \cup \mathbb{N}) \tag{15}$$

holds.

5. Proof of the Main Theorem. Following Zygmund (see details, [28], p. 50), we have:

$$T_r(x) - \ell = \sum_{m=0}^{r} \frac{p_m}{P_r} \{s_m(\rho) - \ell\} =$$

$$= \sum_{m=0}^{r} \frac{p_m}{P_r} \left(\frac{1}{L} \int\limits_0^L \phi(\rho) \frac{\sin\left[\frac{(m+1/2)\rho\pi}{L}\right]}{\sin\left(\frac{\pi\rho}{2L}\right)} \right) d\rho =$$

$$= \sum_{m=0}^{r} \frac{p_m}{P_r} \left\{ \int\limits_0^{1/r} + \int\limits_{1/r}^{\delta} + \int\limits_{\delta}^{L} \right\} \phi(\rho) \frac{\sin\left[\frac{(m+1/2)\rho\pi}{L}\right]}{\sin\left(\frac{\pi\rho}{2L}\right)} d\rho =$$

$$= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \quad \text{(say)}. \tag{16}$$

In order to establish the theorem, it is enough to prove

$$\sum_{r=0}^{n} |T_r(x) - \ell|^2 = o(n) \quad (n \to \infty). \tag{17}$$

Using the results of Lemma 1 and Lemma 2, equation (16) leads to

$$|T_r(x) - \ell| =$$

$$= \frac{1}{LP_r} \int\limits_{1/r}^{\delta} \phi(\rho)\alpha(\rho) d\rho + \frac{1}{LP_r} \int\limits_{1/r}^{\delta} \phi(\rho)\beta(\rho) 2\cot\left(\frac{\pi\rho}{2L}\right) d\rho + o(1) \leqq$$

$$\leqq \frac{1}{LP_r} \int\limits_{1/n}^{\delta} \phi(\rho)\alpha(\rho) d\rho + \frac{1}{LP_r} \int\limits_{1/n}^{\delta} \phi(\rho)\beta(\rho) 2\cot\left(\frac{\pi\rho}{2L}\right) d\rho + o(1) \ (r \leqq n). \tag{18}$$

Moreover, $\alpha(\rho) = O(P_\tau) = \beta(\rho)$; so, using monotonicity of $[l(1/\rho)]^{-1}$, we have:

$$\left| \frac{1}{LP_r} \int\limits_{1/n}^{\delta} \phi(\rho)\alpha(\rho) d\rho \right| = O\left(\frac{1}{LP_r} \int\limits_{1/n}^{\delta} |\phi(\rho)| P(1/\rho) \right) d\rho =$$

$$= O\left(\frac{1}{Ll(r)}\right) \int\limits_{1/n}^{\delta} |\phi(\rho)| l(1/\rho) d\rho =$$

$$= O\left(\frac{1}{Ll(r)}\right) \left[\left(\Phi(\rho) l(1/\rho)\right)_{1/n}^{\delta} - \int\limits_{1/n}^{\delta} \frac{\Phi(\rho) d\rho}{\rho \prod_{m=1}^{k} l_m(1/\rho)} \right] =$$

$$= O\left(\frac{1}{l(r)}\right) \left[o\left(\frac{\rho l(1/\rho)}{\sqrt{\prod_{m=1}^{k+1} l_m(1/\rho)}} \right)_{1/n}^{\delta} \right] +$$

$$+\,O\left(\frac{1}{l(r)}\right)\left[\int_{1/n}^{\delta} o\left(\frac{\rho}{\sqrt{\prod_{m=1}^{k+1} l_m(1/\rho)}}\right)\frac{d\rho}{\rho\prod_{m=1}^{k} l_m(1/\rho)}\right]=$$

$$=o\left(\frac{1}{r}\right)+o\left(\frac{1}{l(r)\left\{\prod_{m=1}^{k} l_m(r)\right\}^{3/2}}\right)\int_{1/n}^{\delta} d\rho=o(1)\ \text{ as } \frac{1}{n}\to 0.$$

Similarly,

$$\left|\frac{1}{LP_r}\int_{1/n}^{\delta}\phi(\rho)\beta(\rho)2\cot\left(\frac{\pi\rho}{2L}\right)d\rho\right|=o(1)\text{ as }\frac{1}{n}\to 0.$$

Now, using Minkowski's inequality, we get:

$$\left\{\sum_{r=0}^{n}|T_r(x)-\ell|^2\right\}^{1/2}\leqq$$

$$\leqq\left[\sum_{r=0}^{n}\left\{\frac{1}{LP_r}\left(\int_{1/n}^{\delta}\phi(\rho)\alpha(\rho)d\rho+\int_{1/n}^{\delta}\phi(\rho)\beta(\rho)2\cot\left(\frac{\pi\rho}{2L}\right)d\rho\right)\right\}\times\right.$$

$$\left.\times\left\{\frac{1}{LP_r}\left(\int_{1/n}^{\delta}\phi(v)\alpha(v)dv+\int_{1/n}^{\delta}\phi(v)\beta(v)2\cot\left(\frac{\pi v}{2L}\right)dv\right)\right\}\right]^{1/2}+o(n)^{1/2}=$$

$$=[j_1+j_2+j_3+j_4]^{1/2}+o(n)^{1/2}. \quad (19)$$

Thus, to complete the proof, we need to verify that j_1, j_2, j_3, and j_4 individually tend to 0 as $n\to\infty$.

Let

$$j_1=\frac{1}{L^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta}\phi(\rho)\alpha(\rho)\phi(v)\alpha(v)\left\{\sum_{r=0}^{n}\frac{1}{P_r^2}\right\}d\rho dv.$$

As we have

$$\int_{1/n}^{\delta}|\phi(x)|l(1/x)dx=[\Phi(\rho)l(1/\rho)]_{1/n}^{\delta}+\int_{1/n}^{\delta}\Phi(\rho)\frac{1}{\rho\prod_{m=1}^{k} l_m(1/\rho)}d\rho=$$

$$= o\left(\frac{\rho l(1/\rho)}{\sqrt{\prod_{m=1}^{k+1} l_m(1/\rho)}}\right)_{1/n}^{\delta} + \int_{1/n}^{\delta}\left(\frac{1}{\sqrt{\prod_{m=1}^{k+1} l_m(1/\rho)}}\right)\left(\frac{1}{\prod_{m=1}^{k} l_m(1/\rho)}\right)d\rho =$$

$$= o\left(\frac{\rho l(1/\rho)}{\sqrt{\prod_{m=1}^{k+1} l_m(1/\rho)}}\right)_{1/n}^{\delta} + \left(\frac{1}{\sqrt{\prod_{m=1}^{k+1} l_m(1/\rho)}\prod_{m=1}^{k} l_m(1/\rho)}\right)\int_{1/n}^{\delta} d\rho =$$

$$= o(1) \quad \left(\frac{1}{n} \to 0\right).$$

So, for $\alpha(\rho) = O(P_\tau)$ and using (6), we get:

$$j_1 \leqq \sum_{r=0}^{n} \frac{1}{P_r^2} \int_{1/n}^{\delta} |\phi(\rho)| l(1/\rho) d\rho \int_{1/n}^{\delta} |\phi(v)| l(1/v) dv = o(1) \quad \left(\frac{1}{n} \to 0\right).$$

Next, as

$$\int_{1/n}^{\delta} |\phi(v)|\alpha(v)\left|2\cot\left(\frac{\pi v}{2L}\right)\right| dv = \int_{1/n}^{\delta} \frac{|\phi(v)|\alpha(v)}{v} dv =$$

$$= \left[\Phi(v)\frac{l(v)}{v}\right]_{1/n}^{\delta} - \int_{1/n}^{\delta}\left[\Phi(v)\frac{d}{dv}\left(\frac{l(1/v)}{v}\right)\right] dv =$$

$$= \left[\Phi(v)\frac{l(v)}{v}\right]_{1/n}^{\delta} + \int_{1/n}^{\delta} \Phi(v)\frac{1}{v}\left(\frac{1}{v\prod_{m=1}^{k} l_m(1/v)}\right) dv =$$

$$= o\left(\frac{l(1/v)}{\sqrt{\prod_{m=1}^{k+1} l_m(1/v)}}\right)_{1/n}^{\delta} +$$

$$+ o\left(\frac{1}{\left(\sqrt{\prod_{m=1}^{k+1} l_m(1/v)}\right)\left(v\prod_{m=1}^{k} l_m(1/v)\right)}\right)\int_{1/n}^{\delta} dv = o(1) \quad \left(\frac{1}{n} \to 0\right),$$

this implies

$$j_2 = \frac{1}{L^2}\int_{1/n}^{\delta}\int_{1/n}^{\delta} \phi(\rho)\alpha(\rho)\phi(v)\beta(v) 2\cot\left(\frac{\pi v}{2L}\right)\left\{\sum_{r=0}^{n}\frac{1}{P_r^2}\right\} d\rho dv \leqq$$

$$\leqq \frac{1}{L^2}\sum_{r=0}^{n}\frac{1}{P_r^2}\left(\int_{1/n}^{\delta}|\phi(\rho)|\alpha(\rho)d\rho\int_{1/n}^{\delta}|\phi(v)|\alpha(v)\left|\cot\left(\frac{\pi v}{2L}\right)\right|dv\right)=$$

$$=o(1)\int_{1/n}^{\delta}|\phi(\rho)|\alpha(\rho)d\rho\int_{1/n}^{\delta}|\phi(v)|\alpha(v)\left|\cot\left(\frac{\pi v}{2L}\right)\right|dv=o(1)\ \left(\frac{1}{n}\to 0\right).$$

In the similar way, we can obtain:

$$j_3=o(1)\quad\text{and}\quad j_4=o(1)\text{ as }\frac{1}{n}\to 0.$$

Hence, the proof of the theorem is completed.

6. Concluding Remarks and Discussion. In this final section of our investigation, we offer a number of additional remarks and observations regarding our findings that we have proved here.

Remark 1. *Let (a_n) and (b_n) be sequences of non-negative integers and satisfy the condition (6) of Theorem 1; then the series $\sum_{n=0}^{\infty} u_n(x)$ is strongly $[D\bar{N}, p_n^{(1)}, 2]$-summable to ℓ, for $x = t$, provided that*

$$p_n=\left[\prod_{m=0}^{k} l_m(n)\right]^{-1}\qquad (k\in\{0\}\cup\mathbb{N})\tag{20}$$

holds.

Remark 2. *If $(a_n) = 0$ and $(b_n) = n$ in Theorem 3, then, under the conditions (14) and (15), $\sum_{n=0}^{\infty} u_n(t)$ is strongly $[\bar{N}, p_n^{(1)}, 2]$-summable to ℓ.*

Remark 3. *If the condition (14) of Theorem 3 and the condition (1.8) of [17] are satisfied, then the series $\sum_{n=0}^{\infty} u_n(x)$ is $[DC, 1, 2]$-summable to the sum ℓ at $x = t$.*

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Jyotiranjan Sahoo
Department of Mathematics, Veer Surendra Sai University of Technology
Burla 768018, Odisha, India
E-mail: jsahoo520@gmail.com

Bidu Bhusan Jena
Faculty of Science (Mathematics), Sri Sri University
Cuttack 754006, Odisha, India
E-mail: bidumath.05@gmail.com

Susanta Kumar Paikray
Department of Mathematics, Veer Surendra Sai University of Technology
Burla 768018, Odisha, India
E-mail: skpaikray_math@vssut.ac.in