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MILNE-TYPE INTEGRAL INEQUALITIES FOR MODIFIED (h, m)-CONVEX FUNCTIONS ON FRACTAL SETS

Abstract. In the article, new versions of integral inequalities of Milne type are derived for (h, m)-convex modified functions of the second type on fractal sets. Based on a new generalized local fractional weighted integral operator, an identity is established as the foundation for subsequently obtained inequalities. Throughout our study, we obtained certain results known in the literature, which include particular cases of our findings.

Key words: *local fractional derivatives, local fractional integrals, fractal sets, Milne inequality, (h, m)-convex modified functions of second type, Hölder inequality, power mean inequality*

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1. Introduction. A function $\phi\colon [\rho_1, \rho_2] \to \mathbb{R}$ is said to be convex if $\phi\big(\tau u + (1-\tau)v\big) \leqslant \tau\phi(u) + (1-\tau)\phi(v)$ holds for all $u, v \in [\rho_1, \rho_2]$ and $\tau \in [0, 1]$. A function ϕ is said to be concave if $-\phi$ is convex.

Convex functions have been widely generalized, including the m-convex function, r-convex function, h-convex function, (h, m)-convex function, s-convex function, and many others. Readers interested in exploring these extensions and generalizations of the classical notion of convexity can refer to [26].

In literature, various integral inequalities, such as Simpson's, trapezoidal, midpoint, and others, are presented. Numerous studies are dedicated to extending and generalizing these integral inequalities. An example includes the derivation of several variations of these inequalities for different classes of functions, such as differentiable convex, bounded, and Lipschitz functions, see references [19], [16], [5], [20], [8] and the works cited therein.

In the works [27], [22], [9], [10] and the literature cited therein, the main attention is paid to fractional versions of trapezoid-type inequalities and midpoint-type inequalities.

Studies [8], [18], [25] and the references therein emphasize the establishment of Simpson-type inequalities.

Milne-type integral inequalities constitute a class of mathematical inequalities associated with integrals. These inequalities are named after Edward Arthur Milne, a distinguished mathematician recognized for his contributions to various areas of mathematics.

In general, a Milne-type integral inequality involves the integration of functions and establishes bounds or inequalities for these integrals based on specific conditions or assumptions regarding the integrands and the integration domain. These inequalities are frequently employed in mathematical analysis, particularly in integral calculus and related fields.

The classical Milne-type inequality in the literature is represented as follows ([13], [14], [28], [15]):

$$\left|\frac{\rho_2-\rho_1}{3}\left(2\phi\left(\rho_1\right)-\phi\left(\frac{\rho_1+\rho_2}{2}\right)+2\phi\left(\rho_2\right)\right)-\int\limits_{\rho_1}^{\rho_2}\phi\left(x\right)dx\right|\leqslant$$
$$\leqslant\quad \frac{7(\rho_2-\rho_1)^5}{23040}\sup_{x\in[\rho_1,\rho_2]}\left|\phi^{(4)}(x)\right|.$$

Numerical integration methods, specifically Milne's and Simpson's formulas, demonstrate both similarities and distinctive features. Both of these methods employ a composite quadrature rule to approximate the definite integral of a function and require the use of a uniformly distributed grid of sampling points.

As can be seen from Milne's inequality, the integrand function must be continuously differentiable up to the fourth order inclusive. Recently presented studies [13], [14], [28] provide an estimate for Milne's formula for a continuously differentiable function using fractional integral operators.

These inequalities are useful for estimating the magnitude of integrals in terms of other integrals, facilitating the analysis of various mathematical problems.

Research on and development of local fractional functions within fractal sets, encompassing such aspects as local fractional calculus, function continuity, and monotonicity, is thoroughly examined in [30].

Following the above work, the real line number in the fractal set $\mathbb{R}^\delta$ has the following properties:

If r_1^δ, r_2^δ, and $r_3^\delta \in \mathbb{R}^\delta$, $0 < \delta \leqslant 1$, then:

- $r_1^\delta + r_2^\delta \in \mathbb{R}^\delta$, $r_1^\delta r_2^\delta \in \mathbb{R}^\delta$.
- $r_1^\delta + r_2^\delta = r_2^\delta + r_1^\delta = (r_1 + r_2)^\delta = (r_2 + r_1)^\delta$.
  From this, we have the following conclusion.
  Given the $r_1^\delta + r_2^\delta = (r_1 + r_2)^\delta$, $-r_1^\delta$ is such that $r_1^\delta + (-r_1^\delta) = 0^\delta$ (for example, $(1^\delta - 2^\delta) = 1^\delta + (-2^\delta) = (1 + (-2))^\delta = (-1)^\delta)$ and, since $r_1^\delta + (-r_1^\delta) = 0^\delta$, we have $(-1)^\delta = -1^\delta$, then $1^\delta - 2^\delta = -1^\delta$.
- $r_1^\delta + (r_2^\delta + r_3^\delta) = (r_1^\delta + r_2^\delta) + r_3^\delta$.
- $r_1^\delta r_2^\delta = r_2^\delta r_1^\delta = (r_1 r_2)^\delta = (r_2 r_1)^\delta$.
- $(r_1^\delta r_2^\delta) r_3^\delta = r_1^\delta (r_2^\delta r_3^\delta)$.
- $r_1^\delta (r_2^\delta + r_3^\delta) = (r_1^\delta r_2^\delta) + (r_1^\delta r_3^\delta)$.
- $r_1^\delta + 0^\delta = 0^\delta + r_1^\delta = r_1^\delta$, and $r_1^\delta 1^\delta = 1^\delta r_1^\delta = r_1^\delta$.

Local fractional integral Hölder inequality, which was established by Yang [31] and used, for example, in [17], is as follows: Let $\phi, \psi \in C_\delta(I)$. Then

$$\frac{1}{\Gamma(\alpha+1)} \int_I |\phi(t)\psi(t)|\, dt^\delta \leqslant$$

$$\leqslant \left(\frac{1}{\Gamma(\alpha+1)} \int_I |\phi(t)|^p dt^\delta \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_I |\psi(t)|^q dt^\delta \right)^{\frac{1}{q}},$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 1. *Let $\psi \in L_1[\rho_1, \rho_2]$. Then the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ are defined by (right and left, respectively):*

$$I_{\nu_1+}^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_{\rho_1}^{x} (x-t)^{\alpha-1} \phi(t)\, dt, \quad x > \rho_1,$$

$$I_{\rho_2-}^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\nu_2} (t-x)^{\alpha-1} \phi(t)\, dt, \quad x < \rho_2.$$

Definition 2. *A non-differentiable mapping $\phi\colon \mathbb{R} \to \mathbb{R}^{\delta}$ is called local fractional continuous at $x = x_0$, if for any $\varepsilon > 0$, there exists $\tau > 0$, satisfying*

$$|\phi(x) - \phi(x_0)| < \varepsilon^{\delta}$$

for $|x - x_0| < \tau$. If $\phi(x)$ is local continuous on some interval (ρ_1, ρ_2), we denote $\phi(x) \in C_{\delta}(\rho_1, \rho_2)$.

Definition 3. *The local fractional derivative of $\phi(x)$, where $x \in [\rho_1, \rho_2]$ of order δ at $x = x_0$ is given by*

$$\phi^{(\delta)}(x_0) = \frac{d^{\delta}\phi}{dx^{\delta}}(x_0) = \lim_{x \to x_0} \frac{\Gamma(\delta + 1)(\phi(x) - \phi(x_0))}{(x - x_0)^{\delta}}.$$

Denote $\phi \in D^{\delta}[\rho_1, \rho_2]$, $\Gamma(\cdot)$ is the Euler gamma function.

Definition 4. *Let $\phi \in D^{\delta}[\rho_1, \rho_2]$. The local fractional integral of $\phi(x)$ of order δ is given by*

$$_{\rho_1}J^{\delta}_{\rho_2}\phi(x) = \frac{1}{\Gamma(\delta + 1)} \int_{\rho_1}^{\rho_2} \phi(x)dx^{\delta}.$$

Here, if $\rho_1 = \rho_2$, then $_{\rho_1}J^{\delta}_{\rho_2}\phi(x) = 0$, if $\rho_1 < \rho_2$, then $_{\rho_1}J^{\delta}_{\rho_2}\phi(x) = -_{\rho_2}J^{\delta}_{\rho_1}\phi(x)$. If $_{\rho_1}J^{\delta}_{x}\phi(x)$ exists for all $x \in [\rho_1, \rho_2]$, then we say that $\phi(x)$ belongs to the class of δ-integrable functions in $[a, b]$, i.e., $\phi(x) \in J^{\delta}_{x}[\rho_1, \rho_2]$.

The following lemma establishes two fundamental properties for the operators defined above [30]:

Lemma 1. *The following results are true:*

(1) If $\phi^{(\delta)} \in C_{\delta}(\rho_1, \rho_2)$, then $_{\rho_1}J^{\delta}_{\rho_2}\phi^{(\delta)}(x) = \phi(\rho_2) - \phi(\rho_1)$.

(2) (Integration by parts rule) Let $u(x), v(x) \in D^{\delta}[\rho_1, \rho_2]$ and $u^{(\delta)}, v^{(\delta)} \in C_{\delta}(\rho_1, \rho_2)$; then we have

$$_{\rho_1}J^{\delta}_{x}(u(x)v^{(\delta)}(x)) = u(x)v(x)\Big|_{\rho_1}^{\rho_2} - {}_{\rho_1}J^{\delta}_{x}((u^{(\delta)}(x)v(x)).$$

Based on the previous definition, we present the integral operators that will be used in our work.

Definition 5. *Let ϕ be a local fractional continuous on $[\rho_1, \rho_2]$ and let $w(x) \in J^{\delta}_{x}[\rho_1, \rho_2]$. The right and left local fractional weighted integral of*

ϕ of order δ are given by

$$ {}^{w}J^{\delta}_{\rho_1+}\phi(\rho_2) = \frac{1}{\Gamma(\delta+1)} \int_{\rho_1}^{\rho_2} w^{(\delta)} \left(\frac{\rho_2 - \lambda}{\rho_2 - \rho_1} \right) \phi(\lambda) d\lambda^{\delta}, $$

and

$$ {}^{w}J^{\delta}_{\rho_2-}\phi(\rho_1) = \frac{1}{\Gamma(\delta+1)} \int_{\rho_1}^{\rho_2} w^{(\delta)} \left(\frac{\lambda - \rho_1}{\rho_2 - \rho_1} \right) \phi(\lambda) d\lambda^{\delta}. $$

Let I be a real interval and $\phi\colon I \to \mathbb{R}^{\delta}$. If $\forall \lambda \in I$, $\phi(\lambda) \geqslant 0^{\delta}$, then ϕ is said to be non-negative function.

In this work, we will use the following notion of convexity on $I = [0, +\infty)$ (which has as its starting point the definition of [1], [7] and [21]):

Definition 6. *Let $\phi\colon I \to \mathbb{R}^{\delta}$ and $h\colon [0,1] \to (0,1]$. If inequality*

$$ \phi\left(\sigma\xi + m(1-\sigma)\varsigma\right) \leqslant h^{s\delta}(\sigma)\phi(\xi) + m^{\delta}(1 - h^{s\delta}(\sigma))\phi(\varsigma) \tag{1} $$

is fulfilled $\forall \xi, \varsigma \in I$ and $\sigma \in [0,1]$, where $s \in (0,1]$ and $m^{\delta} \in [0^{\delta}, 1^{\delta}]$, then the function ϕ is a generalized $(h,m)-$convex of the first kind on I. Let us denote this class of functions by $N^{s,1}_{h,m}(I)$.

Definition 7. *Let $\phi\colon I \to \mathbb{R}^{\delta}$ and $h\colon [0,1] \to (0,1]$. If inequality*

$$ \phi\left(\sigma\xi + m(1-\sigma)\varsigma\right) \leqslant h^{s\delta}(\sigma)\phi(\xi) + m^{\delta}(1 - h(\sigma))^{s\delta}\phi(\varsigma) \tag{2} $$

is fulfilled $\forall \xi, \varsigma \in I$ and $\sigma \in [0,1]$, where $s \in (0,1]$ and $m^{\delta} \in [0^{\delta}, 1^{\delta}]$, then the function ϕ is a generalized (h,m)-convex of second type on I. Let us denote this class of functions by $N^{s,2}_{h,m}(I)$.

Remark 1. *Interested readers can easily verify that from Definition 7 we have many of the notions of convexity reported in the literature. For example, putting*

- *$h(z) = z$, $s = 1$, $m = 1$, and $\delta = 1$, we see that ϕ is a convex function on $[0, +\infty)$ [12], [26];*
- *$h(z) = z$, $s = 1$, and $\delta = 1$, we have the m-convexity [29];*
- *$h(z) = z$, $m = 1$, and $\delta = 1$, then we obtain the $s-$convex function on $[0, +\infty)$ [11];*

- *$s = 1$ and $h(z) = z$, then we get the Definition of generalized $m-$convex functions [21];*
- *$m = s = 1$ and $h(z) = z$: we have the generalized convex function [24];*
- *$m = 1$ and $h(z) = z$: we have the generalized $s-$convex function [23];*
- *$h(z) = z$ we obtain the concept of generalized (s, m)-convex functions on a fractal space [1].*

It is obvious that, under the consideration $\delta = 1$, other known definitions of convexity can be reproduced.

In this work, we obtain new variants of the classical Milne Inequality for generalized (h, m)-convex modified functions the second type, via local integral operators of the Definition 5.

2. Main Results. As the first result, we obtain an equality that will serve as the basis for subsequent results.

Lemma 2. *Let $\phi\colon [0,\infty) \to \mathbb{R}$ and $\phi \in D^{\delta}[\rho_1, \rho_2]$, and $w(x) \in {}_{\rho_1}J_x^{\delta}[\rho_1, \rho_2]$. If $\phi^{(\delta)} \in L_1[\rho_1, \rho_2]$ with $\rho_1 \geqslant 0$, then we have*

$$\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{\delta}\Bigg\{w(1)\Big[\phi\Big(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\Big)+\phi\Big(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\Big)\Big]- \tag{3}$$
$$-w(0)\Big[\phi\left(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\right)+\phi\Big(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\Big)\Big]\Bigg\}-$$
$$-\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{2\delta}\Gamma(\delta+1)\left[{}^{w}J^{\delta}_{\left(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\right)-}\phi\Big(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\Big)+\right.$$
$$\left.+\,{}^{w}J^{\delta}_{\left(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\right)+}\phi\Big(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\Big)\right]=$$
$$=\int_0^1 w(\lambda)\Big[\phi^{(\delta)}\Big(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\Big)-\phi^{(\delta)}\Big(\frac{1+\lambda}{\kappa+2}\rho_1+\frac{\kappa+1-\lambda}{\kappa+2}\rho_2\Big)\Big]d\lambda^{\delta}.$$

Proof. Let us denote

$$\int_0^1 w(\lambda)\Big[\phi^{(\delta)}\Big(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\Big)-\phi^{(\delta)}\Big(\frac{1+\lambda}{\kappa+2}\rho_1+\frac{\kappa+1-\lambda}{\kappa+2}\rho_2\Big)\Big]d\lambda^{\delta}=$$
$$=I_1-I_2.$$

Integrating by parts fractionally in I_1, we have:

$$I_1=\int_0^1 w(\lambda)\phi^{(\delta)}\Big(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\Big)d\lambda^\delta=$$
$$=\Big(\frac{\kappa+2}{\rho_2-\rho_1}\Big)^\delta w(\lambda)\phi\Big(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\Big)\Big|_0^1-$$
$$-\Big(\frac{\kappa+2}{\rho_2-\rho_1}\Big)^\delta\int_0^1 w^{(\delta)}(\lambda)\phi\Big(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\Big)d\lambda^\delta=$$
$$=\Big(\frac{\kappa+2}{\rho_2-\rho_1}\Big)^\delta\Big[w(1)\phi\Big(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\Big)-w(0)\phi\Big(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\Big)\Big]-$$
$$-\Big(\frac{\kappa+2}{\rho_2-\rho_1}\Big)^\delta\int_0^1 w^{(\delta)}(\lambda)\phi\Big(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\Big)d\lambda^\delta.$$

Making a change of variables in this last integral and taking into account that $\frac{\rho_2-\rho_1}{\kappa+2}=\frac{\kappa\rho_1+2\rho_2}{\kappa+2}-\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}$, we obtain:

$$z=\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2;$$
$$\text{if } \lambda=0, \text{ then } z=\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}; \text{ if } \lambda=1, \text{ then } z=\frac{\kappa\rho_1+2\rho_2}{\kappa+2};$$
$$\lambda=\frac{\kappa+2}{\rho_2-\rho_1}z-\frac{(\kappa+1)\rho_1+\rho_2}{\rho_2-\rho_1},\ d\lambda^\delta=\Big(\frac{\kappa+2}{\rho_2-\rho_1}\Big)^\delta dz^\delta,$$
$$w^{(\delta)}(\lambda)=w^{(\delta)}\Big(\frac{\kappa+2}{\rho_2-\rho_1}z-\frac{(\kappa+1)\rho_1+\rho_2}{\rho_2-\rho_1}\Big)=$$
$$=w^{(\delta)}\Big(\frac{(\kappa+2)z-(\kappa+1)\rho_1-\rho_2}{\rho_2-\rho_1}\Big)=$$
$$=w^{(\delta)}\Big(\frac{z-\frac{(\kappa+1)\rho_1-\rho_2}{\kappa+2}}{\frac{\rho_2-\rho_1}{\kappa+2}}\Big)=w^{(\delta)}\Big(\frac{z-\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}}{\frac{\kappa\rho_1+2\rho_2}{\kappa+2}-\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}}\Big).$$
$$\int_0^1 w^{(\delta)}(\lambda)\phi\Big(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\Big)d\lambda^\delta=$$

$$= \left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{\delta} \int_{\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}}^{\frac{\kappa\rho_1+2\rho_2}{\kappa+2}} w^{(\delta)}\left(\frac{z-\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}}{\frac{\kappa\rho_1+2\rho_2}{\kappa+2}-\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}}\right)\phi(z)dz^{\delta} =$$

$$= \frac{(\kappa+2)^{\delta}\Gamma(\delta+1)}{(\rho_2-\rho_1)^{\delta}} {}^{w}J^{\delta}_{\left(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\right)-}\phi\left(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\right).$$

Thus we have

$$\int_0^1 w^{(\delta)}(\lambda)\phi\left(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\right)d\lambda^{\delta} =$$

$$= \left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{\delta} \int_{\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}}^{\frac{\kappa\rho_1+2\rho_2}{\kappa+2}} w^{(\delta)}\left(\frac{z-\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}}{\frac{\kappa\rho_1+2\rho_2}{\kappa+2}-\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}}\right)\phi(z)\,dz^{\delta} =$$

$$= \frac{(\kappa+2)^{\delta}\Gamma(\delta+1)}{(\rho_2-\rho_1)^{\delta}} {}^{w}J^{\delta}_{\left(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\right)-}\phi\left(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\right).$$

So, for I_1 we obtain

$$I_1 = \left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{\delta}\left[w(1)\phi\left(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\right)-w(0)\phi\left(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\right)\right]- \tag{4}$$
$$-\Gamma(\delta+1)\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{2\delta} {}^{w}J^{\delta}_{\left(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\right)-}\phi\left(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\right).$$

In the same way, for I_2 we get

$$I_2 = (-1)^{\delta}\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{\delta}\left[w(1)\phi\left(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\right)-w(0)\phi\left(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\right)\right]-$$
$$-(-1)^{\delta}\Gamma(\delta+1)\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{2\delta} {}^{w}J^{\delta}_{\left(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\right)-}\phi\left(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\right) =$$
$$= -1^{\delta}\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{\delta}\left[w(1)\phi\left(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\right)-w(0)\phi\left(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\right)\right]+$$
$$+1^{\delta}\Gamma(\delta+1)\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{2\delta} {}^{w}J^{\delta}_{\left(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\right)+}\phi\left(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\right). \tag{5}$$

Subtracting (5) from (4) and rearranging, we have the desired equality. This ends the proof. □

Corollary 1. *Under the assumptions of Lemma 2, if we take $w(\lambda) = \left(\lambda^\alpha + \frac{1}{3}\right)$, then for $\kappa = 0$, $m = 1$, and $\delta = 1$ with $\alpha > 0$, we get the identity:*

$$\begin{aligned}
&\frac{1}{3}\left\{2\left[\phi\left(\rho_2\right)+\phi\left(\rho_1\right)\right]-\phi\left(\frac{\rho_1+\rho_2}{2}\right)\right\}- \qquad (6)\\
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\rho_2-\rho_1\right)^\alpha}\left[J^\alpha_{\rho_2-}\phi\left(\frac{\rho_1+\rho_2}{2}\right)+J^\alpha_{\rho_1+}\phi\left(\frac{\rho_1+\rho_2}{2}\right)\right]=\\
&=\frac{\rho_2-\rho_1}{4}\int_0^1\left(\lambda^\alpha+\frac{1}{3}\right)\left[\phi'\left(\frac{1-\lambda}{2}\rho_1+\frac{1+\lambda}{2}\rho_2\right)-\right.\\
&\left.-\phi'\left(\frac{1+\lambda}{2}\rho_1+\frac{1-\lambda}{2}\rho_2\right)\right]d\lambda.
\end{aligned}$$

Proof. Indeed, for the first expression from the right-hand side of (3), we have

$$\begin{aligned}
&\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^\delta\left\{w(1)\left[\phi\left(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\right)+\phi\left(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\right)\right]-\right. \qquad (7)\\
&\qquad\left.-w(0)\left[\phi\left(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\right)+\phi\left(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\right)\right]\right\}=\\
&=\frac{4}{3\left(\rho_2-\rho_1\right)}\left\{2\phi\left(\rho_2\right)-\phi\left(\frac{\rho_1+\rho_2}{2}\right)+2\phi\left(\rho_1\right)\right\},
\end{aligned}$$

and for generalized fractional operators, we get

$$\begin{aligned}
&\left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{2\delta}\Gamma\left(\delta+1\right)\left[{}^wJ^\delta_{\left(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\right)-}\phi\left(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\right)+\right.\\
&\left.+{}^wJ^\delta_{\left(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\right)+}\phi\left(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\right)\right]=\\
&=-\left(\frac{2}{\rho_2-\rho_1}\right)^2\Gamma\left(2\right)\left[{}^wJ^1_{\rho_2-}\phi\left(\frac{\rho_1+\rho_2}{2}\right)+{}^wJ^1_{\rho_1+}\phi\left(\frac{\rho_1+\rho_2}{2}\right)\right]=\\
&=-\left(\frac{2}{\rho_2-\rho_1}\right)^2\left[\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}w'\left(\frac{\lambda-\frac{\rho_1+\rho_2}{2}}{\rho_2-\frac{\rho_1+\rho_2}{2}}\right)\phi(\lambda)d\lambda+\right.\\
&\qquad\left.+\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}w'\left(\frac{\frac{\rho_1+\rho_2}{2}-\lambda}{\frac{\rho_1+\rho_2}{2}-\rho_1}\right)\phi(\lambda)d\lambda\right]=
\end{aligned}$$

$$= -\Big(\frac{2}{\rho_2-\rho_1}\Big)^2\Big[\int\limits_{\frac{\rho_1+\rho_2}{2}}^{\rho_2} w'\Big(\frac{\lambda-\frac{\rho_1+\rho_2}{2}}{\frac{\rho_2-\rho_1}{2}}\Big)\phi(\lambda)d\lambda+\int\limits_{\rho_1}^{\frac{\rho_1+\rho_2}{2}} w'\Big(\frac{\frac{\rho_1+\rho_2}{2}-\lambda}{\frac{\rho_2-\rho_1}{2}}\Big)\phi(\lambda)d\lambda\Big].$$

Since $w'(t) = \alpha t^{\alpha-1}$, we have

$$w'\left(\frac{\lambda-\frac{\rho_1+\rho_2}{2}}{\frac{\rho_2-\rho_1}{2}}\right) = \alpha\left(\frac{\lambda-\frac{\rho_1+\rho_2}{2}}{\frac{\rho_2-\rho_1}{2}}\right)^{\alpha-1} = \frac{2^{\alpha-1}\alpha}{(\rho_2-\rho_1)^{\alpha-1}}\left(\lambda-\frac{\rho_1+\rho_2}{2}\right)^{\alpha-1}$$

and

$$w'\left(\frac{\frac{\rho_1+\rho_2}{2}-\lambda}{\frac{\rho_2-\rho_1}{2}}\right) = \alpha\left(\frac{\frac{\rho_1+\rho_2}{2}-\lambda}{\frac{\rho_2-\rho_1}{2}}\right)^{\alpha-1} = \frac{2^{\alpha-1}\alpha}{(\rho_2-\rho_1)^{\alpha-1}}\left(\frac{\rho_1+\rho_2}{2}-\lambda\right)^{\alpha-1}.$$

And, finally, the weighted operators become Riemann-Liouville fractional:

$$-\left(\frac{2}{\rho_2-\rho_1}\right)^2\Big[\int\limits_{\frac{\rho_1+\rho_2}{2}}^{\rho_2} w'\Big(\frac{\lambda-\frac{\rho_1+\rho_2}{2}}{\frac{\rho_2-\rho_1}{2}}\Big)\phi(\lambda)d\lambda+ \tag{8}$$

$$+\int\limits_{\rho_1}^{\frac{\rho_1+\rho_2}{2}} w'\Big(\frac{\frac{\rho_1+\rho_2}{2}-\lambda}{\frac{\rho_2-\rho_1}{2}}\Big)\phi(\lambda)d\lambda\Big] =$$

$$= -\Big(\frac{2}{\rho_2-\rho_1}\Big)^{\alpha+1}\alpha\Gamma(\alpha)\Big[\frac{1}{\Gamma(\alpha)}\int\limits_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}\Big(\lambda-\frac{\rho_1+\rho_2}{2}\Big)^{\alpha-1}\phi(\lambda)d\lambda+$$

$$+\frac{1}{\Gamma(\alpha)}\int\limits_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\Big(\frac{\rho_1+\rho_2}{2}-\lambda\Big)^{\alpha-1}\phi(\lambda)d\lambda\Big] =$$

$$= -\Big(\frac{2}{\rho_2-\rho_1}\Big)^{\alpha+1}\Gamma(\alpha+1)\left[J^{\alpha}_{\rho_2-}\phi\left(\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha}_{\rho_1+}\phi\left(\frac{\rho_1+\rho_2}{2}\right)\right].$$

From the integrals in the right-hand side in (3), we get

$$\int\limits_0^1 w(\lambda)\Big[\phi^{(\delta)}\Big(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1+\frac{1+\lambda}{\kappa+2}\rho_2\Big)-\phi^{(\delta)}\Big(\frac{1+\lambda}{\kappa+2}\rho_1+\frac{\kappa+1-\lambda}{\kappa+2}\rho_2\Big)\Big]d\lambda = \tag{9}$$

$$= \int_0^1 \left(\lambda^\alpha + \frac{1}{3}\right) \left[\phi' \left(\frac{1-\lambda}{2}\rho_1 + \frac{1+\lambda}{2}\rho_2\right) - \phi^{(\delta)} \left(\frac{1+\lambda}{2}\rho_1 + \frac{1-\lambda}{2}\rho_2\right)\right] d\lambda.$$

Taking into account (7)–(9), it is not difficult to obtain (6). The proof is complete. □

Remark 2. *Identity (6) was obtained by Budak et al. (Lemma 1 in [14]).*

Remark 3. *In the case $\kappa = 0$ and $m = 1$*

a) If we take $w(\lambda) = \left(\left(\frac{1-(1-\lambda)^\alpha}{\alpha}\right)^\beta + \frac{1}{3\alpha^\beta}\right)$, we get Lemma 2.1 of [15] with $\delta = 1$;

b) If we take $w(\lambda) = (\lambda + \frac{1}{3})^\delta$, we have Lemma 2.1 of [6].

For convenience, we denote by $L(\delta,\kappa,w)$ the left-hand side of (3), so

$$\begin{aligned} L(\delta,\kappa,w) = \left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^\delta \Big\{ w(1)\Big[\phi\Big(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\Big) + \phi\Big(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\Big)\Big] - \\ - w(0)\Big[\phi\Big(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\Big) + \phi\Big(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\Big)\Big]\Big\} - \\ - \left(\frac{\kappa+2}{\rho_2-\rho_1}\right)^{2\delta} \Gamma(\delta+1) \Big[{}^{w}J^{\delta}_{\left(\frac{\kappa\rho_1+2\rho_2}{\kappa+2}\right)-}\phi\Big(\frac{(\kappa+1)\rho_1+\rho_2}{\kappa+2}\Big) + \\ + {}^{w}J^{\delta}_{\left(\frac{2\rho_1+\kappa\rho_2}{\kappa+2}\right)+}\phi\Big(\frac{\rho_1+(\kappa+1)\rho_2}{\kappa+2}\Big)\Big]. \end{aligned} \tag{10}$$

Theorem 1. *Let $\phi\colon [0,\infty) \to \mathbb{R}$ and $\phi \in D^\delta[\rho_1,\rho_2]$, and $\phi^{(\delta)} \in L_1[\rho_1,\rho_2]$, and $w(x) \in {}_{\rho_1}J^\delta_x[\rho_1,\rho_2]$ with $0 \leqslant \rho_1 < \rho_2$. If $\phi \in N^{s,2}_{h,m}[0,\infty)$ and $\frac{\rho_1}{m} \in [\rho_1,\rho_2]$; then*

$$\begin{aligned} \big|L(\delta,\kappa,w)\big| \leqslant \Big|(w(1)-w(0))\phi\Big(\frac{x+y}{2}\Big)\Big| \leqslant \\ \leqslant \left(\frac{\kappa+2}{y-x}\right)^\delta \Gamma(\delta+1)\Big| h^{s\delta}\Big(\frac{1}{2}\Big) {}^{w}J^{\delta}_{\left(\frac{\kappa x+2y}{\kappa+2}\right)-}\phi\Big(\frac{(\kappa+1)x+y}{\kappa+2}\Big) + \\ + m^{2\delta}\Big(1-h\Big(\frac{1}{2}\Big)\Big)^{s\delta} {}^{w}J^{\delta}_{\left(\frac{2x+\kappa y}{m(\kappa+2)}\right)+}\phi\Big(\frac{x+(\kappa+1)y}{m(\kappa+2)}\Big)\Big| \leqslant \\ \leqslant h^{s\delta}\Big(\frac{1}{2}\Big)\,|\phi(x)| \int_0^1 w^{(\delta)}(\lambda) h^{s\delta}\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big) d\lambda^\delta + \\ + m^\delta\Big(1-h\Big(\frac{1}{2}\Big)\Big)^{s\delta}\Big|\phi\left(\frac{y}{m}\right)\Big| \int_0^1 w^{(\delta)}(\lambda)\Big(1-h\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big)\Big)^{s\delta} d\lambda^\delta + \end{aligned} \tag{11}$$

$$+ h^{s\delta}\Big(\frac{1}{2}\Big)\left|\phi\left(\frac{x}{m}\right)\right|\int_0^1 w^{(\delta)}(\lambda)h^{s\delta}\Big(\frac{1+\lambda}{\kappa+2}\Big)d\lambda^\delta +$$

$$+ m^\delta\Big(1-h\Big(\frac{1}{2}\Big)\Big)^{s\delta}\left|\phi\Big(\frac{y}{m^2}\Big)\right|\int_0^1 w^{(\delta)}(\lambda)\Big(1-h\Big(\frac{1+\lambda}{\kappa+2}\Big)\Big)^{s\delta}d\lambda^\delta.$$

Proof. Putting $\lambda = \frac{1}{2}$, we have, from the second-type generalized (h, m) convexity of ϕ, the following:

$$\phi\left(\frac{\rho_1+\rho_2}{2}\right) \leqslant h^{s\delta}\Big(\frac{1}{2}\Big)\phi(\rho_1) + m^\delta\Big(1-h\Big(\frac{1}{2}\Big)\Big)^{s\delta}\phi\left(\frac{\rho_2}{m}\right).$$

Putting $\rho_1 = \frac{\kappa+1-\lambda}{\kappa+2}x + \frac{1+\lambda}{\kappa+2}y$ and $\rho_2 = \frac{1+\lambda}{\kappa+2}x + \frac{\kappa+1-\lambda}{\kappa+2}y$, we have

$$\phi\left(\frac{x+y}{2}\right) \leqslant h^{s\delta}\Big(\frac{1}{2}\Big)\phi\Big(\frac{\kappa+1-\lambda}{\kappa+2}x + \frac{1+\lambda}{\kappa+2}y\Big) + \\ + m^\delta\Big(1-h\Big(\frac{1}{2}\Big)\Big)^{s\delta}\phi\Big(\frac{1+\lambda}{\kappa+2}\frac{x}{m} + \frac{\kappa+1-\lambda}{\kappa+2}\frac{y}{m}\Big).$$

Multiplying this inequality by $w^{(\delta)}(\lambda)$ and integrating between 0 and 1 with respect to λ, we obtain

$$\int_0^1 w^{(\delta)}(\lambda)\phi\left(\frac{x+y}{2}\right)d\lambda^\delta \leqslant h^{s\delta}\Big(\frac{1}{2}\Big)\int_0^1 w^{(\delta)}\phi\Big(\frac{\kappa+1-\lambda}{\kappa+2}x + \frac{1+\lambda}{\kappa+2}y\Big)d\lambda^\delta + \\ + m^\delta\Big(1-h\Big(\frac{1}{2}\Big)\Big)^{s\delta}\int_0^1 w^{(\delta)}(\lambda)\phi\Big(\frac{1+\lambda}{\kappa+2}\frac{x}{m} + \frac{\kappa+1-\lambda}{\kappa+2}\frac{y}{m}\Big)d\lambda^\delta.$$

From this, we have

$$(w(1)-w(0))\phi\left(\frac{x+y}{2}\right) \leqslant h^{s\delta}\left(\frac{1}{2}\right)I_1 + m^\delta\left(1-h\left(\frac{1}{2}\right)\right)^{s\delta}I_2 \qquad (12)$$

Change of variables $z = \frac{\kappa+1-\lambda}{\kappa+2}x + \frac{1+\lambda}{\kappa+2}y$ in I_1 and $z = \frac{1+\lambda}{\kappa+2}\frac{x}{m} + \frac{\kappa+1-\lambda}{\kappa+2}\frac{y}{m}$ in I_2 leads us to the following result:

$$z = \frac{\kappa+1-\lambda}{\kappa+2}x + \frac{1+\lambda}{\kappa+2}y \Longrightarrow \begin{cases} \text{if } \lambda = 0, \text{ then } z = \frac{(\kappa+1)x+y}{\kappa+2}; \\ \text{if } \lambda = 1, \text{ then } z = \frac{\kappa x+2y}{\kappa+2}; \end{cases}$$

$$\lambda = \frac{(\kappa+2)z}{y-x} - \frac{(\kappa+1)x+y}{y-x}, d\lambda = \frac{\kappa+2}{y-x}dz \text{ and } d\lambda^\delta = \left(\frac{\kappa+2}{y-x}\right)^\delta dz^\delta;$$

$$I_1 = \int_0^1 w^{(\delta)}(\lambda)\phi\Big(\frac{\kappa+1-\lambda}{\kappa+2}x + \frac{1+\lambda}{\kappa+2}y\Big)d\lambda^\delta =$$

$$= \int_{\frac{(\kappa+1)x+y}{\kappa+2}}^{\frac{\kappa x+2y}{\kappa+2}} w^{(\delta)}\Big(\frac{(\kappa+2)z}{y-x} - \frac{(\kappa+1)x+y}{y-x}\Big)\phi(z)\left(\frac{\kappa+2}{y-x}\right)^\delta dz^\delta =$$

$$= \left(\frac{\kappa+2}{y-x}\right)^\delta \int_{\frac{(\kappa+1)x+y}{\kappa+2}}^{\frac{\kappa x+2y}{\kappa+2}} w^{(\delta)}\left(\frac{z-\frac{(\kappa+1)x+y}{\kappa+2}}{\frac{\kappa x+2y}{\kappa+2}-\frac{(\kappa+1)x+y}{\kappa+2}}\right)\phi(z)\,dz^\delta =$$

$$= \left(\frac{\kappa+2}{y-x}\right)^\delta \Gamma(\delta+1)\ {}^{w}J^{\delta}_{\left(\frac{\kappa x+2y}{\kappa+2}\right)-}\phi\Big(\frac{(\kappa+1)x+y}{\kappa+2}\Big).$$

Analogously, for I_2 we have

$$z = \frac{1+\lambda}{\kappa+2}\frac{x}{m} + \frac{\kappa+1-\lambda}{\kappa+2}\frac{y}{m} \Longrightarrow \begin{cases} \text{if } \lambda=0, \text{ then } z = \frac{x+(\kappa+1)y}{m(\kappa+2)}; \\ \text{if } \lambda=1, \text{ then } z = \frac{2x+\kappa y}{m(\kappa+2)}; \end{cases}$$

$$\lambda = \frac{x+\kappa y - m(\kappa+2)z}{y-x} \Longrightarrow d\lambda = -\frac{m(\kappa+2)}{y-x}dz$$

$$\Longrightarrow d\lambda^\delta = (-1)^\delta\Big(\frac{m(\kappa+2)}{y-x}\Big)^\delta dz^\delta = -1^\delta\Big(\frac{m(\kappa+2)}{y-x}\Big)^\delta dz^\delta$$

$$I_2 = \int_0^1 w^{(\delta)}(\lambda)\phi\Big(\frac{1+\lambda}{\kappa+2}\frac{x}{m} + \frac{\kappa+1-\lambda}{\kappa+2}\frac{y}{m}\Big)d\lambda^\delta =$$

$$= -1^\delta \int_{\frac{x+(\kappa+1)y}{m(\kappa+2)}}^{\frac{2x+\kappa y}{m(\kappa+2)}} w^{(\delta)}\Big(\frac{x+\kappa y-m(\kappa+2)z}{y-x}\Big)\phi(z)\left(\frac{m(\kappa+2)}{y-x}\right)^\delta dz^\delta =$$

$$= \left(\frac{m(\kappa+2)}{y-x}\right)^\delta \int_{\frac{2x+\kappa y}{m(\kappa+2)}}^{\frac{x+(\kappa+1)y}{m(\kappa+2)}} w^{(\delta)}\left(\frac{\frac{x+\kappa y}{m(\kappa+2)}-z}{\frac{x+(\kappa+1)y}{m(\kappa+2)}-\frac{2x+\kappa y}{m(\kappa+2)}}\right)\phi(z)\,dz^\delta \ =$$

$$= \left(\frac{m(\kappa+2)}{y-x}\right)^{\delta}\Gamma(\delta+1)^{w}J^{\delta}_{\frac{2x+\kappa y}{m(\kappa+2)}+}\phi\left(\frac{x+(\kappa+1)y}{m\,(\kappa+2)}\right).$$

Taking into account the last two results in (12), we can see the first inequality of (11).

To obtain the right-hand side member, using the generalized (h,m)-convexity of the second type of ϕ, we have, successively,

$$\phi\left(\frac{\kappa+1-\lambda}{\kappa+2}x+\frac{1+\lambda}{\kappa+2}y\right)=\phi\left(\frac{\kappa+1-\lambda}{\kappa+2}x+\left(1-\frac{\kappa+1-\lambda}{\kappa+2}\right)y\right)\leqslant$$

$$\leqslant h^{s\delta}\left(\frac{\kappa+1-\lambda}{\kappa+2}\right)\phi(x)+m^{\delta}\left(1-h\left(\frac{\kappa+1-\lambda}{\kappa+2}\right)\right)^{s\delta}\phi\left(\frac{y}{m}\right)$$

and

$$\phi\left(\frac{1+\lambda}{\kappa+2}\frac{x}{m}+\frac{\kappa+1-\lambda}{\kappa+2}\frac{y}{m}\right)=\phi\left(\frac{1+\lambda}{\kappa+2}\frac{x}{m}+\left(1-\frac{1+\lambda}{\kappa+2}\right)\frac{y}{m}\right)\leqslant$$

$$\leqslant h^{s\delta}\left(\frac{1+\lambda}{\kappa+2}\right)\phi\left(\frac{x}{m}\right)+m^{\delta}\left(1-h\left(\frac{1+\lambda}{\kappa+2}\right)\right)^{s\delta}\phi\left(\frac{y}{m^2}\right).$$

Multiplying the first inequality by $h^{s\delta}\left(\frac{1}{2}\right)w^{(\delta)}(\lambda)$ and the second by $m^{\delta}\left(1-h\left(\frac{1}{2}\right)\right)^{s\delta}w^{(\delta)}(\lambda)$, after integrating between 0 and 1 we obtain

$$h^{s\delta}\left(\frac{1}{2}\right)\int_0^1 w^{(\delta)}(\lambda)\phi\left(\frac{\kappa+1-\lambda}{\kappa+2}x+\frac{1+\lambda}{\kappa+2}y\right)d\lambda^{\delta}\leqslant \tag{13}$$

$$\leqslant h^{s\delta}\left(\frac{1}{2}\right)\Big[\phi(x)\int_0^1 w^{(\delta)}(\lambda)h^{s\delta}\left(\frac{\kappa+1-\lambda}{\kappa+2}\right)d\lambda^{\delta}+$$

$$+m^{\delta}\phi\left(\frac{y}{m}\right)\int_0^1 w^{(\delta)}(\lambda)\left(1-h\left(\frac{\kappa+1-\lambda}{\kappa+2}\right)\right)^{s\delta}d\lambda^{\delta}\Big]$$

and

$$m^{\delta}\left(1-h\left(\frac{1}{2}\right)\right)^{s\delta}\int_0^1 w^{(\delta)}(\lambda)\phi\left(\frac{1+\lambda}{\kappa+2}\frac{x}{m}+\frac{\kappa+1-\lambda}{\kappa+2}\frac{y}{m}\right)d\lambda^{\delta}\leqslant \tag{14}$$

$$\leqslant m^{\delta}\left(1-h\left(\frac{1}{2}\right)\right)^{s\delta}\Big[\phi\left(\frac{x}{m}\right)\int_0^1 w^{(\delta)}(\lambda)h^{s\delta}\left(\frac{1+\lambda}{\kappa+2}\right)d\lambda^{\delta}+$$

$$+\phi\left(\frac{y}{m^2}\right)\int_0^1 w^{(\delta)}(\lambda)\Big(1-h\Big(\frac{1+\lambda}{\kappa+2}\Big)\Big)^{s\delta}d\lambda^\delta\Big].$$

Making the change of variables $z=\frac{\kappa+1-\lambda}{\kappa+2}x+\frac{1+\lambda}{\kappa+2}y$ in the integral of the left-hand side of (13) and $z=\frac{1+\lambda}{\kappa+2}\frac{x}{m}+\frac{\kappa+1-\lambda}{\kappa+2}\frac{y}{m}$ in the integral of the left-hand side of (14), the required inequality is obtained.

This ends the proof. □

Corollary 2. *Under the assumptions of Theorem 1, if we take $h(t)=t$, then for $\kappa=0$, $m=s=1$, we get the inequalities*

$$\frac{2^\delta}{(\rho_2-\rho_1)^\delta}\Big|\Big\{\Big[w(1)\phi(\rho_2)-2w(0)\phi\left(\frac{\rho_1+\rho_2}{2}\right)+w(1)\phi(\rho_1)\Big]-$$
$$-\frac{2^\delta\Gamma(\delta+1)}{(\rho_2-\rho_1)^\delta}\left[{}^wJ^\delta_{\rho_2-}\phi\left(\frac{\rho_1+\rho_2}{2}\right)+{}^wJ^\delta_{\rho_1+}\phi\left(\frac{\rho_1+\rho_2}{2}\right)\right]\Big\}\Big|\leqslant$$
$$\leqslant\left|(w(1)-w(0))\phi\left(\frac{x+y}{2}\right)\right|\leqslant$$
$$\leqslant\frac{\Gamma(\delta+1)}{(y-x)^\delta}\left|{}^wJ^\delta_{y-}\phi\left(\frac{x+y}{2}\right)+1^{2\delta}\ {}^wJ^\delta_{x+}\phi\left(\frac{x+y}{2}\right)\right|\leqslant$$
$$\leqslant\frac{1^\delta\left[|\phi(x)|+1^\delta|\phi(y)|\right]}{2^\delta}\int_0^1 w^{(\delta)}(\lambda)d\lambda^\delta.$$

Remark 4. *If we take $w(\lambda)=\left(\lambda^\alpha+\frac{1}{3}\right)$ and $\delta=1$ with $\alpha>0$ in Corollary 2, then we get the inequalities:*

$$\frac{4}{\rho_2-\rho_1}\Big|\frac{1}{3}\Big\{2\phi(\rho_2)-\phi\left(\frac{\rho_1+\rho_2}{2}\right)+2\phi(\rho_1)\Big\}-$$
$$-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^{\alpha-1}}\left[J^\alpha_{\rho_2-}\phi\left(\frac{\rho_1+\rho_2}{2}\right)+J^\alpha_{\rho_1+}\phi\left(\frac{\rho_1+\rho_2}{2}\right)\right]\Big|\leqslant$$
$$\leqslant\left|\phi\left(\frac{x+y}{2}\right)\right|\leqslant\frac{1}{y-x}\left|\ J^\alpha_{y-}\phi\left(\frac{x+y}{2}\right)+\ J^\alpha_{x+}\phi\left(\frac{x+y}{2}\right)\right|\leqslant$$
$$\leqslant\frac{|\phi(x)|+|\phi(y)|}{2}.$$

Here $J^\alpha_{y-}\phi$ and $J^\alpha_{x+}\phi$ are Riemann-Liouville fractional integral operators. It should be noted that the last two inequalities are a variant of the Hermite-Hadamard inequality for fractional integration operators.

Remark 5. *If we put $h(\lambda) = \lambda$, $\kappa = 0$ and $w^{(\delta)}(\lambda) = 1$ in the above result, we obtain an extension of the Theorem 3.1 of [21].*

By imposing more restrictive conditions on $\phi^{(\delta)}$ on the right-hand side of (3), we can obtain more refined inequalities.

So, we have this first result:

Theorem 2. *Let $\phi\colon [0,\infty) \to \mathbb{R}^{\delta}$ and $\phi^{(\delta)} \in L_1[\rho_1,\rho_2]$, and $w(x) \in {}_{\rho_1}J_x^{\delta}[\rho_1,\rho_2]$ with $0 \leqslant \rho_1 < \rho_2$. If $\left|\phi^{(\delta)}\right| \in N_{h,m}^{s,2}[0,\infty)$, with $m \in (0,1]$ and $\frac{\rho_1}{m} \in [\rho_1,\rho_2]$, then*

$$
\begin{aligned}
&|L(\delta,\kappa,w)| \leqslant \\
&\leqslant \left|\phi^{(\delta)}(\rho_1)\right| \left[\int_0^1 w(\lambda)\left(h^{s\delta}\left(\frac{\kappa+1-\lambda}{\kappa+2}\right) + h^{s\delta}\left(\frac{1+\lambda}{\kappa+2}\right)\right) d\lambda^{\delta}\right] + \qquad (15)\\
&+m^{\delta}\left|\phi^{(\delta)}\left(\frac{\rho_2}{m}\right)\right| \left[\int_0^1 w(\lambda)\left(\left(1-h\left(\frac{1+\lambda}{\kappa+2}\right)\right)^{s\delta} + \left(1-h\left(\frac{\kappa+1-\lambda}{\kappa+2}\right)\right)^{s\delta}\right) d\lambda^{\delta}\right].
\end{aligned}
$$

Proof. By using properties of the fractal integral, for the right-side of (3) we can write:

$$
\begin{aligned}
&\left|\int_0^1 w(\lambda)\left[\phi^{(\delta)}\left(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1 + \frac{1+\lambda}{\kappa+2}\rho_2\right) - \right.\right. \qquad (16)\\
&\qquad \left.\left. - \phi^{(\delta)}\left(\frac{1+\lambda}{\kappa+2}\rho_1 + \frac{\kappa+1-\lambda}{\kappa+2}\rho_2\right)\right] d\lambda^{\delta}\right| \leqslant \\
&\leqslant \int_0^1 w(\lambda)\left|\phi^{(\delta)}\left(\frac{\kappa+1-\lambda}{\kappa+2}\rho_1 + \frac{1+\lambda}{\kappa+2}\rho_2\right)\right| d\lambda^{\delta} + \\
&\qquad + \int_0^1 w(\lambda)\left|\phi^{(\delta)}\left(\frac{1+\lambda}{\kappa+2}\rho_1 + \frac{\kappa+1-\lambda}{\kappa+2}\rho_2\right)\right| d\lambda^{\delta} = \\
&= |I_1| + |I_2|.
\end{aligned}
$$

And using the $(h,m)-$convexity of $\phi^{(\delta)}$ in both integrals leads us to

$$
|I_1| + |I_2| \leqslant \left|\phi^{(\delta)}(\rho_1)\right| \int_0^1 w(\lambda) h^{s\delta}\left(\frac{\kappa+1-\lambda}{\kappa+2}\right) d\lambda +
$$

$$+ m^{\delta}\left|\phi^{(\delta)}\Big(\frac{\rho_2}{m}\Big)\right|\int_0^1 w(\lambda)\Big(1-h\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big)\Big)^{s\delta} d\lambda +$$

$$+ \left|\phi^{(\delta)}(\rho_1)\right|\int_0^1 w(\lambda)h^{s\delta}\Big(\frac{1+\lambda}{\kappa+2}\Big)d\lambda +$$

$$+ m^{\delta}\left|\phi^{(\delta)}\left(\frac{\rho_2}{m}\right)\right|\int_0^1 w(\lambda)\Big(1-h\Big(\frac{1+\lambda}{\kappa+2}\Big)\Big)^{s\delta} d\lambda \leqslant$$

$$\leqslant \left|\phi^{(\delta)}(\rho_1)\right|\Big[\int_0^1 w(\lambda)\Big(h^{s\delta}\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big)+h^{s\delta}\Big(\frac{1+\lambda}{\kappa+2}\Big)\Big)d\lambda^{\delta}\Big]+$$

$$+ m^{\delta}\left|\phi^{(\delta)}\left(\frac{\rho_2}{m}\right)\right|\Big[\int_0^1 w(\lambda)\Big(\Big(1-h\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big)\Big)^{s\delta}+$$

$$+\Big(1-h\Big(\frac{1+\lambda}{\kappa+2}\Big)\Big)^{s\delta}\Big)d\lambda^{\delta}\Big].$$

Which is the desired result. □

Corollary 3. *Under the assumptions of Theorem 2, if we take $h(t) = t$, then, for $\kappa = 0$, we get the inequalities*

$$\frac{2^{\delta}}{(\rho_2-\rho_1)^{\delta}}\Big|\Big\{\Big[w(1)\phi(\rho_2) - 2w(0)\phi\left(\frac{\rho_1+\rho_2}{2}\right) + w(1)\phi(\rho_1)\Big]-$$

$$-\frac{2^{\delta}\Gamma(\delta+1)}{(\rho_2-\rho_1)^{\delta}}\left[{}^{w}J^{\delta}_{\rho_2-}\phi\left(\frac{\rho_1+\rho_2}{2}\right) + {}^{w}J^{\delta}_{\rho_1+}\phi\left(\frac{\rho_1+\rho_2}{2}\right)\right]\Big\}\Big| \leqslant$$

$$\leqslant \Big[\left|\phi^{(\delta)}(\rho_1)\right| + m^{\delta}\left|\phi^{(\delta)}\left(\frac{\rho_2}{m}\right)\right|\Big]\Big[\int_0^1 w(\lambda)\Big(\Big(\frac{1-\lambda}{2}\Big)^{s\delta}+\Big(\frac{1+\lambda}{2}\Big)^{s\delta}\Big)d\lambda^{\delta}\Big].$$

Remark 6. *If we take $w(\lambda) = \left(\lambda^{\alpha}+\frac{1}{3}\right)$ and $m = s = \delta = 1$ with $\alpha > 0$ in Corollary 3, then we get the inequality*

$$\Big|\frac{1}{3}\Big\{2\phi(\rho_2) - \phi\left(\frac{\rho_1+\rho_2}{2}\right) + 2\phi(\rho_1)\Big\} -$$

$$-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho_2-\rho_1)^{\alpha}}\left[J^{\alpha}_{\rho_2-}\phi\left(\frac{\rho_1+\rho_2}{2}\right) + J^{\alpha}_{\rho_1+}\phi\left(\frac{\rho_1+\rho_2}{2}\right)\right]\Big| \leqslant$$

$$\leqslant \frac{\rho_2-\rho_1}{12}\left(\frac{\alpha+4}{\alpha+1}\right)\left[\left|\phi'(\rho_1)\right|+\left|\phi'(\rho_2)\right|\right].$$

Here $J^{\alpha}_{y-}\phi$ and $J^{\alpha}_{x+}\phi$ are Riemann-Liouville fractional integral operators. This inequality was obtained by Budak et al. (see Theorem 1 in [14]).

Theorem 3. *Let $\phi\colon [0,\infty) \to \mathbb{R}^{\delta}$ and $\left|\phi^{(\delta)}\right|^q \in L_1[\rho_1,\rho_2]$, and $w(x) \in {}_{\rho_1}J^{\delta}_x[\rho_1,\rho_2]$ with $0 \leqslant \rho_1 < \rho_2$. If $\left|\phi^{(\delta)}\right|^q \in N^{s,2}_{h,m}[0,\infty)$, with $m \in (0,1]$ and $\frac{\rho_1}{m} \in [\rho_1,\rho_2]$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$|L(\delta,\kappa,w)| \leqslant \Big(\int_0^1 w^p(\lambda)d\lambda^{\delta}\Big)^{\frac{1}{p}}\Bigg\{\Bigg[\left|\phi^{(\delta)}(\rho_1)\right|^q\int_0^1 h^{\delta}\left(\frac{\kappa+1-\lambda}{\kappa+2}\right)d\lambda^{\delta}+ \quad (17)$$

$$+m^{\delta}\left|\phi^{\delta}\left(\frac{\rho_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{\kappa+1-\lambda}{\kappa+2}\right)\right)^{\delta}d\lambda^{\delta}\Bigg]^{\frac{1}{q}}+$$

$$+\Bigg[\left|\phi^{(\delta)}(\rho_1)\right|^q\int_0^1 h^{\delta}\Big(\frac{1+\lambda}{\kappa+2}\Big)d\lambda^{\delta}+m^{\delta}\left|\phi^{(\delta)}\left(\frac{\rho_2}{m}\right)\right|^q\int_0^1\Big(1-h\Big(\frac{1+\lambda}{\kappa+2}\Big)\Big)^{\delta}d\lambda^{\delta}\Bigg]^{\frac{1}{q}}\Bigg\}.$$

Proof. Using the (h,m)-convexity of $\left|\phi^{(\delta)}\right|^q$ and the well-known Hölder inequality from (16), we obtain:

$$|I_1|+|I_2| \leqslant \Big(\int_0^1 w^p(\lambda)d\lambda^{\delta}\Big)^{\frac{1}{q}}\Bigg[\left|\phi^{(\delta)}(\rho_1)\right|^q\int_0^1 h^{\delta}\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big)d\lambda^{\delta}+$$

$$+m^{\delta}\left|\phi^{\delta}\left(\frac{\rho_2}{m}\right)\right|^q\int_0^1\Big(1-h\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big)\Big)^{\delta}d\lambda^{\delta}\Bigg]^{\frac{1}{q}}+$$

$$+\Big(\int_0^1 w^p(\lambda)d\lambda^{\delta}\Big)^{\frac{1}{q}}\Bigg[\left|\phi^{(\delta)}(\rho_1)\right|^q\int_0^1 h^{\delta}\left(\frac{1+\lambda}{\kappa+2}\right)d\lambda^{\delta}+$$

$$+m^{\delta}\left|\phi^{\delta}\left(\frac{\rho_2}{m}\right)\right|^q\int_0^1\Big(1-h\Big(\frac{1+\lambda}{\kappa+2}\Big)\Big)^{\delta}d\lambda^{\delta}\Bigg]^{\frac{1}{q}}.$$

From the last inequality, taking into account (3), it obviously follows that (17). The proof is complete. □

Theorem 4. *Under the assumptions of the previous theorem, if $q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then we have the following inequality:*

$$
\begin{aligned}
|L(\delta,\kappa,w)| \leqslant \Big(\int_0^1 w(\lambda)d\lambda^\delta\Big)^{1-\frac{1}{q}} \Big\{\Big[\big|\phi^{(\delta)}(\rho_1)\big|^q \int_0^1 w(\lambda)h^{s\delta}\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big)d\lambda^\delta + \\
+ m^\delta \Big|\phi^\delta\Big(\frac{\rho_2}{m}\Big)\Big|^q \int_0^1 w(\lambda)\Big(1-h\Big(\frac{\kappa+1-\lambda}{\kappa+2}\Big)^{s\delta} d\lambda^\delta\Big]^{\frac{1}{q}} + \\
+ \Big[\big|\phi^{(\delta)}(\rho_1)\big|^q \int_0^1 w(\lambda)h^{s\delta}\left(\frac{1+\lambda}{\kappa+2}\right)d\lambda^\delta + \\
+ m^\delta \Big|\phi^{(\delta)}\Big(\frac{\rho_2}{m}\Big)\Big|^q \int_0^1 w(\lambda)\left(1-h\left(\frac{1+\lambda}{\kappa+2}\right)\right)^{s\delta} d\lambda^\delta\Big]^{\frac{1}{q}}\Big\}. \qquad (18)
\end{aligned}
$$

Proof. The proof follows the same path as the previous one, only a different form of the Hölder's inequality is used: the power mean one. □

Remark 7. *Under the conditions of Theorems 3 and 4, if we take $w(\lambda) = \left(\lambda^\alpha + \frac{1}{3}\right)$ with $\alpha > 0$ and $h(t) = t$, then for $k = 0$, and $\delta = m = s = 1$, we get the Theorem 1 and 2 from [14].*

Remark 8. *Other refinements can be obtained using other known inequalities, such as Young's.*

3. Conclusions. In this work, we present a generalized formulation of the fractal weighted integral, which contains, as a particular case, many of the integral operators reported in the literature. In this context, we present several integral inequalities that generalize several known results.

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