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LITTLEWOOD–PALEY g_λ^*-FUNCTION CHARACTERIZATIONS OF MUSIELAK–ORLICZ HARDY SPACES ON SPACES OF HOMOGENEOUS TYPE

Abstract. Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, in the sense of Coifman and Weiss, and $\varphi\colon\ \mathcal{X} \times [0, \infty) \to [0, \infty)$ satisfy that, for almost every $x \in \mathcal{X}$, $\varphi(x, \cdot)$ is an Orlicz function and that $\varphi(\cdot, t)$ is a Muckenhoupt weight uniformly in $t \in [0, \infty)$. In this article, by using the aperture estimate of Littlewood–Paley auxiliary functions on the Musielak–Orlicz space $L^\varphi(\mathcal{X})$, we obtain the Littlewood–Paley g_λ^*-function characterization of Musielak–Orlicz Hardy space $H^\varphi(\mathcal{X})$. Particularly, the range of λ coincides with the best-known one.

Key words: *space of homogeneous type, Musielak–Orlicz Hardy space, Littlewood–Paley auxiliary function, g_λ^*-function*

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1. Introduction. It is well known that the real-variable theory of Hardy-type spaces on $\mathbb{R}^n$, including various equivalent characterizations and the boundedness of singular integral operators, plays a fundamental role in harmonic analysis and partial differential equations (see, for instance, [26]). Recall that the classical Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ was originally introduced by Stein and Weiss [27]; this initiated the study of the real-variable theory of $H^p(\mathbb{R}^n)$. Particularly, Calderón and Torchinsky [4] established Littlewood–Paley function characterizations of $H^p(\mathbb{R}^n)$. Up to now, many new variants of classical Hardy spaces have sprung up and their real-variable theories have been well developed in order to meet the increasing demand from harmonic analysis, partial differential equations, and geometric analysis (see, for instance, [14], [25]).

The bilinear decompositions of the product of Hardy spaces and their dual spaces play key roles in improving the estimates of many nonlinear quantities, such as div-curl products (see, for instance, [31]), weak Jacobians (see, for instance, [18]), and commutators (see, for instance, [20]). Bonami et al. [3] showed that, for any given $f \in H^1(\mathbb{R}^n)$, there exist two bounded linear operators $S_f \colon \mathrm{BMO}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $T_f \colon \mathrm{BMO}(\mathbb{R}^n) \to H^{\Phi}_w(\mathbb{R}^n)$ such that, for any $g \in \mathrm{BMO}(\mathbb{R}^n)$, $f \times g = S_f g + T_f g$, where $H^{\Phi}_w(\mathbb{R}^n)$ denotes the weighted Orlicz–Hardy space associated to the weight function $w(x) := 1/\log(e + |x|)$ for any $x \in \mathbb{R}^n$ and to the Orlicz function

$$\Phi(t) := \frac{t}{\log(e + t)}, \ \forall\, t \in [0, \infty).$$

This result was essentially improved by Bonami et al. in [2], where they further proved the following bilinear decomposition:

$$H^1(\mathbb{R}^n) \times \mathrm{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\log}(\mathbb{R}^n),$$

where $H^{\log}(\mathbb{R}^n)$ denotes the Musielak–Orlicz Hardy space related to the Musielak–Orlicz function

$$\varphi(x, t) := \frac{t}{\log(e + t) + \log(e + |x|)}, \ \forall\, x \in \mathbb{R}^n, \forall\, t \in [0, \infty).$$

Bonami et al. in [2] also concluded that $H^{\log}(\mathbb{R}^n)$ is the smallest space in the dual sense. Motivated by these, Ky [21] introduced the Musielak–Orlicz Hardy space $H^{\varphi}(\mathbb{R}^n)$ with φ being a Musielak–Orlicz function, which generalizes both the Orlicz–Hardy space of Janson [19] and the weighted Hardy space of Strömberg and Torchinsky [28], and established both the atomic and the grand maximal function characterizations of $H^{\varphi}(\mathbb{R}^n)$. Since then, the real-variable theory of Musielak–Orlicz Hardy spaces has rapidly been developed. Precisely, Hou et al. [15] characterized $H^{\varphi}(\mathbb{R}^n)$ by the Lusin-area function and the molecule; Liang et al. [24] further established several other real-variable characterizations, respectively, in terms of various maximal functions and Littlewood–Paley g-function and g^*_λ-function.

On the other hand, there has been an increasing interest in extending the above results of Musielak–Orlicz Hardy spaces from the Euclidean space to more general underlying spaces, such as the anisotropic Euclidean space; see, for instance, [22], [23]. In particular, Coifman and Weiss [5] originally introduced the concept of the space $\mathcal{X}$ of homogeneous type and the atomic Hardy space $H^{p,q}_{\mathrm{cw}}(\mathcal{X})$ with $p \in (0, 1]$ and $q \in (p, \infty] \cap [1, \infty]$, and proved that $H^{p,q}_{\mathrm{cw}}(\mathcal{X})$ is independent of the choice of q. In the same article, they also posed a question: to what extent the geometrical condition of $\mathcal{X}$ is necessary for the validity of the radial maximal function characterization of $H^1_{\mathrm{cw}}(\mathcal{X})$. Since then, lots of efforts have been made to establish various real-variable characterizations of the atomic Hardy spaces on $\mathcal{X}$ with few geometrical assumptions. However, due to the lack of Calderón reproducing formulae on $\mathcal{X}$, many existing results of both function spaces and boundedness of operators require some additional geometrical assumptions on $\mathcal{X}$, such as the reverse doubling condition of μ (see, for instance, [6]).

Recently, He et al. [12] first introduced a kind of approximations of the identity with exponential decay and then obtained new Calderón reproducing formulae on $\mathcal{X}$. Later, He et al. completely answered the aforementioned question of Coifman and Weiss by developing a quite complete real-variable theory of the Hardy space and its localized version on $\mathcal{X}$, respectively, in [11] and [13]. Fu et al. [7] further generalized the corresponding results in [11] to Musielak–Orlicz Hardy spaces $H^{\varphi}(\mathcal{X})$. Indeed, let $\eta \in (0,1)$, ω be the upper dimension of $\mathcal{X}$, and φ a growth function, with uniformly lower type $p \in (0,1]$, satisfying that

$$\frac{p}{q(\varphi)} \in \Big(\frac{\omega}{\omega+\eta}, 1\Big],$$

where $q(\varphi)$ is the critical weight index of φ. Fu et al. in [7, Theorem 6.16] characterized $H^{\varphi}(\mathcal{X})$ via the Littlewood–Paley g_λ^*-function with

$$\lambda \in \bigg(\omega\Big[\frac{2q(\varphi)}{p} + 1\Big], \infty\bigg).$$

In this article, we first establish an aperture estimate of Littlewood–Paley auxiliary functions on the Musielak–Orlicz space $L^{\varphi}(\mathcal{X})$, and then obtain the Littlewood–Paley g_λ^*-function characterizations of $H^{\varphi}(\mathcal{X})$ with $\lambda \in (2\omega q(\varphi)/p, \infty)$, which improves the corresponding results in [7, Theorem 6.16] via widening the range of λ into the best-known one.

The organization of the remainder of this article is as follows.

In Section 2, we recall some notation and concepts that are used throughout this article. More precisely, in Subsection 2.1, we recall the definition of a space $\mathcal{X}$ of homogeneous type and state some basic properties of $\mathcal{X}$. In Subsection 2.2, we introduce the concepts of the uniformly Muckenhoupt condition, the Musielak–Orlicz space $L^{\varphi}(\mathcal{X})$, the spaces of test functions and distributions, the system of dyadic cubes, and approximations of the identity with exponential decay on $\mathcal{X}$. Then, via the

Lusin-area function S_α with $\alpha \in (0,\infty)$, we introduce the Musielak–Orlicz Hardy space $H^\varphi(\mathcal{X})$.

In Section 3, we first recall the concepts of Littlewood–Paley g_λ^*-function and auxiliary function $S_\alpha^{(a)}$. Then, by an argument similar to that used in the proof of [11, Lemma 5.11], we establish an aperture estimate of $S_\alpha^{(a)}$ on $L^\varphi(\mathcal{X})$ (see Lemma 5 below). Finally, from Lemma 5 and the fact that g_λ^* and $S_\alpha^{(a)}$ are pointwisely comparable, we further obtain the Littlewood–Paley g_λ^*-function characterizations of $H^\varphi(\mathcal{X})$ with the best known range $\lambda \in (2\omega q(\varphi)/p, \infty)$.

At the end of this section, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We denote by C a *positive constant* which is independent of the main parameters, but may vary from line to line. We use $C_{(\alpha,\ldots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \ldots$. The symbol $f \lesssim g$ means $f \leqslant Cg$ and, if $f \lesssim g \lesssim f$, then we write $f \sim g$. If $f \leqslant Cg$ and $g = h$ or $g \leqslant h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, rather than $f \lesssim g = h$ or $f \lesssim g \leqslant h$. If E is a subset of $\mathcal{X}$, we denote by $\mathbf{1}_E$ its *characteristic function* and by $E^\complement$ the set $\mathcal{X} \backslash E$. For any $\alpha \in \mathbb{R}$, we denote by $\lfloor \alpha \rfloor$ the biggest integer not greater than α. For any index $q \in [1,\infty]$, we denote by q' its *conjugate index*, namely, $1/q + 1/q' = 1$. For any $x, x_0, \in \mathcal{X}$ and $r, \vartheta \in (0, \infty)$, let $V_r(x) := \mu(B(x, r))$,

$$V(x, y) := \begin{cases} \mu(B(x, d(x, y))), & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases}$$

and

$$P_\vartheta(x_0, x; r) := \frac{1}{V_r(x_0) + V(x_0, x)} \left[\frac{r}{r + d(x_0, x)}\right]^\vartheta. \tag{1}$$

2. Preliminaries. In this section, we recall some basic concepts about the space $\mathcal{X}$ of homogeneous type and Musielak–Orlicz Hardy spaces.

2.1. Spaces of Homogeneous Type. In this subsection, we recall the concept of spaces of homogeneous type and some related basic estimates.

Definition 1. *A* quasi-metric space $(\mathcal{X}, d)$ *is a non-empty set* $\mathcal{X}$ *equipped with a* quasi-metric d*, namely, a non-negative function defined on* $\mathcal{X} \times \mathcal{X}$ *such that for any* $x, y, z \in \mathcal{X}$*:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*

(ii) $d(x, y) = d(y, x)$;

(iii) *there exists a constant* $A_0 \in [1, \infty)$, *independent of* x, y, *and* z, *such that*

$$d(x, z) \leqslant A_0[d(x, y) + d(y, z)]. \tag{2}$$

The *ball* B of $\mathcal{X}$, centered at $x_0 \in \mathcal{X}$ with radius $r \in (0, \infty)$, is defined by setting

$$B := B(x_0, r) := \{x \in \mathcal{X} : d(x, x_0) < r\}.$$

For any ball B and any $\tau \in (0, \infty)$, we denote $B(x_0, \tau r)$ by τB if $B := B(x_0, r)$ for some $x_0 \in \mathcal{X}$ and $r \in (0, \infty)$.

Definition 2. *Let* $(\mathcal{X}, d)$ *be a quasi-metric space and* μ *a non-negative measure on* $\mathcal{X}$. *The triple* $(\mathcal{X}, d, \mu)$ *is called* a space of homogeneous type *if* μ *satisfies the following* doubling condition: *there exists a constant* $C_{(\mu)} \in [1, \infty)$, *such that for any ball* $B \subset \mathcal{X}$:

$$\mu(2B) \leqslant C_{(\mu)}\mu(B).$$

The above doubling condition implies that for any ball $B \subset \mathcal{X}$ and any $\lambda \in [1, \infty)$,

$$\mu(\lambda B) \leqslant C_{(\mu)}\lambda^{\omega}\mu(B), \tag{3}$$

where $\omega := \log_2 C_{(\mu)}$ is called the *upper dimension* of $\mathcal{X}$.

Throughout this article, according to [5, pp. 587–588], we always make the following assumptions on $(\mathcal{X}, d, \mu)$:

(i) for any $x \in \mathcal{X}$, the balls $\{B(x, r)\}_{r \in (0, \infty)}$ form a basis of open neighborhoods of x;

(ii) μ is *Borel regular*, which means that all open sets are μ-measurable and every set $A \subset \mathcal{X}$ is contained in a Borel set E, such that $\mu(A) = \mu(E)$;

(iii) for any $x \in \mathcal{X}$ and $r \in (0, \infty)$, $\mu(B(x, r)) \in (0, \infty)$;

(iv) $\operatorname{diam} \mathcal{X} := \sup_{x, y \in \mathcal{X}} d(x, y) = \infty$, and $(\mathcal{X}, d, \mu)$ is *non-atomic*, which means $\mu(\{x\}) = 0$ for any $x \in \mathcal{X}$.

Notice that $\operatorname{diam} \mathcal{X} = \infty$ implies that $\mu(\mathcal{X}) = \infty$ (see, for instance, [1, p. 284]). From this, it follows that, under the above assumptions, $\mu(\mathcal{X}) = \infty$ if and only if $\operatorname{diam} \mathcal{X} = \infty$.

The following basic estimates are from [9, Lemma 2.1], which can be proved by using (3).

Lemma 1. *Let $x, y \in \mathcal{X}$ and $r \in (0,\infty)$. Then $V(x,y) \sim V(y,x)$ and*

$$V_r(x) + V_r(y) + V(x,y) \sim V_r(x) + V(x,y) \sim \\ \sim V_r(y) + V(x,y) \sim \mu(B(x, r + d(x,y))).$$

Moreover, if $d(x,y) \leqslant r$, then $V_r(x) \sim V_r(y)$. Here the positive equivalence constants are independent of x, y, and r.

2.2. Musielak–Orlicz Hardy Spaces. Throughout this article, we always let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type. In this subsection, we recall the concept of Musielak–Orlicz Hardy spaces and state some known results.

A function $\Phi\colon\ [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is non-decreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0,\infty)$, and $\lim_{t\to\infty} \Phi(t) = \infty$.

Now, we recall the concept of uniformly upper and lower types, which was introduced in [16, p. 1924].

Definition 3. *For a given function $\varphi\colon \mathcal{X} \times [0,\infty) \to [0,\infty)$, such that, for almost every $x \in \mathcal{X}$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of* uniformly upper (resp., *lower*) type p *for some $p \in (0,\infty)$ if there exists a positive constant $C_{(p)}$, depending on p, such that, for almost every $x \in \mathcal{X}$, $s \in [1,\infty)$ (resp., $s \in [0,1]$), and $t \in [0,\infty)$,*

$$\varphi(x, st) \leqslant C_{(p)} s^p \varphi(x,t).$$

Next, we recall the concept of the uniformly Muckenhoupt condition from [16, Definition 2.6].

Definition 4. *A function $\varphi\colon\ \mathcal{X} \times [0,\infty) \to [0,\infty)$ is said to satisfy the* uniformly Muckenhoupt condition *for some $q \in [1,\infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathcal{X})$, if, when $q \in (1,\infty)$,*

$$[\varphi]_{\mathbb{A}_q(\mathcal{X})} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathcal{X}} \frac{1}{[\mu(B)]^q} \int_B \varphi(x,t) d\mu(x) \left\{ \int_B [\varphi(y,t)]^{-\frac{1}{(q-1)}} d\mu(y) \right\}^{q-1} < \infty,$$

or

$$[\varphi]_{\mathbb{A}_1(\mathcal{X})} := \sup_{t\in(0,\infty)} \sup_{B\subset\mathcal{X}} \frac{1}{\mu(B)} \int_B \varphi(x,t)\, d\mu(x) \left(\operatorname{ess\,sup}_{y\in B} [\varphi(y,t)]^{-1} \right) < \infty,$$

where the first suprema are taken over all $t \in (0,\infty)$ and the second ones over all balls $B \subset \mathcal{X}$.

Throughout this article, let

$$\mathbb{A}_\infty(\mathcal{X}) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathcal{X}).$$

For any $\varphi \in \mathbb{A}_\infty(\mathcal{X})$, μ-measurable set $E \subset \mathcal{X}$, and $t \in [0,\infty)$, let

$$\varphi(E,t) := \int_E \varphi(x,t)\, d\mu(x).$$

For any given $p \in (0,\infty)$, a function f is said to be *locally p-integrable* if, for any $x \in \mathcal{X}$, there exists an $r \in (0,\infty)$, such that

$$\int_{B(x,r)} |f(y)|^p \, d\mu(y) < \infty.$$

Denote by $L^p_{\mathrm{loc}}(\mathcal{X})$ the set of all the *locally p-integrable functions* on $\mathcal{X}$. In what follows, we always let $\mathcal{M}$ denote the *Hardy–Littlewood maximal operator* defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $x \in \mathcal{X}$,

$$\mathcal{M}(f)(x) := \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y), \tag{4}$$

where the supremum is taken over all balls B of $\mathcal{X}$ containing x.

Now, we state some basic properties of $\mathbb{A}_q(\mathcal{X})$ with $q \in [1,\infty)$, which are just parts of [7, Lemma 2.6] (see also [30, Lemma 1.1.3] for the corresponding Euclidean case).

Lemma 2. *The following conclusions hold true:*

(i) $\mathbb{A}_1(\mathcal{X}) \subset \mathbb{A}_p(\mathcal{X}) \subset \mathbb{A}_q(\mathcal{X})$ *for any p, q satisfying $1 \leqslant p \leqslant q < \infty$.*

(ii) *If $\varphi \in \mathbb{A}_q(\mathcal{X})$ with $q \in [1,\infty)$, then there exists a positive constant C, such that, for any ball $B \subset \mathcal{X}$, μ-measurable set $E \subset B$, and $t \in (0,\infty)$,*

$$\frac{\varphi(B,t)}{\varphi(E,t)} \leqslant C \left[\frac{\mu(B)}{\mu(E)}\right]^q.$$

(iii) *If $q \in (1,\infty)$ and $\varphi \in \mathbb{A}_q(\mathcal{X})$, then there exists a positive constant C, such that, for any $f \in L^1_{\mathrm{loc}}(\mathcal{X})$ and $t \in [0,\infty)$,*

$$\int_{\mathcal{X}} [\mathcal{M}(f)(x)]^q \varphi(x,t)\, d\mu(x) \leqslant C \int_{\mathcal{X}} |f(x)|^q \varphi(x,t)\, d\mu(x),$$

where $\mathcal{M}$ is the same as in (4).

The *critical weight index* $q(\varphi)$ of $\varphi \in \mathbb{A}_\infty(\mathcal{X})$ is defined by setting

$$q(\varphi) := \inf\left\{q \in [1,\infty) : \varphi \in \mathbb{A}_q(\mathcal{X})\right\}. \tag{5}$$

The following concept of growth functions was first introduced in [16, Definition 2.7].

Definition 5. *A function $\varphi\colon\ \mathcal{X} \times [0,\infty) \to [0,\infty)$ is called a* growth function *if the following conditions are satisfied:*

(i) *φ is a* Musielak–Orlicz function, *namely,*

(i)$_1$ *the function $\varphi(x,\cdot)\colon [0,\infty) \to [0,\infty)$ is an Orlicz function for almost every $x \in \mathcal{X}$;*

(i)$_2$ *the function $\varphi(\cdot,t)$ is μ-measurable for any $t \in [0,\infty)$.*

(ii) *$\varphi \in \mathbb{A}_\infty(\mathcal{X})$.*

(iii) *φ is of uniformly lower type p for some $p \in (0,1]$ and of uniformly upper type 1.*

Next, we recall the definition of Musielak–Orlicz spaces, which was first introduced in [16, Definition 2.8].

Definition 6. *Let φ be a growth function in Definition 5. The* Musielak–Orlicz space *$L^\varphi(\mathcal{X})$ is defined to be the set of all the μ-measurable functions f, such that*

$$\int_{\mathcal{X}} \varphi(x, |f(x)|)\, d\mu(x) < \infty,$$

equipped with the Luxemburg *(also called the* Luxemburg–Nakano*)* (quasi-)norm

$$\|f\|_{L^\varphi(\mathcal{X})} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathcal{X}} \varphi\Big(x, \frac{|f(x)|}{\lambda}\Big)\, d\mu(x) \leqslant 1\right\}.$$

Now, we recall some basic properties of $L^{\varphi}(\mathcal{X})$, which were first given in [7, Lemma 2.8] (see also [30, Lemmas 1.1.6 and 1.1.10] for the corresponding Euclidean case).

Lemma 3. *Let φ be a growth function in Definition 5. Then the following conclusions hold true.*

(i) *φ is uniformly σ-quasi-subadditive on $\mathcal{X}\times[0,\infty)$, namely, there exists a positive constant C, such that, for any $(x,t_j)\in\mathcal{X}\times[0,\infty)$ with $j\in\mathbb{N}$,*

$$\varphi\Big(x,\sum_{j\in\mathbb{N}}t_j\Big)\leqslant C\sum_{j\in\mathbb{N}}\varphi(x,t_j).$$

(ii) *For any $f\in L^{\varphi}(\mathcal{X})\backslash\{\mathbf{0}\}$,*

$$\int_{\mathcal{X}}\varphi\Big(x,\frac{|f(x)|}{\|f\|_{L^{\varphi}(\mathcal{X})}}\Big)\,d\mu(x)=1.$$

(iii) *For any $(x,t)\in\mathcal{X}\times[0,\infty)$,*

$$\widetilde{\varphi}(x,t):=\int_0^t\frac{\varphi(x,s)}{s}\,ds$$

is a growth function and equivalent to φ, namely, there exists a positive constant C, such that, for any $(x,t)\in\mathcal{X}\times[0,\infty)$,

$$\frac{1}{C}\varphi(x,t)\leqslant\widetilde{\varphi}(x,t)\leqslant C\varphi(x,t).$$

Moreover, for almost every $x\in\mathcal{X}$, $\widetilde{\varphi}(x,\cdot)$ is continuous and strictly increasing.

Next, we introduce the Musielak–Orlicz Hardy spaces via the Lusin-area functions. To this end, we first recall the concept of spaces of test functions on $\mathcal{X}$, which was originally introduced by Han et al. [9, Definition 2.2] (see also [10, Definition 2.8]).

Definition 7. *Let $x_0\in\mathcal{X}$, $r\in(0,\infty)$, $\varrho\in(0,1]$, and $\vartheta\in(0,\infty)$. A function f on $\mathcal{X}$ is called a* test function of type $(x_0,r,\varrho,\vartheta)$, *denoted by $f\in\mathcal{G}(x_0,r,\varrho,\vartheta)$, if there exists a positive constant C, such that*

(T1) *for any* $x \in \mathcal{X}$,

$$|f(x)| \leqslant CP_\vartheta(x_0, x; r), \tag{6}$$

here and thereafter, P_ϑ *is the same as in* (1)*;*

(T2) *for any* $x, y \in \mathcal{X}$ *satisfying* $d(x,y) \leqslant [r + d(x_0, x)]/(2A_0)$ *with* A_0 *the same as in* (2),

$$|f(x) - f(y)| \leqslant C\left[\frac{d(x,y)}{r + d(x_0,x)}\right]^\varrho P_\vartheta(x_0, x; r). \tag{7}$$

Moreover, for any $f \in \mathcal{G}(x_0, r, \varrho, \vartheta)$*, define*

$$\|f\|_{\mathcal{G}(x_0, r, \varrho, \vartheta)} := \inf\{C \colon C \text{ satisfies (6) and (7)}\}.$$

The subspace $\mathring{\mathcal{G}}(x_0, r, \varrho, \vartheta)$ *is defined by setting*

$$\mathring{\mathcal{G}}(x_0, r, \varrho, \vartheta) := \left\{f \in \mathcal{G}(x_0, r, \varrho, \vartheta) \colon \int_{\mathcal{X}} f(x)\, d\mu(x) = 0\right\}$$

equipped with the norm $\|\cdot\|_{\mathring{\mathcal{G}}(x_0,r,\varrho,\vartheta)} := \|\cdot\|_{\mathcal{G}(x_0,r,\varrho,\vartheta)}$.

Fix an $x_0 \in \mathcal{X}$. Denote $\mathring{\mathcal{G}}(x_0, 1, \varrho, \vartheta)$ simply by $\mathring{\mathcal{G}}(\varrho, \vartheta)$. Obviously, $\mathring{\mathcal{G}}(\varrho, \vartheta)$ is a Banach space. Note that, for any fixed $x \in \mathcal{X}$ and $r \in (0, \infty)$, $\mathring{\mathcal{G}}(x, r, \varrho, \vartheta) = \mathring{\mathcal{G}}(\varrho, \vartheta)$ with equivalent norms, but the positive equivalence constants may depend on x and r.

Fix $\epsilon \in (0, 1]$ and $\varrho, \vartheta \in (0, \epsilon]$. Let $\mathring{\mathcal{G}}_0^\epsilon(\varrho, \vartheta)$ be the completion of the the set $\mathring{\mathcal{G}}(\epsilon, \epsilon)$ in $\mathring{\mathcal{G}}(\varrho, \vartheta)$. Furthermore, the norm of $\mathring{\mathcal{G}}_0^\epsilon(\varrho, \vartheta)$ is defined by setting $\|\cdot\|_{\mathring{\mathcal{G}}_0^\epsilon(\varrho,\vartheta)} := \|\cdot\|_{\mathcal{G}(\varrho,\vartheta)}$. The space $\mathring{\mathcal{G}}_0^\epsilon(\varrho, \vartheta)$ is called the *space of test functions*. The *dual space* $(\mathring{\mathcal{G}}_0^\epsilon(\varrho, \vartheta))'$ is defined to be the set of all continuous linear functionals from $\mathring{\mathcal{G}}_0^\epsilon(\varrho, \vartheta)$ to $\mathbb{C}$, equipped with the weak-$*$ topology. The space $(\mathring{\mathcal{G}}_0^\epsilon(\varrho, \vartheta))'$ is called the *space of distributions.*

The following system of dyadic cubes of $(\mathcal{X}, d, \mu)$ was established by Hytönen and Kairema in [17, Theorem 2.2].

Lemma 4. *Suppose that constants* $0 < c_0 \leqslant C_0 < \infty$ *and* $\delta \in (0, 1)$ *satisfy* $12A_0^3C_0\delta \leqslant c_0$ *with* A_0 *the same as in* (2). *Assume that a set of points* $\{z_\alpha^k \colon k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\} \subset \mathcal{X}$ *with* $\mathcal{A}_k$, *for any* $k \in \mathbb{Z}$, *being a set of indices, has the following properties: for any* $k \in \mathbb{Z}$,

$$d\left(z_\alpha^k, z_\beta^k\right) \geqslant c_0\delta^k \text{ if } \alpha \neq \beta, \text{ and } \min_{\alpha\in\mathcal{A}_k} d\left(x, z_\alpha^k\right) < C_0\delta^k \text{ for any } x \in \mathcal{X}.$$

Then there exists a family of sets $\{Q_\alpha^k \colon k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\}$*, satisfying*

(i) *for any* $k \in \mathbb{Z}$, $\bigcup_{\alpha\in\mathcal{A}_k} Q_\alpha^k = \mathcal{X}$ *and* $\{Q_\alpha^k\colon \alpha \in \mathcal{A}_k\}$ *is disjoint;*

(ii) *if* $k, l \in \mathbb{Z}$ *and* $k \leqslant l$ *then, for any* $\alpha \in \mathcal{A}_k$ *and* $\beta \in \mathcal{A}_l$, *either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \varnothing$;

(iii) *for any* $k\in\mathbb{Z}$ *and* $\alpha \in\mathcal{A}_k$, $B(z_\alpha^k, (3A_0^2)^{-1}c_0\delta^k) \subset Q_\alpha^k \subset B(z_\alpha^k, 2A_0C_0\delta^k)$.

Throughout this article, for any $k \in \mathbb{Z}$, define

$$\mathcal{G}_k := \mathcal{A}_{k+1}\backslash\mathcal{A}_k \text{ and } \mathcal{Y}^k := \left\{z_\alpha^{k+1}\right\}_{\alpha\in\mathcal{G}_k} =: \left\{y_\alpha^k\right\}_{\alpha\in\mathcal{G}_k},$$

and, for any $x \in \mathcal{X}$, define

$$d\left(x, \mathcal{Y}^k\right) := \inf_{y\in\mathcal{Y}^k} d(x, y).$$

Now, recall the concept of approximations of the identity with exponential decay introduced in [12, Definition 2.7].

Definition 8. *Let* δ *be the same as in Lemma 4. A sequence* $\{Q_k\}_{k\in\mathbb{Z}}$ *of bounded linear integral operators on* $L^2(\mathcal{X})$ *is called an* approximation of the identity with exponential decay *(for short, exp-ATI) if there exist constants* $C,\nu \in (0, \infty)$, $a \in (0, 1]$, *and* $\eta \in (0, 1)$, *such that, for any* $k \in \mathbb{Z}$, *the kernel of the operator* Q_k, *a function on* $\mathcal{X} \times \mathcal{X}$, *which is still denoted by* Q_k, *has the following properties:*

(i) *(the* identity condition*)* $\sum_{k\in\mathbb{Z}} Q_k = I$ *in* $L^2(\mathcal{X})$, *where* I *denotes the* identity operator *on* $L^2(\mathcal{X})$;

(ii) *(the* size condition*) for any* $x, y \in \mathcal{X}$,

$$|Q_k(x, y)| \leqslant CE_k(x, y),$$

here and thereafter,

$$E_k(x, y) := \frac{1}{\sqrt{V_{\delta^k}(x)V_{\delta^k}(y)}} \exp\left\{-\nu\left[\frac{d(x, y)}{\delta^k}\right]^a\right\}\times \\ \times \exp\left\{-\nu\left[\frac{\max\{d(x, \mathcal{Y}^k), d(y, \mathcal{Y}^k)\}}{\delta^k}\right]^a\right\};$$

(iii) *(the* regularity condition*) for any* $x, x', y \in \mathcal{X}$ *with* $d(x, x') \leqslant \delta^k$,

$$|Q_k(x, y) - Q_k(x', y)|+|Q_k(y, x) - Q_k(y, x')| \leqslant \\ \leqslant C\left[\frac{d(x, x')}{\delta^k}\right]^\eta E_k(x, y);$$

(iv) *(the* second difference regularity condition*) for any $x, x', y, y' \in \mathcal{X}$ with $d(x,x') \leqslant \delta^k$ and $d(y,y') \leqslant \delta^k$,*

$$|[Q_k(x,y) - Q_k(x',y)] - [Q_k(x,y') - Q_k(x',y')]| \leqslant \\ \leqslant C\Big[\frac{d(x,x')}{\delta^k}\Big]^\eta \Big[\frac{d(y,y')}{\delta^k}\Big]^\eta E_k(x,y);$$

(v) *(the* cancellation condition*) for any $x, y \in \mathcal{X}$,*

$$\int\limits_{\mathcal{X}} Q_k(x,y')\,d\mu(y') = 0 = \int\limits_{\mathcal{X}} Q_k(x',y)\,d\mu(x').$$

Next, we recall the concept of the Lusin-area function (see, for instance, [11, Section 5]).

Definition 9. *Let δ and η be the same, respectively, as in Lemma 4 and Definition 8, and let $\varrho, \vartheta \in (0,\eta)$. Assume that $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ and $\{Q_k\}_{k\in\mathbb{Z}}$ is an* exp-ATI. *For any $\alpha \in (0,\infty)$, the* Lusin-area function $S_\alpha(f)$ of f with aperture α *is defined by setting, for any $x \in \mathcal{X}$,*

$$S_\alpha(f)(x) := \Bigg\{\sum_{k\in\mathbb{Z}} \int\limits_{B(x,\alpha\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\alpha\delta^k}(x)}\Bigg\}^{\frac{1}{2}},$$

here and thereafter, for any $y \in \mathcal{X}$,

$$Q_k f(y) := \int\limits_{\mathcal{X}} Q_k(y,x) f(x)\,d\mu(x).$$

When $\alpha := 1$, we simply write $S := S_1$.

Now, we recall the concept of Musielak–Orlicz Hardy spaces, which was first introduced in [7, Definition 6.2].

Definition 10. *Let η be the same as in Definition 8 and φ a growth function in Definition 5 with uniformly lower type $p \in (0,1]$ satisfying*

$$\frac{p}{q(\varphi)} > \frac{\omega}{\omega+\eta},$$

and let

$$\varrho, \vartheta \in \Big(\omega\Big[\frac{q(\varphi)}{p} - 1\Big], \eta\Big),$$

where $q(\varphi)$ *and* ω *are the same, respectively, as in* (5) *and* (3). *The* Musielak–Orlicz Hardy space $H^{\varphi}(\mathcal{X})$ *is defined by setting*

$$H^{\varphi}(\mathcal{X}) := \left\{ f \in \left(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta) \right)' : \ \|\mathcal{S}(f)\|_{L^{\varphi}(\mathcal{X})} < \infty \right\}$$

and, moreover, for any $f \in H^{\varphi}(\mathcal{X})$, *let*

$$\|f\|_{H^{\varphi}(\mathcal{X})} := \|\mathcal{S}(f)\|_{L^{\varphi}(\mathcal{X})}.$$

Remark 1.

(i) *As was proved in [7, Theorem 6.3], the space* $H^{\varphi}(\mathcal{X})$ *in Definition 10 is independent of the choices of exp-ATIs in* $S(f)$.

(ii) *Combining [29, Remark 3.17(iii)], [7, Theorems 5.4 and 6.15], and [7, Proposition 6.12], we conclude that the space* $H^{\varphi}(\mathcal{X})$ *in Definition 10 is independent of the choices of* $(\mathring{\mathcal{G}}_0^{\eta}(\varrho,\vartheta))'$ *whenever*

$$\varrho, \vartheta \in (\omega[q(\varphi)/p - 1], \eta).$$

3. Littlewood–Paley g_λ^*-Function Characterizations of $H^{\varphi}(\mathcal{X})$. In this section, we establish Littlewood–Paley g_λ^*-function characterizations of $H^{\varphi}(\mathcal{X})$, which improves the corresponding results in [7, Theorem 6.16] by widening the range of the parameter λ into the best-known one. To this end, we first recall the concept of Littlewood–Paley g_λ^*-function (see, for instance, [11, Section 5]).

Definition 11. *Let* δ *and* η *be the same, respectively, as in Lemma 4 and Definition 8, and let* $\varrho, \vartheta \in (0, \eta)$. *Assume that* $f \in (\mathring{\mathcal{G}}_0^{\eta}(\varrho, \vartheta))'$ *and* $\{Q_k\}_{k\in\mathbb{Z}}$ *is an* exp-ATI. *The* Littlewood–Paley g_λ^*-function $g_\lambda^*(f)$ *of* f, *with any given* $\lambda \in (0, \infty)$, *is defined by setting, for any* $x \in \mathcal{X}$,

$$g_\lambda^*(f)(x) := \left\{ \sum_{k\in\mathbb{Z}} \int_{\mathcal{X}} |Q_k f(y)|^2 \left[\frac{\delta^k}{\delta^k + d(x,y)} \right]^{\lambda} \frac{d\mu(y)}{V_{\delta^k}(x) + V_{\delta^k}(y)} \right\}^{\frac{1}{2}}.$$

The following theorem is the main result of this section.

Theorem 1. *Let* η *be the same as in Definition 8 and* φ *a growth function in Definition 5 with uniformly lower type* $p \in (0, 1]$ *satisfying*

$$\frac{p}{q(\varphi)} > \frac{\omega}{\omega + \eta},$$

and let

$$\varrho, \vartheta \in \Big(\omega\Big[\frac{q(\varphi)}{p} - 1\Big], \eta\Big),$$

where $q(\varphi)$ and ω are the same, respectively, as in (5) and (3). Further, assume that

$$\lambda \in \Big(\frac{2\omega q(\varphi)}{p}, \infty\Big).$$

Then $f \in H^{\varphi}(\mathcal{X})$ if and only if $f \in (\mathring{\mathcal{G}}_0^{\eta}(\varrho, \vartheta))'$ and $g_\lambda^*(f) \in L^{\varphi}(\mathcal{X})$. Moreover, there exists a constant $C \in [1, \infty)$, such that, for any $f \in H^{\varphi}(\mathcal{X})$,

$$C^{-1} \|g_\lambda^*(f)\|_{L^{\varphi}(\mathcal{X})} \leqslant \|f\|_{H^{\varphi}(\mathcal{X})} \leqslant C \|g_\lambda^*(f)\|_{L^{\varphi}(\mathcal{X})}.$$

To prove Theorem 1, we need more preparations. Let δ and η be the same as in Theorem 1. For any $\varrho, \vartheta \in (0, \eta)$, $\alpha \in (0, \infty)$, and $f \in (\mathring{\mathcal{G}}_0^{\eta}(\varrho, \vartheta))'$, recall that the *Littlewood–Paley auxiliary function $S_\alpha^{(a)}$ of f with aperture* α is defined by setting, for any $x \in \mathcal{X}$,

$$S_\alpha^{(a)}(f)(x) := \Bigg\{\sum_{k\in\mathbb{Z}} \int\limits_{B(x,\alpha\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(y)}\Bigg\}^{\frac{1}{2}}. \tag{8}$$

Particularly, when $\alpha = 1$, by Lemma 1, we conclude that, for any $f \in (\mathring{\mathcal{G}}_0^{\eta}(\varrho, \vartheta))'$ and $x \in \mathcal{X}$,

$$S_1^{(a)}(f)(x) \sim S(f)(x), \tag{9}$$

where the implicit positive constant is independent of both f and x.

The following conclusion, which shows an aperture estimate of $S_\alpha^{(a)}(f)$ on $L^{\varphi}(\mathcal{X})$, plays a key role in the proof of Theorem 1.

Lemma 5. *Let $q \in (1, \infty)$ and $\varphi \in \mathbb{A}_q(\mathcal{X})$ with uniformly lower type $p \in (0, 1]$. If $\alpha \in [1, \infty)$, then there exists a positive constant C, such that*

$$\int\limits_{\mathcal{X}} \varphi\left(x, S_\alpha^{(a)}(f)(x)\right) d\mu(x) \leqslant C\alpha^{\omega q} \int\limits_{\mathcal{X}} \varphi\left(x, S_1^{(a)}(f)(x)\right) d\mu(x), \tag{10}$$

where ω is the same as in (3).

Proof. For any non-negative function g and any $x \in \mathcal{X}$, define

$$\widetilde{\mathcal{M}}(g)(x) := \sup_{k\in\mathbb{Z}} \sup_{d(x,y)<\alpha\delta^k} \frac{1}{V_{\delta^k}(y)} \int\limits_{B(y,\delta^k)} g(z)\, d\mu(z),$$

where δ is the same as in Lemma 4. Moreover, for any $t \in (0, \infty)$ and $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho, \vartheta))'$ with η being the same as in Definition 8 and $\varrho, \vartheta \in (0, \eta)$, define

$$E_t := \left\{x \in \mathcal{X} \colon S_1^{(a)}(f)(x) > t\right\} \text{ and } \widetilde{E}_t := \left\{x \in \mathcal{X} \colon \widetilde{\mathcal{M}}(\mathbf{1}_{E_t})(x) > \frac{1}{2}\right\}.$$

On the one hand, by [11, p. 2252], we conclude that, for any non-negative function g and any $x \in \mathcal{X}$,

$$\widetilde{\mathcal{M}}(g)(x) \lesssim \alpha^\omega \mathcal{M}(g)(x)$$

with $\mathcal{M}$ as in (4), which, combined with Lemma 2(iii), further implies that, for any $t \in (0, \infty)$,

$$\begin{aligned} \varphi\left(\widetilde{E}_t, t\right) &= \varphi\left(\left\{x \in \mathcal{X} \colon \widetilde{\mathcal{M}}(\mathbf{1}_{E_t})(x) > \frac{1}{2}\right\}, t\right) \leqslant \\ &\leqslant \varphi\left(\left\{x \in \mathcal{X} \colon C\alpha^\omega \mathcal{M}(\mathbf{1}_{E_t})(x) > \frac{1}{2}\right\}, t\right) \leqslant \\ &\leqslant C\alpha^{\omega q} \int_{\mathcal{X}} \left[\mathcal{M}(\mathbf{1}_{E_t})(x)\right]^q \varphi(x, t)\, d\mu(x) \leqslant C\alpha^{\omega q} \varphi\left(E_t, t\right), \end{aligned} \tag{11}$$

where C is a positive constant independent of both α and t.

On the other hand, fix $t \in (0, \infty)$ and, for any $y \in \mathcal{X}$, let

$$\rho(y) := \inf_{x \in \widetilde{E}_t^\complement} d(x, y).$$

Obviously, for any $k \in \mathbb{Z}$ and $x, y \in \mathcal{X}$, $x \in \widetilde{E}_t^\complement \cap B(y, \alpha\delta^k)$ implies that $\rho(y) < \alpha\delta^k$. Moreover, by an argument similar to that used in [11, p. 2253], we conclude that, for any $k \in \mathbb{Z}$ and any $y \in \mathcal{X}$ satisfying $\rho(y) < \alpha\delta^k$,

$$\mu\left(E_t^\complement \cap B(y, \delta^k)\right) \geqslant \frac{1}{2}\mu\left(B(y, \delta^k)\right).$$

From these, (8), Tonelli theorem, (3), and Lemma 2(ii), we deduce that, for any $t \in (0, \infty)$,

$$\int_{\widetilde{E}_t^\complement} \left[S_\alpha^{(a)}(f)(x)\right]^2 \varphi(x, t)\, d\mu(x) =$$

$$= \int_{\widetilde{E}_t^\complement} \sum_{k\in\mathbb{Z}} \int_{B(x,\alpha\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(y)} \varphi(x,t)\, d\mu(x) \leqslant$$

$$\leqslant \sum_{k\in\mathbb{Z}} \int_{\rho(y)<\alpha\delta^k} |Q_k f(y)|^2 \varphi\left(\widetilde{E}_t^\complement \cap B(y,\alpha\delta^k), t\right) \frac{d\mu(y)}{V_{\delta^k}(y)} \leqslant$$

$$\leqslant \sum_{k\in\mathbb{Z}} \int_{\rho(y)<\alpha\delta^k} |Q_k f(y)|^2 \varphi\left(B(y,\alpha\delta^k), t\right) \frac{d\mu(y)}{V_{\delta^k}(y)} \lesssim$$

$$\lesssim \alpha^{\omega q} \sum_{k\in\mathbb{Z}} \int_{\rho(y)<\alpha\delta^k} |Q_k f(y)|^2 \varphi\left(B(y,\delta^k), t\right) \frac{d\mu(y)}{V_{\delta^k}(y)} \lesssim$$

$$\lesssim \alpha^{\omega q} \sum_{k\in\mathbb{Z}} \int_{\mathcal{X}} |Q_k f(y)|^2 \varphi\left(E_t^\complement \cap B(y,\delta^k), t\right) \frac{d\mu(y)}{V_{\delta^k}(y)} \sim$$

$$\sim \alpha^{\omega q} \sum_{k\in\mathbb{Z}} \int_{E_t^\complement} \int_{B(x,\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(y)} \varphi(x,t)\, d\mu(x) \sim$$

$$\sim \alpha^{\omega q} \int_{E_t^\complement} \left[S_1^{(a)}(f)(x)\right]^2 \varphi(x,t)\, d\mu(x).$$

This, together with Tonelli theorem and the fact that φ is of uniformly upper type 1, further implies that, for any $t \in (0,\infty)$,

$$\varphi\left(\widetilde{E}_t^\complement \cap \left\{x \in \mathcal{X} \colon S_\alpha^{(a)}(f)(x) > t\right\}, t\right) =$$

$$= \int_{\widetilde{E}_t^\complement \cap \{x\in\mathcal{X}\colon S_\alpha^{(a)}(f)(x)>t\}} \varphi(x,t)\, d\mu(x) \leqslant$$

$$\leqslant \int_{\widetilde{E}_t^\complement \cap \{x\in\mathcal{X}\colon S_\alpha^{(a)}(f)(x)>t\}} \left[\frac{S_\alpha^{(a)}(f)(x)}{t}\right]^2 \varphi(x,t)\, d\mu(x) \leqslant$$

$$\leqslant t^{-2} \int_{\widetilde{E}_t^\complement} \left[S_\alpha^{(a)}(f)(x)\right]^2 \varphi(x,t)\, d\mu(x) \lesssim$$

$$\lesssim \alpha^{\omega q} t^{-2} \int_{E_t^\complement} \left[S_1^{(a)}(f)(x)\right]^2 \varphi(x,t)\, d\mu(x) \sim$$

$$\sim \alpha^{\omega q} t^{-2} \int_{\mathcal{X}} \left[S_1^{(a)}(f)(x) \mathbf{1}_{\{y \in \mathcal{X}:\, S_1^{(a)}(f)(y) \leqslant t\}}(x) \right]^2 \varphi(x,t)\, d\mu(x) \sim$$

$$\sim \alpha^{\omega q} t^{-2} \int_{\mathcal{X}} \int_0^{S_1^{(a)}(f)(x) \mathbf{1}_{\{y \in \mathcal{X}:\, S_1^{(a)}(f)(y) \leqslant t\}}(x)} s\varphi(x,t)\, ds\, d\mu(x) \sim$$

$$\sim \alpha^{\omega q} t^{-2} \int_0^{\infty} \int_{\{x \in \mathcal{X}:\, S_1^{(a)}(f)(x) \mathbf{1}_{\{y \in \mathcal{X}:\, S_1^{(a)}(f)(y) \leqslant t\}}(x) > s\}} s\varphi(x,t)\, d\mu(x)\, ds \sim$$

$$\sim \alpha^{\omega q} t^{-2} \int_0^{t} \int_{\{x \in \mathcal{X}:\, S_1^{(a)}(f)(x) \mathbf{1}_{\{y \in \mathcal{X}:\, S_1^{(a)}(f)(y) \leqslant t\}}(x) > s\}} s\varphi(x,t)\, d\mu(x)\, ds \lesssim$$

$$\lesssim \alpha^{\omega q} t^{-2} \int_0^{t} \int_{\{x \in \mathcal{X}:\, S_1^{(a)}(f)(x) \mathbf{1}_{\{y \in \mathcal{X}:\, S_1^{(a)}(f)(y) \leqslant t\}}(x) > s\}} s \cdot \frac{t}{s} \cdot \varphi(x,s)\, d\mu(x)\, ds \sim$$

$$\sim \alpha^{\omega q} t^{-1} \int_0^{t} \varphi\left(\left\{ x \in \mathcal{X}:\, S_1^{(a)}(f)(x) \mathbf{1}_{\{y \in \mathcal{X}:\, S_1^{(a)}(f)(y) \leqslant t\}}(x) > s \right\}, s \right) ds \lesssim$$

$$\lesssim \alpha^{\omega q} t^{-1} \int_0^{t} \varphi\left(\left\{ x \in \mathcal{X}:\, S_1^{(a)}(f)(x) > s \right\}, s \right) ds. \tag{12}$$

Combining (11) and (12), we find that, for any $t \in (0,\infty)$,

$$\varphi\left(\left\{ x \in \mathcal{X}:\, S_\alpha^{(a)}(f)(x) > t \right\}, t \right) \leqslant$$

$$\leqslant \varphi\left(\widetilde{E}_t, t \right) + \varphi\left(\widetilde{E}_t^{\complement} \cap \left\{ x \in \mathcal{X}:\, S_\alpha^{(a)}(f)(x) > t \right\}, t \right) \lesssim$$

$$\lesssim \alpha^{\omega q} \left[\varphi\left(E_t, t\right) + t^{-1} \int_0^{t} \varphi\left(\left\{ x \in \mathcal{X}:\, S_1^{(a)}(f)(x) > s \right\}, s \right) ds \right]. \tag{13}$$

Moreover, from Lemma 3(iii) and Tonelli theorem, we deduce that, for any $\alpha \in (0,\infty)$,

$$\int_{\mathcal{X}} \varphi\left(x, S_\alpha^{(a)}(f)(x)\right) d\mu(x) \sim \int_{\mathcal{X}} \int_0^{S_\alpha^{(a)}(f)(x)} \frac{\varphi(x,t)}{t} dt\, d\mu(x) \sim$$

$$\sim \int_0^\infty t^{-1} \int_{\{y\in\mathcal{X}:\, S_\alpha^{(a)}(f)(y)>t\}} \varphi(x,t)d\mu(x)\,dt \sim$$

$$\sim \int_0^\infty t^{-1}\varphi\left(\left\{x\in\mathcal{X}:\, S_\alpha^{(a)}(f)(x)>t\right\},t\right)dt,$$

which, together with (13) and Tonelli theorem, further implies that

$$\int_{\mathcal{X}} \varphi\left(x, S_\alpha^{(a)}(f)(x)\right)\, d\mu(x) \sim$$

$$\sim \int_0^\infty t^{-1}\varphi\left(\left\{x\in\mathcal{X}:\, S_\alpha^{(a)}(f)(x)>t\right\},t\right)dt \lesssim$$

$$\lesssim \alpha^{\omega q}\Bigg[\int_0^\infty t^{-1}\varphi\left(E_t,t\right)dt + \int_0^\infty t^{-2}\int_0^t \varphi\Big(\Big\{x\in\mathcal{X}:\, S_1^{(a)}(f)(x)>s\Big\},s\Big)dsdt\Bigg] \sim$$

$$\sim \alpha^{\omega q}\Bigg[\int_0^\infty t^{-1}\varphi\left(E_t,t\right)dt + \int_0^\infty\int_s^\infty t^{-2}\varphi\Big(\Big\{x\in\mathcal{X}:\, S_1^{(a)}(f)(x)>s\Big\},s\Big)dtds\Bigg] \sim$$

$$\sim \alpha^{\omega q}\Bigg[\int_0^\infty t^{-1}\varphi\left(E_t,t\right)\,dt + \int_0^\infty s^{-1}\varphi\left(E_s,s\right)\,ds\Bigg] \sim$$

$$\sim \alpha^{\omega q}\int_0^\infty t^{-1}\varphi\left(\Big\{x\in\mathcal{X}:\, S_1^{(a)}(f)(x)>t\Big\},t\right)\,dt \sim$$

$$\sim \alpha^{\omega q}\int_{\mathcal{X}} \varphi\left(x, S_1^{(a)}(f)(x)\right)\, d\mu(x).$$

This finishes the proof of (10) and, hence, of Lemma 5. □

Now, we prove Theorem 1.

Proof of Theorem 1. We first prove the sufficiency. Let $f \in (\mathring{\mathcal{G}}_0^\eta(\varrho,\vartheta))'$ and $g_\lambda^*(f) \in L^\varphi(\mathcal{X})$. By Definition 11, Lemma 1, and Definition 9, we conclude that, for any $x\in\mathcal{X}$,

$$g_\lambda^*(f)(x) \geqslant \left\{\sum_{k\in\mathbb{Z}}\int_{B(x,\delta^k)} |Q_k f(y)|^2\left[\frac{\delta^k}{\delta^k+d(x,y)}\right]^\lambda \frac{d\mu(y)}{V_{\delta^k}(x)+V_{\delta^k}(y)}\right\}^{1/2} \sim$$

$$\sim \left\{ \sum_{k\in\mathbb{Z}} \int\limits_{B(x,\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(x)} \right\}^{1/2} \sim S(f)(x),$$

which, combined with Definition 10, further implies that

$$\|f\|_{H^\varphi(\mathcal{X})} = \|S(f)\|_{L^\varphi(\mathcal{X})} \lesssim \|g^*_\lambda(f)\|_{L^\varphi(\mathcal{X})}.$$

This shows $f \in H^\varphi(\mathcal{X})$ and, hence, finishes the proof of the sufficiency.

Next, we prove the necessity. By Definition 11 and (8), we find that, for any $f \in H^\varphi(\mathcal{X})$ and $x \in \mathcal{X}$,

$$\begin{aligned}
[g^*_\lambda(f)(x)]^2 &= \Bigg[\sum_{k\in\mathbb{Z}} \int\limits_{B(x,\delta^k)} + \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \int\limits_{B(x,2^{j+1}\delta^k)\setminus B(x,2^j\delta^k)} \Bigg] |Q_k f(y)|^2 \times \\
&\quad \times \left[\frac{\delta^k}{\delta^k + d(x,y)}\right]^\lambda \frac{d\mu(y)}{V_{\delta^k}(x) + V_{\delta^k}(y)} \lesssim \\
&\lesssim \sum_{k\in\mathbb{Z}} \int\limits_{B(x,\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(y)} + \\
&\quad + \sum_{j=0}^{\infty} 2^{-(j+1)\lambda} \sum_{k\in\mathbb{Z}} \int\limits_{B(x,2^{j+1}\delta^k)} |Q_k f(y)|^2 \frac{d\mu(y)}{V_{\delta^k}(y)} \sim \\
&\sim \left[S_1^{(a)}(f)(x)\right]^2 + \sum_{j=1}^{\infty} 2^{-j\lambda} \left[S_{2^j}^{(a)}(f)(x)\right]^2 \sim \\
&\sim \sum_{j=0}^{\infty} 2^{-j\lambda} \left[S_{2^j}^{(a)}(f)(x)\right]^2,
\end{aligned}$$

which further implies that

$$g^*_\lambda(f)(x) \lesssim \left\{ \sum_{j=0}^{\infty} 2^{-j\lambda} \left[S_{2^j}^{(a)}(f)(x)\right]^2 \right\}^{\frac{1}{2}} \lesssim \sum_{j=0}^{\infty} 2^{-\frac{j\lambda}{2}} S_{2^j}^{(a)}(f)(x). \tag{14}$$

Moreover, from $\lambda > \frac{2\omega q(\varphi)}{p}$, (5), and Lemma 2(i), it follows that there exists a $q > q(\varphi)$, such that $\lambda > \frac{2\omega q}{p}$ and $\varphi \in \mathbb{A}_q(\mathcal{X})$. By this, (14), the facts that $\varphi(x,\cdot)$ is non-decreasing for almost every $x \in \mathcal{X}$ and φ is of uniformly lower type p, Lemma 3(i), and Lemma 5, we conclude that, for any $f \in H^\varphi(\mathcal{X})$,

$$\int_{\mathcal{X}} \varphi\Big(x, g_\lambda^*(f)(x)\Big)\, d\mu(x) \lesssim \int_{\mathcal{X}} \varphi\Big(x, \sum_{j=0}^{\infty} 2^{-\frac{j\lambda}{2}} S_{2^j}^{(a)}(f)(x)\Big)\, d\mu(x) \lesssim$$
$$\lesssim \sum_{j=0}^{\infty} \int_{\mathcal{X}} \varphi\left(x, 2^{-\frac{j\lambda}{2}} S_{2^j}^{(a)}(f)(x)\right)\, d\mu(x) \lesssim$$
$$\lesssim \sum_{j=0}^{\infty} 2^{-\frac{j\lambda p}{2}} \cdot 2^{j\omega q} \int_{\mathcal{X}} \varphi\left(x, S_1^{(a)}(f)(x)\right)\, d\mu(x) \sim$$
$$\sim \int_{\mathcal{X}} \varphi\left(x, S_1^{(a)}(f)(x)\right)\, d\mu(x),$$

which, together with (9), Lemma 3(ii), the positive homogeneity of both g_λ^* and $S_1^{(a)}$, and the fact that φ is of uniformly upper type 1, further implies that

$$\int_{\mathcal{X}} \varphi\Big(x, \frac{g_\lambda^*(f)(x)}{\|f\|_{H^\varphi(\mathcal{X})}}\Big)\, d\mu(x) = \int_{\mathcal{X}} \varphi\Big(x, g_\lambda^*\Big(\frac{f}{\|f\|_{H^\varphi(\mathcal{X})}}\Big)(x)\Big) \lesssim$$
$$\lesssim \int_{\mathcal{X}} \varphi\Big(x, S_1^{(a)}\Big(\frac{f}{\|f\|_{H^\varphi(\mathcal{X})}}\Big)(x)\Big)\, d\mu(x) \sim$$
$$\sim \int_{\mathcal{X}} \varphi\Big(x, \frac{S(f)(x)}{\|S(f)\|_{L^\varphi(\mathcal{X})}}\Big)\, d\mu(x) \sim 1.$$

Thus, there exists a positive constant C, such that, for any $f \in H^\varphi(\mathcal{X})$,

$$\|g_\lambda^*(f)\|_{L^\varphi(\mathcal{X})} \leqslant C\|f\|_{H^\varphi(\mathcal{X})}.$$

This finishes the proof of the necessity and, hence, of Theorem 1. □

Remark 2. *Let ω and η be the same, respectively, as in (3) and Definition 8.*

(i) *Assume that $p \in (\omega/(\omega + \eta), 1]$ and*

$$\varphi(x, t) := t^p,\ \forall\, x \in \mathcal{X},\ \forall\, t \in [0, \infty).$$

Then $H^\varphi(\mathcal{X})$ is just the classical Hardy space $H^p(\mathcal{X})$. In this case, Theorem 1 shows the Littlewood–Paley g_λ^*-function characterization of $H^p(\mathcal{X})$ with the best known range $\lambda \in (2\omega/p, \infty)$, which coincides with [11, Theorem 5.12].

(ii) *Recall that Fu et al. established the Littlewood–Paley g_λ^*-function characterization of $H^\varphi(\mathcal{X})$ with $\lambda \in (\omega[\frac{2q(\varphi)}{p}+1], \infty)$ in [7, Theorem 6.16]. Thus, Theorem 1 improves the conclusion of [7, Theorem 6.16] by widening the range of λ into $\lambda \in (\frac{2\omega q(\varphi)}{p}, \infty)$.*

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