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HYPERELLIPTIC INTEGRALS AND SPECIAL FUNCTIONS FOR THE SPATIAL VARIATIONAL PROBLEM

Abstract. The study of the properties of special functions plays an important role in solving many problems in geometric function theory. We study the properties of hyperelliptic integrals and special functions, which definition includes a parameter that depends on the dimension of the space. The appearance of these functions is associated with the solution of a specific variational problem of finding in n-dimensional Euclidean space a surface that has the smallest area in a given metric among the hypersurfaces formed by rotation around the polar axis of a plane curve connecting two fixed points in the upper half-plane.

Key words: *special functions, hyperelliptic integrals, modulus of a family of surfaces, variational problem*

2020 Mathematical Subject Classification: *33E20, 30C65, 30C70, 49Q05*

1. Introduction. The study of the properties of special functions and their application to solving extremal problems of geometric function theory and, in particular, quasiconformal mappings is the subject of many works by G. D. Anderson, M.K. Vamanamuthy, M. Vuorinen, T. Sugawa, X. Zhang, and others (see, for example, [1]– [4]). Dependence on dimension of the volume of an n-dimensional ball of unit radius expressed in terms of the gamma function, and various relationships associated with this quantity, was studied in [5], [6], [9].

In the work of the authors [8], a solution was obtained to the variational problem; it arose when studying the change in the modulus of a family of surfaces that separate the boundary components of a spherical ring in the n-dimensional Euclidean space E^n ($n \geqslant 3$), upon transition to its subfamily consisting of surfaces enveloping the continuum (obstacle) belonging to the ring.

Let $x = (x_1, x_2, \dots x_n)$ be a point in E^n, $n \geqslant 3$, $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ be the length of x. Let us choose the Ox_1-axis as the polar axis in the system of spherical coordinates in E^n and define a complex structure on the two-dimensional plane Ox_1x_2, identifying it with the complex plane $\mathbb{C}_z$.

We consider a family of plane piecewise-smooth curves $\gamma = \gamma_L(\theta, \psi)$ given by the parametric equation $z(\tau) = e^{\rho(\tau) + i\varphi(\tau)}$, $\tau \in [\theta, \psi]$, and connecting the points $z_0 = rLe^{i\theta}$ and $z_1 = rle^{i\psi}$ $(1 \leqslant l < L)$ in the closed set

$$\overline{B_r} = \{z \colon r \leqslant |z| \leqslant Lr, \arg z \in [\theta, \psi], 0 < \theta < \psi < \pi, r > 0\}.$$

It is assumed that at the points of differentiability $\rho'(\tau) \leqslant 0$ and $\varphi'(\tau) \geqslant 0$. The choice of curve representation is due to the convenience for further analysis.

The variational problem mentioned above is to find, among the surfaces formed by rotation in E^n around the polar axis of the curves $\gamma_L(\theta, \psi)$, the surface of the smallest area calculated in the metric $\rho_0(x) = |x|^{1-n}\omega_{n-1}^{-1}$, where ω_{n-1} is the surface area of the hypersphere unit. This metric is extremal for the modulus of the family of surfaces separating the boundary components of the spherical ring in E^n [7]. The description of optimal trajectories of the variational problem and the calculation of the areas of minimal surfaces leads to a class of special functions, the representation of which involves hyperelliptic integrals of a standardized forms:

$$\Theta_n(a, b, c) = \int\limits_a^b \frac{dx}{\sqrt{c^2 - x^2}\sqrt{x^{2(n-2)} - 1}}, \quad E_n(a, b, c) = \int\limits_a^b \frac{\sqrt{x^{2(n-2)} - 1}\, dx}{\sqrt{c^2 - x^2}}, \tag{1}$$

where $1 \leqslant a < b \leqslant c < \infty$ and $n \geqslant 3$ is a natural number.

The purpose of this work is to study the general properties of such hyperelliptic integrals and apply them to study the extremal properties of the special functions under consideration.

In Section 2, we study the behavior of hyperelliptic integrals (1) depending on the parameters. These results are used in Section 3 to study the properties of special functions that describe optimal trajectories of the variational problem. Section 4 is devoted to the study of extremal properties of special functions related to the area of minimal surfaces formed by rotation around the polar axis of optimal trajectories. The established properties have application to the solution of some variational and extremal problems for capacities and modules of spatial condensers. In Section 5, we present the results of numerical experiments carried out using PTC Mathcad Prime to construct graphs of special functions and calculate the values of extremal functions and constants defined in the statements proven in the work for various values of dimension.

2. General properties of hyperelliptic integrals Θ_n and E_n.

Along with the general representation of functionals (1), we also consider the following particular forms:

$$\Theta_n(a,b) = \int\limits_a^b \frac{dx}{\sqrt{b^2-x^2}\sqrt{x^{2(n-2)}-1}}, \tag{2}$$

$$\Theta_n(b) = \int\limits_1^b \frac{dx}{\sqrt{b^2-x^2}\sqrt{x^{2(n-2)}-1}}, \tag{3}$$

$$E_n(a,b) = \int\limits_a^b \frac{\sqrt{x^{2(n-2)}-1}\,dx}{\sqrt{b^2-x^2}}, \tag{4}$$

$$E_n(b) = \int\limits_1^b \frac{\sqrt{x^{2(n-2)}-1}\,dx}{\sqrt{b^2-x^2}}. \tag{5}$$

Note that for $n = 3$ functionals (2)–(5) are elliptic integrals that can be reduced to the Legendre normal form.

Lemma 1. *1) $\Theta_n(a,b,c)$ is strictly decreases as a function of n and function of $a \in [1,b)$, is strictly increases as a function of $b \in [1;c]$, and strictly decreases as a function of $c \in [b,\infty)$ for fixed values of other variables; $\Theta_n(a)$ strictly decreases.*

2) For natural $n \geqslant 3$ and any values $1 \leqslant a < b \leqslant c < \infty$, the following chains of inequalities hold:

$$\Theta_n(a,b,c) \leqslant \Theta_n(1,b,c) \leqslant \Theta_n(c) \leqslant \Theta_n(b) \leqslant \Theta_n(a) < \\ < \Delta_n := \lim_{a\to 1} \Theta_n(a) = \frac{\pi}{2\sqrt{n-2}};$$

$$\Theta_n(a,b,c) \leqslant \Theta_n(a,b) \leqslant \Theta_n(b) \leqslant \Theta_n(a) < \Delta_n.$$

3) $\lim\limits_{b\to a}\Theta_n(a,b)=\lim\limits_{b\to\infty}\Theta_n(a,b)=0$ *and* $\max\limits_{b\in(a;\infty)}\Theta_n(a,b)=\Theta_n(a,b_n(a))$, *where* $b_n(a)$ *is a solution of equality*

$$\int\limits_a^b \frac{(n-2)x^{2(n-2)}dx}{\sqrt{b^2-x^2}(x^{2(n-2)}-1)^{3/2}}=\frac{a}{\sqrt{b^2-a^2}\sqrt{a^{2(n-2)}-1}}, \tag{6}$$

or equality

$$\Theta_n(a,b)=\frac{a}{(n-2)\sqrt{b^2-a^2}\sqrt{a^{2(n-2)}-1}}-\int\limits_a^b\frac{dx}{\sqrt{b^2-x^2}[x^{2(n-2)}-1]^{3/2}}. \tag{7}$$

Proof. Direct calculations show that

$$\frac{\partial\Theta_n(a,b,c)}{\partial n}<0,\ \frac{\partial\Theta_n(a,b,c)}{\partial a}<0,\ \frac{\partial\Theta_n(a,b,c)}{\partial b}>0,$$
$$\frac{\partial\Theta_n(a,b,c)}{\partial c}<0,\quad \frac{\partial\Theta_n(a,b)}{\partial a}<0,$$

whence follows the strict monotonicity of the function $\Theta_n(a,b,c)$ in each of the variables, as well as the strict monotonicity of the function $\Theta_n(a,b)$ in the variable a. Hence, $\Theta_n(a,b,c)\leqslant\Theta_n(1,b,c)\leqslant\Theta_n(c)$ and $\Theta_n(a,b,c)\leqslant$ $\leqslant\Theta_n(a,b)\leqslant\Theta_n(b)$.

Setting $x=1+(a-1)y$, we replace the variable in the integral

$$\Theta_n(a)=\int\limits_1^a\frac{dx}{\sqrt{a^2-x^2}\sqrt{x^{2(n-2)}-1}}.$$

After simple transformations, we get

$$\Theta_n(a)=\int\limits_0^1\frac{dy}{\sqrt{a+1-2y-(a-1)y^2}\sqrt{\Sigma_{k=1}^{2(n-2)}C_{2(n-2)}^k(a-1)^{k-1}y^k}}. \tag{8}$$

From this it follows that $\frac{d\Theta_n(a)}{da}<0$, i.e. $\Theta_n(a)$ strictly decreases, and

$$\sup_{a\in(1;\infty)}\Theta_n(a)=\lim_{a\to1}\Theta_n(a)=\frac{1}{2\sqrt{n-2}}\int\limits_0^1\frac{dy}{\sqrt{y}\sqrt{1-y}}=\Delta_n=\frac{\pi}{2\sqrt{n-2}}.$$

Similarly, setting $x = 1 + (b-1)y$, replacing the variable in the integral $\Theta_n(a,b)$, we find:

$$\Theta_n(a,b) = \int\limits_{\frac{a-1}{b-1}}^{1} \frac{dy}{\sqrt{b+1-2y-(b-1)y^2}\sqrt{\Sigma_{k=1}^{2(n-2)} C_{2(n-2)}^{k} (b-1)^{k-1} y^k}}. \quad (9)$$

This implies $\lim_{b\to a} \Theta_n(a,b) = \lim_{b\to\infty} \Theta_n(a,b) = 0$.

On the other hand, setting $x = b\sin t$ in the integral $\Theta_n(a,b)$, we find:

$$\Theta_n(a,b) = \int\limits_{\arcsin\frac{a}{b}}^{\pi/2} \frac{dt}{\sqrt{(b\sin t)^{2(n-2)} - 1}}.$$

Hence,

$$\frac{\partial\Theta_n(a,b)}{\partial b} = \frac{a}{b\sqrt{b^2-a^2}\sqrt{a^{2(n-2)}-1}} - \int\limits_{\arcsin\frac{a}{b}}^{\pi/2} \frac{(n-2)b^{2n-5}\sin^{2(n-2)} t dt}{[(b\sin t)^{2(n-2)}-1]^{3/2}},$$

or, after the reverse replace,

$$\frac{\partial\Theta_n(a,b)}{\partial b} = \frac{a}{b\sqrt{b^2-a^2}\sqrt{a^{2(n-2)}-1}} - \frac{n-2}{b}\int\limits_{a}^{b} \frac{x^{2(n-2)}dx}{\sqrt{b^2-x^2}(x^{2(n-2)}-1)^{3/2}}. \quad (10)$$

Therefore, the maximum value of $\Theta_n(a,b)$ on the interval $b \in (a,\infty)$ is achieved at $b = b_n(a)$, which is the root of equation (6) or equation (7), obtained from (6) using simple transformations. □

Lemma 2. *1) $E_n(a,b,c)$ is strictly increases as a function of n, strictly decreases as a function of $a \in [1,b)$, strictly increases as a function of $b \in [1,c]$, and strictly decreases as a function of $c \in [b,\infty)$; $E_n(a,b)$ strictly decreases as a function of $a \in [1,b)$ and strictly increases as a function of $b \in (a,\infty)$ for fixed values of other variables; $E_n(b)$ strictly increases. Moreover, for natural $n \geqslant 3$ and any values $1 \leqslant a < b \leqslant c < \infty$, the following chains of inequalities hold:*

$$E_n(a,b,c) \leqslant E_n(a,b) \leqslant E_n(b) \leqslant E_n(c) < \lim_{c\to\infty} E_n(c) = \infty;$$

$$E_n(a,b,c) \leqslant E_n(1,b,c) \leqslant E_n(c) < \infty.$$

2) The following functional equation holds:

$$\frac{\partial E_n(b)}{\partial b} = \frac{n-2}{b}\left[E_n(b) + \Theta_n(b)\right]. \tag{11}$$

Proof. 1) Direct calculations show that $\dfrac{\partial E_n(a,b,c)}{\partial n} > 0$, $\dfrac{\partial E_n(a,b,c)}{\partial a} < 0$, $\dfrac{\partial E_n(a,b,c)}{\partial b} > 0$, $\dfrac{\partial E_n(a,b,c)}{\partial c} < 0$, and $\dfrac{\partial E_n(a,b)}{\partial a} < 0$. Replacing the variable in the integral $E_n(a,b)$ setting $x = b\sin t$, we find:

$$E_n(a,b) = \int\limits_a^b \frac{\sqrt{x^{2(n-2)}-1}dx}{\sqrt{b^2-x^2}} = \int\limits_{\arcsin\frac{a}{b}}^{\pi/2} \sqrt{(b\sin t)^{2(n-2)}-1}dt. \tag{12}$$

Hence,

$$\frac{\partial E_n(a,b)}{\partial b} = \frac{a}{b}\frac{\sqrt{a^{2(n-2)}-1}}{\sqrt{b^2-a^2}} + \int\limits_{\arcsin\frac{a}{b}}^{\pi/2} \frac{(n-2)b^{2n-5}\sin^{2(n-2)}t dt}{\sqrt{(b\sin t)^{2(n-2)}-1}} > 0, \tag{13}$$

whence follows the strict monotonicity of functions $E_n(a,b,c)$, $E_n(a,b)$ and $E_n(b)$ in each of the variables, as well as the chain of inequalities given in the formulation of the lemma.

2) From (13), it follows that

$$\frac{\partial E_n(b)}{\partial b} = \frac{n-2}{b}\int\limits_{\arcsin\frac{1}{b}}^{\pi/2} \frac{(b\sin t)^{2(n-2)}}{\sqrt{(b\sin t)^{2(n-2)}-1}}dt.$$

Taking into account the definitions of functions $E_n(b)$ and $\Theta_n(b)$, after simple transformations we obtain (11). □

3. Hyperelliptic integrals in the representation of optimal trajectories for the variational problem.

In the metric $\rho_0(x) = |x|^{1-n}\omega_{n-1}^{-1}$, the area $F(\gamma)$ of the surface formed by the rotation in E^n of the curve $\gamma = \gamma_L(\theta,\psi)$ around the polar axis in a polar coordinate system, has the form:

$$F(\gamma) = \frac{\omega_{n-2}}{\omega_{n-1}}\int\limits_\theta^\psi \sin^{n-2}\varphi(\tau)\sqrt{\left(\varphi'(\tau)\right)^2 + \left(\rho'(\tau)\right)^2}d\tau. \tag{14}$$

The description of the optimal trajectories, providing the minimum value of the functional (14) in the considered class of curves (see [8]), leads to the following functions of the variables θ and ψ:

$$h_0\left(\theta,\psi\right)=\int\limits_{\theta}^{\psi}\frac{\sin^{n-2}\theta dt}{\sqrt{\sin^{2(n-2)}t-\sin^{2(n-2)}\theta}}, \tag{15}$$

if $\psi>\theta$ and $\sin\psi\geqslant\sin\theta$ ($\psi<\frac{\pi}{2}$ or $\theta\leqslant\pi-\psi\leqslant\frac{\pi}{2}$);

$$h_1\left(\theta,\psi\right)=\int\limits_{\theta}^{\psi}\frac{\sin^{n-2}\psi dt}{\sqrt{\sin^{2(n-2)}t-\sin^{2(n-2)}\psi}}, \tag{16}$$

if $\psi>\theta$ and $\sin\psi<\sin\theta$ ($\pi-\psi<\theta\leqslant\frac{\pi}{2}$ or $\theta>\frac{\pi}{2}$);

$$h\left(\theta\right)=h_0\Big(\theta,\frac{\pi}{2}\Big). \tag{17}$$

Replacing the variable in the integrals (15)–(17) and assuming

$$b=\frac{1}{\sin\theta},d=\frac{1}{\sin\psi}=\frac{1}{\sin\left(\pi-\psi\right)},$$

we find:

$$h_0\left(\theta,\psi\right)=\Theta_n(1,b\sin\psi,b),\ \text{if}\ 0<\theta<\psi\leqslant{}^{\pi}/_{2}; \tag{18}$$

$$h_0\left(\theta,\psi\right)=\Theta_n(b)+\Theta_n(b\sin(\pi-\psi),b):=\Xi_n(b,\pi-\psi)=\Xi_n(b,\psi), \tag{19}$$

if $0<\theta\leqslant\pi-\psi<{}^{\pi}/_{2}$;

$$h_1\left(\theta,\psi\right)=\Theta_n(1,d\sin(\pi-\theta),d),\ \text{if}\ {}^{\pi}/_{2}\leqslant\theta<\psi<\pi; \tag{20}$$

$$h_1\left(\theta,\psi\right)=\Theta_n(d)+\Theta_n(d\sin\theta,d)=\Xi_n(d,\theta),\ \text{if}\ \pi-\psi<\theta<{}^{\pi}/_{2}; \tag{21}$$

$$h\left(\theta\right)=\Theta_n(b),\ \text{if}\ 0<\theta<{}^{\pi}/_{2}. \tag{22}$$

Lemma 3. *1)* $\lim\limits_{b\to {}^{1}/_{\sin\psi}} \Theta_n(1, b\sin\psi, b) = \lim\limits_{b\to\infty} \Theta_n(1, b\sin\psi, b) = 0$ *and*

$$\max_{b\in(a,\infty)} \Theta_n(1, b\sin\psi, b) = \Theta_n(1, b_n(\psi)\sin\psi, b_n(\psi)),$$

where $b_n(\psi)$ *is a solution of equality*

$$\int\limits_1^{b\sin\psi} \frac{b^2 dx}{(b^2-x^2)^{3/2}\sqrt{x^{2(n-2)}-1}} = \frac{\tan\psi}{\sqrt{(b\sin\psi)^{2(n-2)}-1}}. \tag{23}$$

2) $\Xi_n(b,\psi) = \Theta_n(b) + \Theta_n(b\sin\psi, b)$ *strictly decreases in each of the variables for a fixed value of the other.*

Proof. 1) Change of variable in the integral $\Theta_n(1, b\sin\psi, b)$, setting $x = 1 + (b\sin\psi - 1)y$, leads to the following representation:

$$\Theta_n(1, b\sin\psi, b) =$$
$$= \int\limits_0^1 \frac{\sqrt{b\sin\psi - 1}\, dy}{\sqrt{b^2 - [1 + (b\sin\psi - 1)y]^2}\sqrt{\sum_{k=1}^{2(n-2)} C^k_{2(n-2)}(b\sin\psi - 1)^{k-1}y^k)}},$$

whence follow the equalities

$$\lim_{b\to {}^{1}/_{\sin\psi}} \Theta_n(1, b\sin\psi, b) = \lim_{b\to\infty} \Theta_n(1, b\sin\psi, b) = 0.$$

Therefore, for any $\psi \in (0, {}^{\pi}/_{2})$, the maximum value of $\Theta_n(1, b\sin\psi, b)$ is reached at the point $b_n(\psi)$, which is the solution of the equation

$$\frac{\partial\Theta_n(1, b\sin\psi, b))}{\partial b} = \frac{\operatorname{tg}\psi}{b\sqrt{(b\sin\psi)^{2(n-2)}-1}} - \int\limits_1^{b\sin\psi} \frac{b\, dx}{(b^2-x^2)^{3/2}\sqrt{x^{2(n-2)}-1}} = 0.$$

2) It is easy to see that for $\psi \in (0, {}^{\pi}/_{2})$

$$\frac{\partial\Xi_n(b,\psi)}{\partial\psi} = \left.\frac{\partial\Theta_n(a,b)}{\partial a}\right|_{a=b\sin\psi} b\cos\psi = -\frac{1}{\sqrt{(b\sin\psi)^{2(n-2)}-1}} < 0.$$

Since $\frac{\partial\Theta_n(b\sin\psi,b)}{\partial b} = \left.\frac{\partial\Theta_n(a,b)}{\partial b}\right|_{a=b\sin\theta} + \sin\theta \left.\frac{\partial\Theta_n(a,b)}{\partial a}\right|_{a=b\sin\theta}$, then, taking into account equality (10) and equality

$$\left.\frac{\partial\Theta_n(a,b)}{\partial a}\right|_{a=b\sin\theta} = -\frac{1}{b\cos\theta\sqrt{(b\sin\theta)^{2(n-2)}-1}},$$

we find

$$\frac{\partial\Theta_n(b\sin\psi,b)}{\partial b} = -\frac{(n-2)}{b}\int\limits_{b\sin\theta}^{b}\frac{x^{2(n-2)}dx}{\sqrt{b^2-x^2}(x^{2(n-2)}-1)^{3/2}} < 0.$$

Because $\frac{d\Theta_n(b)}{db} < 0$ (see lemma 1), we have $\frac{\partial\Xi_n(b,\psi)}{\partial b} < 0$. Hence, $\Xi_n(b,\psi)$ strictly decreases as a function of b. □

The properties of hyperelliptic integrals formulated in Lemmas 1–3 imply:

Theorem 1. [8] *1) $h_0(\theta,\psi)$ is strictly increasing as a function of $\theta \in (0,\pi-\psi]$ for fixed $\psi \in (\pi/2,\pi)$ and*

$$\max_{\theta\in(0,\pi-\psi]} h_0(\theta,\psi) = 2h(\pi-\psi) = 2\Theta_n\left({}^1/_{\sin\psi}\right).$$

2) If $0<\theta<\psi<\pi/2$, then $\lim_{\theta\to 0+} h_0(\theta,\psi) = \lim_{\theta\to\psi-0} h_0(\theta,\psi) = 0$ and $h_n(\psi) = \max_{\theta\in(0,\psi)} h_0(\theta,\psi) = h_0(\theta_n,\psi)$, where $\theta=\theta_n(\psi)$ is a solution of the equation

$$\int\limits_{\theta}^{\psi}\frac{dt}{\cos^2 t\sqrt{\sin^{2(n-2)} t-\sin^{2(n-2)}\theta}} = \frac{\operatorname{tg}\psi}{\sqrt{\sin^{2(n-2)}\psi-\sin^{2(n-2)}\theta}}. \qquad (24)$$

3) $h_1(\theta,\psi)$ is strictly decreasing as a function of $\theta \in [\pi-\psi,\pi/2)$ for fixed $\psi\in(\pi/2,\pi)$ and

$$\max_{\theta\in[\pi-\psi,\pi/2)} h_1(\theta,\psi) = 2h(\pi-\psi) = 2\Theta_n\left({}^1/_{\sin\psi}\right),$$

$$\max_{\theta\in[\pi/2,\psi)} h_1(\theta,\psi) = h(\pi-\psi) = \Theta_n\left({}^1/_{\sin\psi}\right).$$

4) $h(\theta)$ is strictly increasing on the interval $(0,\pi/2)$ and

$$\sup_{\theta} h(\theta) = \lim_{\theta\to\pi/2} h(\theta) = \Delta_n = \frac{\pi}{2\sqrt{n-2}}. \qquad (25)$$

4. Extremal properties of special functions for the variational problem.

Calculating the areas of surfaces formed by rotation of the optimal trajectories for functional (14) and comparing these areas with the area of the $(n-1)$-dimensional sphere of unit radius calculated in the same metric, leads to the need to study the properties of a number of special functions that can be represented using of hyperelliptic integrals (1). Let us first consider the functions involved in expressing the areas of surfaces formed by the rotation of the optimal trajectories for the functional (10):

$$H(\theta) = \int_{\theta}^{\pi/2} \frac{\sin^{2(n-2)} t dt}{\sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)} \theta}}, \tag{26}$$

$$H_0(\theta, \psi) = \int_{\theta}^{\psi} \frac{\sin^{2(n-2)} t dt}{\sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)} \theta}}, \tag{27}$$

if $\psi > 0$ and $\sin\psi \geqslant \sin\theta$ ($0 < \theta < \psi \leqslant \frac{\pi}{2}$ or $\theta \leqslant \pi - \psi \leqslant \frac{\pi}{2}$);

$$H_1(\theta, \psi) = \int_{\theta}^{\psi} \frac{\sin^{2(n-2)} t dt}{\sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)} \psi}}, \tag{28}$$

if $\psi > 0$ and $\sin\psi < \sin\theta$ ($\pi - \psi < \theta \leqslant \frac{\pi}{2}$ or $\frac{\pi}{2} \leqslant \theta < \psi < \pi$).

Replacing the variable in the integrals (26)–(28) and assuming

$$b = \frac{1}{\sin\theta}, \quad d = \frac{1}{\sin\psi} = \frac{1}{\sin(\pi - \psi)},$$

we find:

$$H(\theta) = \frac{1}{b^{n-2}} \int_{1}^{b} \frac{x^{2(n-2)} dx}{\sqrt{b^2 - x^2}\sqrt{x^{2(n-2)} - 1}} = \frac{1}{b^{n-2}} \left[E_n(b) + \Theta_n(b)\right]. \tag{29}$$

If $0 < \theta < \psi \leqslant \frac{\pi}{2}$, then

$$H_0(\theta, \psi) = \frac{1}{b^{n-2}} \left[E_n(1, b\sin\psi, b) + \Theta_n(1, b\sin\psi, b)\right]. \tag{30}$$

If $0 < \theta \leqslant \pi - \psi < \frac{\pi}{2}$, then

$$H_0(\theta, \psi) = \frac{1}{b^{n-2}} \left[\mathbb{E}_n(b, \pi - \psi) + \Xi_n(b, \pi - \psi)\right], \tag{31}$$

where

$$\mathbb{E}_n(b, \pi - \psi) = E_n(b) + E_n(b \sin(\pi - \psi), b) = \mathbb{E}_n(b, \psi). \tag{32}$$

If $0 < \pi - \psi \leqslant \theta \leqslant \frac{\pi}{2}$, then

$$H_1(\theta, \psi) = \frac{1}{d^{n-2}} \left[\mathbb{E}_n(d, \theta) + \Xi_n(d, \theta)\right]. \tag{33}$$

If $\frac{\pi}{2} \leqslant \theta < \psi < \pi$, then

$$H_1(\theta, \psi) = \frac{1}{d^{n-2}} \left[\mathbb{E}_n(1, d\sin(\pi - \theta), d) + \Theta_n(1, d\sin(\pi - \theta), d)\right]. \tag{34}$$

Lemma 4. *$\mathbb{E}_n(b, \psi)$ is strictly decreasing as a function of $\psi \in (0, {}^{\pi}/_{2})$, strictly increasing as a function of b, and we have a functional equation*

$$\frac{\partial \mathbb{E}_n(b, \psi)}{\partial b} = \frac{n-2}{b} \left[\mathbb{E}_n(b, \psi) + \Xi_n(b, \psi)\right]. \tag{35}$$

Proof. It is easy to see that $\frac{\partial \mathbb{E}_n(b,\psi)}{\partial \psi} < 0$ at $\psi \in (0, {}^{\pi}/_{2})$. By virtue of (32), we have $\frac{\partial \mathbb{E}_n(b,\psi)}{\partial b} = \frac{\partial E_n(b)}{\partial b} + \left(\sin\psi \frac{\partial E_n(a,b)}{\partial a} + \frac{\partial E_n(a,b)}{\partial b}\right)\Big|_{a = b\sin\psi}$.

Carrying out the necessary calculations and transformations, we find

$$\frac{\partial E_n(a, b)}{\partial a}\bigg|_{a = b\sin\psi} = -\frac{\sqrt{(b\sin\psi)^{2(n-2)} - 1}}{b\cos\psi}$$

and (see (13))

$$\frac{\partial E_n(a, b)}{\partial b}\Big|_{a = b\sin\psi} =$$

$$= \frac{\operatorname{tg}\psi}{b}\sqrt{(b\sin\psi)^{2(n-2)} - 1} + \frac{n-2}{b}\int\limits_{b\sin\psi}^{b} \frac{x^{2(n-2)}dx}{\sqrt{b^2 - x^2}\sqrt{x^{2(n-2)} - 1}}.$$

Because

$$\frac{n-2}{b}\int\limits_{b\sin\psi}^{b} \frac{x^{2(n-2)}dx}{\sqrt{b^2 - x^2}\sqrt{x^{2(n-2)} - 1}} = \frac{n-2}{b}\left[E_n(b\sin\psi, b) + \Theta_n(b\sin\psi, b)\right],$$

taking into account (11), (19), and (32), we get (35), whence it follows that $\frac{\partial \mathbb{E}_n(b,\psi)}{\partial b} > 0$. □

Theorem 2. *1)* $H(\theta)$, $H_0(\theta) = 2H(\theta) - \int_0^{\pi-\theta} \sin^{n-2} t\,dt$ *and* $H_1(\theta) = H(\theta) - \int_0^{\pi-\theta} \sin^{n-2} t dt$ *are strictly increasing on the interval* $\theta \in (0, \pi/2)$ *and*

$$\sup_{\theta\in(0,\pi/2)} H(\theta) = \Delta_n, \quad \inf_{\theta\in(0,\pi/2)} H(\theta) = \frac{\omega_{n-1}}{2\omega_{n-2}};$$

$$\sup_{\theta\in(0,\pi/2)} H_0(\theta) = 2\Delta_n - \frac{\omega_{n-1}}{2\omega_{n-2}} = \Delta_n^0, \quad \inf_{\theta\in(0,\pi/2)} H_0(\theta) = 0; \tag{36}$$

$$\sup_{\theta\in(0,\pi/2)} H_1(\theta) = \Delta_n - \frac{\omega_{n-1}}{2\omega_{n-2}} = \Delta_n^1, \quad \inf_{\theta\in(0,\pi/2)} H_1(\theta) = -\frac{\omega_{n-1}}{2\omega_{n-2}}.$$

2) $\Delta_n^0 > \Delta_n^1 > 0$ *for* $n \geqslant 3$, $\lim_{n\to\infty} \Delta_n^0 = 0$, *and there are inequalities:*

$$\begin{cases} \left(\sqrt{\pi} - \frac{\sqrt{2}}{2}\right)\sqrt{\frac{\pi}{n-2}} \leqslant \Delta_n^0 \leqslant \sqrt{\pi}\left(\frac{\sqrt{\pi}}{\sqrt{n-2}} - \frac{1}{\sqrt{2(n-1)}}\right), \\ \frac{\sqrt{\pi}-\sqrt{2}}{2}\sqrt{\frac{\pi}{n-2}} \leqslant \Delta_n^1 \leqslant \sqrt{\frac{\pi}{2}}\left(\frac{\sqrt{\pi}}{\sqrt{2(n-2)}} - \frac{1}{\sqrt{n-1}}\right). \end{cases} \tag{37}$$

3) The equation

$$H(\theta) = \int_0^{\pi-\theta} \sin^{n-2} t dt \tag{38}$$

has a unique solution $\theta = \chi(n) \in (0, \pi/2)$.

Proof. 1) Because

$$H(\theta) = \int_\theta^{\pi/2} \sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)} \theta} dt + h(\theta) \sin^{n-2}\theta, \tag{39}$$

$$\frac{d}{d\theta} \int_\theta^{\pi/2} \sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)} \theta} dt = - \int_\theta^{\pi/2} \frac{(n-2)\sin^{2n-5}\theta dt}{\sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)}\theta}},$$

then $\frac{\partial H(\theta)}{\partial\theta} = \sin^{n-2}\theta \frac{\partial h(\theta)}{\partial\theta} > 0$, since, according to Theorem 1, $\frac{\partial h(\theta)}{\partial\theta} > 0$. Hence, $H(\theta)$, $H_0(\theta)$, and $H_1(\theta)$ strictly increase on the interval $(0, \pi/2)$.

From here, due to (25) and (39), taking into account the equality $\int\limits_0^{\pi/2} \sin^{n-2} t dt = \frac{\omega_{n-1}}{2\omega_{n-2}}$, we arrive at (36).

2) Inequalities (37) is a consequence of inequality

$$\sqrt{\frac{2\pi}{n-1}} \leqslant \frac{\omega_{n-1}}{\omega_{n-2}} \leqslant \sqrt{\frac{2\pi}{n-2}},$$

which was established by K.H. Borgward [6] (see also [9]). It follows that $\Delta_n^\nu > 0$ and $\lim_{n\to\infty} \Delta_n^\nu = 0$, $\nu = 0, 1$.

3) Since $H_1(\theta)$ strictly increases on the interval $(0, {}^{\pi}/_{2})$ and takes values of different signs, equation (38) has a unique solution $\theta = \chi(n)$. □

For $\nu = 0, 1$ we set:

$$\mathbb{H}_\nu(\theta, \psi) = H_\nu(\theta, \psi) - \int\limits_0^\psi \sin^{n-2} t dt.$$

Theorem 3. *1) $H_0(\theta, \psi)$ and $\mathbb{H}_0(\theta, \psi)$ are strictly increasing as functions of $\theta \in (0, \pi - \psi]$ for fixed $\psi \in ({}^{\pi}/_{2}, \pi - \theta]$, $\mathbb{H}_0(\theta, \psi)$ takes positive values, and*

$$\max_{\theta\in(0,\pi-\psi]} \mathbb{H}_0(\theta, \psi) = H_0(\pi - \psi).$$

2) $\mathbb{H}_0(\theta, \psi)$ is strictly increasing as a function of $\psi \in (\theta, \pi - \theta]$ for fixed $\theta \in (0, {}^{\pi}/_{2})$, $\max_{\psi\in(\theta,\pi-\theta]} \mathbb{H}_0(\theta, \psi) = H_0(\theta)$, and for any $\theta \in (0, {}^{\pi}/_{2})$, there exists a unique solution $\psi = \psi_n^0(\theta)$ of the equation

$$H_0(\theta, \psi) = \int\limits_0^\psi \sin^{n-2} t\, dt. \tag{40}$$

3) $\mathbb{H}_0(\theta, \psi)$ is strictly increasing as a function of θ near the point $\theta = 0$ for fixed $\psi \in (\theta, {}^{\pi}/_{2}]$, $\lim_{\theta\to0} \mathbb{H}_0(\theta, \psi) = 0$ and

$$\max_{\theta\in(0,\psi]} \mathbb{H}_0(\theta, \psi) = \mathbb{H}_0(\theta_n, \psi) > 0,$$

where $\theta_n = \theta_n(\psi)$ is a solution of equation (24). In addition, on the interval $(\theta_n(\psi), \psi)$ there exists a solution $\theta = \theta_n^0(\psi)$ of equation (40).

Proof. 1) Let $0 < \theta \leqslant \pi - \psi < \frac{\pi}{2}$. By virtue of equality (31), we have

$$\frac{\partial H_0(\theta,\psi)}{\partial b} = -\frac{(n-2)}{b^{n-1}}\Big[\mathbb{E}_n(b,\pi-\psi)+\Xi_n(b,\pi-\psi)\Big]+ \\ +\frac{1}{b^{n-2}}\Big[\frac{\partial \mathbb{E}_n(b,\pi-\psi)}{\partial b}+\frac{\partial \Xi_n(b,\pi-\psi)}{\partial b}\Big],$$

where $b=\frac{1}{\sin\theta}$. Taking Lemma 4 and relation (35) into account, we find

$$\frac{\partial H_0(\theta,\psi)}{\partial b} = \frac{1}{b^{n-2}}\frac{\partial \Xi_n(b,\pi-\psi)}{\partial b} < 0.$$

Hence,

$$\frac{\partial H_0(\theta,\psi)}{\partial \theta} = -\frac{\cos\theta}{\sin^n\theta}\frac{\partial \Xi_n(b,\pi-\psi)}{\partial b} > 0,$$

that is, $H_0(\theta,\psi)$ and $\mathbb{H}_0(\theta,\psi)$ are strictly increasing as functions of $\theta\in(0,\pi-\psi]$ for fixed $\psi\in\left(\frac{\pi}{2},\pi-\theta\right]$.

Since

$$\inf_{\theta\in(0,\pi-\psi]}\mathbb{H}_0(\theta,\psi) = \lim_{\theta\to 0+}\mathbb{H}_0(\theta,\psi)=0$$

and by Theorem 2,

$$\max_{\theta\in(0,\pi-\psi]}\mathbb{H}_0(\theta,\psi) = \mathbb{H}_0(\pi-\psi,\psi) = H_0(\pi-\psi) > 0,$$

then the function $H_0(\theta,\psi)$ takes positive values for any $\theta\in(0,\pi-\psi]$.

2) Since

$$\frac{\partial \mathbb{H}_0(\theta,\psi)}{\partial \psi} = \sin^{n-2}\psi\Big(\frac{\sin^{n-2}\psi}{\sqrt{\sin^{2(n-2)}\psi-\sin^{2(n-2)}\theta}}-1\Big) > 0,$$

for $\psi\in(\theta,\pi-\theta]$, then $\mathbb{H}_0(\theta,\psi)$ is strictly increasing as a function of $\psi\in(\theta,\pi-\theta]$ for fixed $\theta\in(0,{}^{\pi}/_{2})$ and

$$\max_{\psi\in(\theta,\pi-\theta]}\mathbb{H}_0(\theta,\psi) = H_0(\theta).$$

Because

$$\inf_{\psi\in(\theta,\pi-\theta]}\mathbb{H}_0(\theta,\psi) = \lim_{\psi\to\theta+0}\mathbb{H}_0(\theta,\psi) = -\int\limits_0^{\theta}\sin^{n-2}t\,dt < 0,$$

and by Theorem 2,

$$\max_{\psi\in(\theta,\pi-\theta]}\mathbb{H}_0(\theta,\psi) = \mathbb{H}_0(\theta,\pi-\theta) = H_0(\theta) > 0,$$

then, for any $\theta \in (0, {}^{\pi}/_{2})$, there exists a unique solution $\psi = \psi_n^0(\theta)$ of equation (40).

3) Let $0 < \theta < \psi \leqslant \frac{\pi}{2}$. It is easy to see that

$$\frac{\partial \mathbb{H}_0(\theta,\psi)}{\partial\theta} = \frac{\partial H_0(\theta,\psi)}{\partial\theta} = \sin^{n-2}\theta \frac{\partial h_0(\theta,\psi)}{\partial\theta}.$$

Therefore, the monotonicity intervals of $\mathbb{H}_0(\theta,\psi)$ and $h_0(\theta,\psi)$ as functions of the variable θ for a fixed value of ψ coincide. By virtue of Theorem 1 (property 2), this implies that

$$\max_{\theta\in(0,\psi]} \mathbb{H}_0(\theta,\psi) = \mathbb{H}_0(\theta_n(\psi),\psi),$$

where $\theta_n(\psi)$ is a solution of the equation (24).

As $\mathbb{H}_0(\theta,\psi)$ increases near the point $\theta = 0$ and $\lim_{\theta\to 0}\mathbb{H}_0(\theta,\psi) = 0$, $\mathbb{H}_0(\theta_n(\psi),\psi) > 0$. From the definition of $\mathbb{H}_0(\theta,\psi)$, equality (30), and the properties of the hyperelliptic integrals $E_n(1, b\sin\psi, b)$ and $\Theta_n(1, b\sin\psi, b)$ (see Lemma 3), it follows that

$$\lim_{\theta\to\psi} \mathbb{H}_0(\theta,\psi) = -\int\limits_0^{\pi-\psi} \sin^{n-2} t dt < 0.$$

Therefore, on the interval $(\theta_n(\psi),\psi)$ there exists a solution $\theta = \theta_n^0(\psi)$ of equation (40). □

Theorem 4. *1) $\mathbb{H}_1(\theta,\psi)$ is strictly decreasing as a function of $\theta \in [\pi-\psi,\psi)$ for fixed $\psi \in [{}^{\pi}/_{2},\pi)$. There are equalities:*

$$\max_{\theta\in[\pi-\psi,\psi)} \mathbb{H}_1(\theta,\psi) = H_0(\pi-\psi);\ \mathbb{H}_1({}^{\pi}/_{2},\psi) = H_1(\pi-\psi);$$

$$\inf_{\theta\in[\pi-\psi,\psi)} \mathbb{H}_1(\theta,\psi) = \lim_{\theta\to\psi-0} \mathbb{H}_1(\theta,\psi) = -\int\limits_0^{\psi} \sin^{n-2} t dt < 0.$$

2) $\mathbb{H}_1(\theta,\psi)$ is strictly decreasing as a function of $\psi \in [{}^{\pi}/_{2},\pi)$ for fixed $\theta \in [\pi-\psi, {}^{\pi}/_{2}]$. There exists a unique solution $\psi = \psi_n^1(\theta)$ of equation

$$H_1(\theta,\psi) = \int\limits_0^{\psi} \sin^{n-2} t dt \qquad (41)$$

and, besides, $\psi_n^1({}^{\pi}/_2) = \pi - \chi(n)$*, where* $\chi(n)$ *is a unique solution of the equation (38). In addition, for any* $\psi \in (\pi - \chi(n), \pi)$ *there exists a unique solution* $\theta = \theta_n^1(\psi)$ *of the equation (41).*

3) $\mathbb{H}_1(\theta, \psi)$ *takes negative value near the ends of the interval* $\psi \in (\theta, \pi)$ *for any fixed* $\theta \in ({}^{\pi}/_2, \psi)$ *and*

$$\max_{\psi \in (\theta, \pi)} \mathbb{H}_1(\theta, \psi) = \mathbb{H}_1(\theta, \psi_n(\theta)),$$

where $\psi = \psi_n(\theta)$ *is a solution of the equation*

$$\int_{\theta}^{\psi} \frac{dt}{\cos^2 t \sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)} \psi}} = \\ = \frac{1}{\sin^{n-3} \psi \cos \psi} - \frac{\operatorname{tg} \theta}{\sqrt{\sin^{2(n-2)} \theta - \sin^{2(n-2)} \psi}}. \quad (42)$$

There exists a solution $\theta = \theta(n)$ *of the equation*

$$\mathbb{H}_1(\theta, \psi_n(\theta)) = 0 \quad (43)$$

and for $\theta \in ({}^{\pi}/_2, \theta(n))$ *there exists two solutions* $\psi = \overline{\psi}_n(\theta) \in (\theta, \psi_n(\theta))$ *and* $\overline{\overline{\psi}}_n(\theta) \in (\psi_n(\theta), \pi)$ *of the equation (41).*

Proof. The statement 1) follows from the fact that

$$\frac{\partial \mathbb{H}_1(\theta, \psi)}{\partial \theta} = -\frac{\sin^{2(n-2)} \theta}{\sqrt{\sin^{2(n-2)} \theta - \sin^{2(n-2)} \psi}} < 0$$

for $\theta \in [\pi - \psi, \psi)$ and fixed $\psi \in [{}^{\pi}/_2, \pi)$. It means that $\mathbb{H}_1(\theta, \psi)$ is strictly decreasing as a function of θ.

2) As

$$H_1(\theta, \psi) = \int_{\theta}^{\psi} \sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)} \psi} dt + \sin^{n-2} \psi h_1(\theta, \psi),$$

$\frac{\partial H_1(\theta,\psi)}{\partial \psi} = \sin^{n-2} \psi \frac{\partial h_1(\theta,\psi)}{\partial \psi}$. By virtue of (20) and Lemma 3 (property 2),

$$\frac{\partial h_1(\theta, \psi)}{\partial \psi} = \frac{\partial \Xi_n(d, \theta)}{\partial \psi} = \frac{\partial \Xi_n(d, \theta)}{\partial d} \frac{\cos(\pi - \psi)}{\sin^2(\pi - \psi)} < 0.$$

It follows that

$$\frac{\partial \mathbb{H}_1(\theta,\psi)}{\partial \psi} = \sin^{n-2}\psi\Big(\frac{\partial h_1(\theta,\psi)}{\partial \psi} - 1\Big) < 0.$$

It means that $\mathbb{H}_1(\theta, \pi - \psi)$ is strictly decreasing as a function of ψ for fixed $\theta \in [\pi - \psi, {}^{\pi}/_2]$. Therefore, by virtue of Theorem 3,

$$\max_{\psi\in[\pi-\theta,\pi)} \mathbb{H}_1(\theta,\psi) = \mathbb{H}_1(\theta,\pi-\theta) = H_0(\theta) > 0,$$

$$\inf_{\psi\in[\pi-\theta,\pi)} \mathbb{H}_1(\theta,\psi) = \lim_{\psi\to\pi} \mathbb{H}_1(\theta,\psi) = -\int\limits_0^\theta \sin^{n-2} t dt < 0.$$

It follows that there exists a unique solution $\psi = \psi_n^1(\theta)$ of the equation (41). Since $H_1({}^{\pi}/_2,\psi) = H(\pi-\psi)$, $\pi - \psi_n^1({}^{\pi}/_2) = \chi(n)$.

By Theorem 2, $H_0(\pi-\psi)$ and $H_1(\pi-\psi)$ are strictly decreasing on the interval $\psi \in ({}^{\pi}/_2, \pi)$. Therefore,

$$\inf_{\theta\in[\pi-\psi,{}^{\pi}/_2)} \mathbb{H}_1(\theta,\psi) = H_1(\pi-\psi) < 0,$$

$$\max_{\theta\in[\pi-\psi,{}^{\pi}/_2)} \mathbb{H}_1(\theta,\psi) = H_0(\pi-\psi) > 0$$

for $\psi \in (\pi - \chi(n), \pi)$.

Consequently, for any $\psi \in (\pi - \chi(n), \pi)$ there exists a unique solution $\theta = \theta_n^1(\psi)$ of equation (41).

3) Let $\theta \in ({}^{\pi}/_2,\psi)$. In this case,

$$\mathbb{H}_1(\theta,\psi) = H_0(\pi-\psi,\pi-\theta) - \int\limits_0^\psi \sin^{n-2} t dt.$$

Therefore,

$$\lim_{\psi\to\theta} \mathbb{H}_1(\theta,\psi) = \lim_{\psi\to\pi} \mathbb{H}_1(\theta,\psi) = -\int\limits_0^\theta \sin^{n-2} t dt < 0.$$

As $H_0(\pi-\psi,\pi-\theta) =$

$$= \int\limits_{\pi-\psi}^{\pi-\theta} \sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)} \psi} dt + \sin^{n-2}(\pi-\psi) h_0(\pi-\psi,\pi-\theta),$$ then

$$\frac{\partial \mathbb{H}_1(\theta,\psi)}{\partial \psi} = \sin^{n-2}\psi\Big(\frac{\partial h_0(\pi-\psi,\pi-\theta)}{\partial \psi} - 1\Big).$$

Therefore, $\max_{\psi\in(\theta,\pi)} \mathbb{H}_1(\theta,\psi) = \mathbb{H}_1(\theta,\psi_n(\theta))$, where $\psi = \psi_n(\theta)$ is a solution of equation $\frac{\partial h_0(\pi-\psi,\pi-\theta)}{\partial\psi} = 1$. Using representation (18) and repeating the calculations performed in the proof of Lemma 1, this equation can be represented in explicit form (42). Because

$$\max_{\theta\in[\pi/2,\psi)} \mathbb{H}_1(\theta,\psi) = \mathbb{H}_1(\pi/2,\psi) = H_1(\pi-\psi)$$

and by Theorem 2

$$\sup_{\psi\in(\pi/2,\pi)} H_1(\pi-\psi) = \Delta_n^1 > 0;$$

then $\mathbb{H}_1(\theta,\psi) > 0$ for values θ and ψ close to $\pi/2$ and $\mathbb{H}_1(\theta,\psi_n(\theta)) > 0$ for θ close to $\pi/2$. Therefore, there exist a solution $\theta = \theta(n)$ of equation (43) and for $\theta \in (\pi/2,\theta(n))$ there exists two solutions $\psi = \overline{\psi}_n(\theta) \in (\theta,\psi_n(\theta))$ and $\overline{\overline{\psi}}_n(\theta) \in (\psi_n(\theta),\pi)$ of the equation (41). □

5. Results of numerical experiments.

Let us present the results of numerical experiments on constructing plots of the functions under study.

Figures 1 and 2 present the results of numerical experiments on constructing plots of the functions $\Theta_3(a,b)$ and $\Theta_5(a,b)$ for specific values $a \in \Big(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{2}}, 2\Big)$.

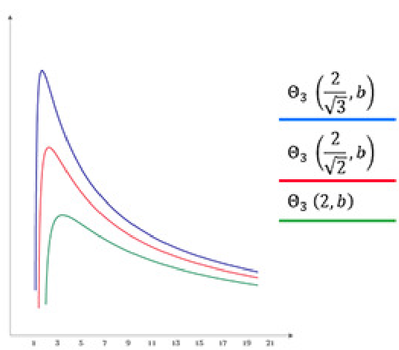

Figure 1: The plots of $\Theta_3(a,b)$

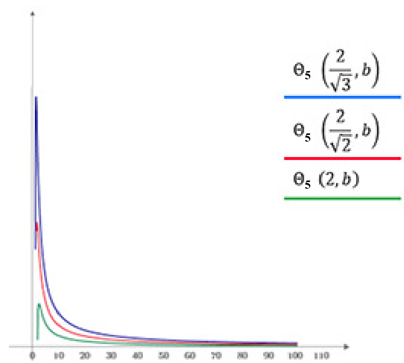

Figure 2: The plots of $\Theta_5(a,b)$

Figures 3 and 4 present the results of numerical experiments on constructing plots of the functions $h_0(\theta,\psi)$ for specific values $\psi \in (\pi/12, \pi/6, \pi/4,$

${}^{\pi}/_{3}, {}^{5\pi}/_{12})$, when $n = 3$ (see Figure 3) and $n = 5$ (see Figure 4). These plots suggest that $\theta_n(\psi)$ is a unique solution to the equation (23) for every $\psi \in (0, {}^{\pi}/_{2})$.

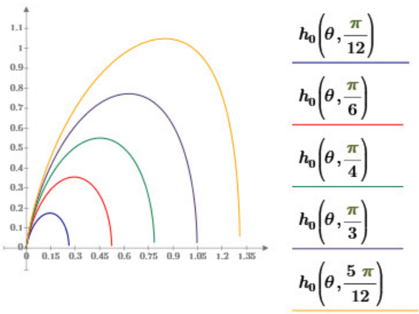

Figure 3: The plots of $h_0(\theta, \psi), n = 3$

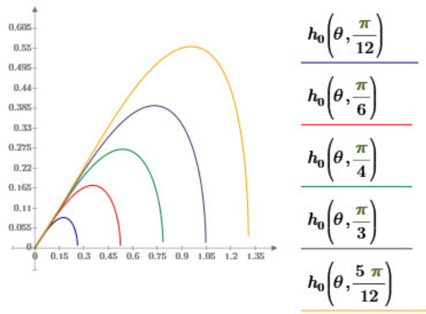

Figure 4: The plots of $h_0(\theta, \psi), n = 5$

Figures 5 and 6 present results of numerical experiments on constructing plots of the functions $\mathbb{H}_0(\theta, \psi)$ for specific values $\psi \in (\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12})$, when $n = 3$ (see Figure 5) and $n = 5$ (see Figure 6).

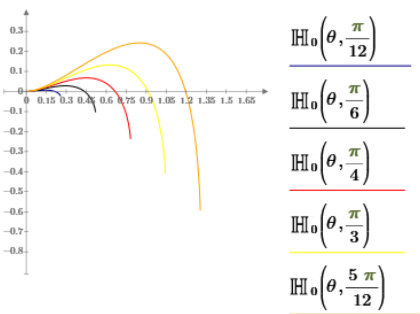

Figure 5: The plots of $\mathbb{H}_0(\theta, \psi), n = 3$

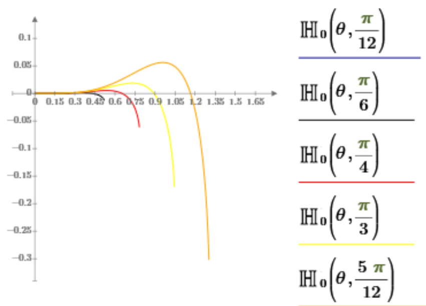

Figure 6: The plots of $\mathbb{H}_0(\theta, \psi), n = 5$

These plots suggest that $\theta_n^0(\psi)$ is a unique solution to the equation (39) for every $\psi \in (0, {}^{\pi}/_{2})$.

Figures 7 and 8 present results of numerical experiments on constructing plots of the functions $\mathbb{H}_1(\theta, \psi)$ for specific values $\theta \in (\theta(n) - {}^{\pi}/_{200}, \theta(n), \theta(n) + {}^{\pi}/_{200})$, when $n = 3, \theta(3) \approx 1.6435$ (see Figure 7) and for $n = 5, \theta(5) \approx 1.5946$ (see Figure 8).

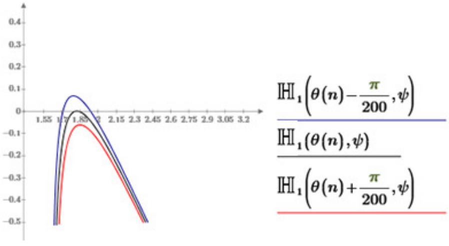

Figure 7: The plots of $\mathbb{H}_1(\theta,\psi)$, $n = 3$

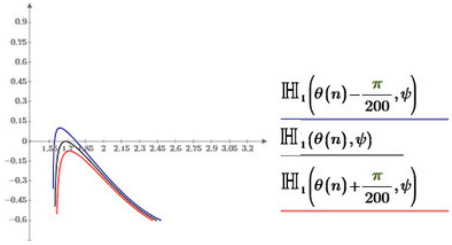

Figure 8: The plots of $\mathbb{H}_1(\theta,\psi)$, $n = 5$

Let us present the results of numerical experiments on studying the dependence on the dimension $n \in (3, 4, 5, 7)$ of the constants defined in Lemma 1 and Theorem 1 (Table 1), as well as the values of extremal functions $\theta_n, \theta_n^0, \theta_n^1$ (Table 2), functions ψ_n^0, ψ_n^1 (Table 3), function ψ_n (Table 4), and functions $\overline{\psi}_n(\theta)$ and $\overline{\overline{\psi}}_n(\theta)$ (Table 5) in specific points.

Table 1: The values of certain constants.

n	Δ_n	Δ_n^0	Δ_n^1	$\chi(n)$	$\theta(n)$
3	1.571	2.142	0.571	1.073	1.6425
4	1.111	1.436	0.325	1.272	1.6057
5	0.907	1.147	0.240	1.347	1.5946
7	0.702	0.872	0.169	1.411	1.5862

Table 2: The values of extremal functions $\theta_n, \theta_n^0, \theta_n^1$.

n	$\theta_n\left(\frac{\pi}{3}\right)$	$\theta_n\left(\frac{\pi}{4}\right)$	$\theta_n^0\left(\frac{\pi}{3}\right)$	$\theta_n^0\left(\frac{\pi}{4}\right)$	$\theta_n^1\left(\pi - \frac{2\chi(n)}{3}\right)$	$\theta_n^1\left(\pi - \frac{\chi(n)}{2}\right)$
3	0.628	0.452	0.914	0.667	1.237	1.001
4	0.691	0.503	0.893	0.652	1.184	0.915
5	0.733	0.538	0.892	0.653	1.153	0.870
7	0.786	0.582	0.901	0.663	1.115	0.824

It is of interest to obtain explicit estimates of the extremal functions and extremal values defined in the work, depending on the dimension.

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Table 3: The values of extremal functions ψ_n^0 and ψ_n^1.

n	$\psi_n^0\left(\frac{\pi}{4}\right)$	$\psi_n^0\left(\frac{\pi}{3}\right)$	$\psi_n^0\left(\frac{\pi}{2}-0.1\right)$	$\psi_n^1\left(\frac{\pi}{4}\right)$	$\psi_n^1\left(\frac{\pi}{3}\right)$	$\psi_n^1\left(\frac{\pi}{2}-0.1\right)$
3	0.914	1.176	1.512	2.749	2.572	2.204
4	0.9332	1.199	1.521	2.601	2.404	2.020
5	0.9333	1.202	1.524	2.533	2.330	1.947
7	0.922	1.196	1.526	2.467	2.256	1.880

Table 4: The values of extremal functions ψ_n.

n	$\psi_n\left(\frac{\pi}{2}+0.1\right)$	$\psi_n\left(\frac{\pi}{4}+\frac{\theta(n)}{2}\right)$	$\psi_n\big(\theta(n)\big)$	$\psi_n\left(\frac{2\pi}{3}\right)$	$\psi_n\left(\pi-0.1\right)$
3	1.868	1.753	1.825	2.294	3.057
4	1.827	1.680	1.722	2.235	3.052
5	1.805	1.652	1.684	2.205	3.050
7	1.780	1.628	1.652	2.173	3.047

Table 5: The values of extremal functions $\overline{\psi}_n(\theta)$ and $\overline{\overline{\psi}}_n(\theta)$.

n	$\overline{\psi}_n\left(\frac{\pi}{2}+0.01\right)$	$\overline{\psi}_n\left(\frac{\pi}{4}+\frac{\theta(n)}{2}\right)$	$\overline{\overline{\psi}}_n\left(\frac{\pi}{2}+0.01\right)$	$\overline{\overline{\psi}}_n\left(\frac{\pi}{4}+\frac{\theta(n)}{2}\right)$
3	1.868	1.753	1.825	2.294
4	1.827	1.680	1.722	2.235
5	1.805	1.652	1.684	2.205
7	1.780	1.628	1.652	2.173

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