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ESTIMATES FOR THE SECOND HANKEL–CLIFFORD TRANSFORM AND TITCHMARSH EQUIVALENCE THEOREM

Abstract. We obtain estimates of integrals containing the second Hankel–Clifford transforms of functions from Sobolev-Hankel–Clifford spaces. As a corollary, we obtain a new variant of Titchmarsh equivalence theorem for the second Hankel–Clifford transform.

Key words: *second Hankel–Clifford transform, Hankel–Clifford translation, Sobolev–Hankel–Clifford spaces, Titchmarsh equivalence theorem*

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1. Introduction. Let $f\colon \mathbb{R} \to \mathbb{C}$ be in $L^1(\mathbb{R})$. The Fourier transform of f is defined by

$$\widehat{f}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} f(t) e^{-itx}\, dt, \quad x \in \mathbb{R}.$$

If $f \in L^p(\mathbb{R})$, $1 < p \leqslant 2$, then Fourier transform $\widehat{f}(x)$ is defined as the limit of $(2\pi)^{-1/2} \int\limits_{-a}^{b} f(x) e^{-itx}\, dx$ in the norm of $L^q(\mathbb{R})$, $q = p/(p-1)$, as $a, b \to +\infty$.

From the definition it follows that $\widehat{f} \in L^q(\mathbb{R})$. The following Hausdorff–Young inequality

$$\|\widehat{f}\|_q \leqslant C\|f\|_p := C\Big(\int\limits_{\mathbb{R}} |f(t)|^p\, dt\Big)^{1/p}, \quad f \in L^p(\mathbb{R}), \quad 1 < p \leqslant 2, \qquad (1)$$

is valid. For $p = q = 2$, we have the Plancherel equality instead of (1). More about these results can be found in [14, Ch. III and IV] or [3, Ch. 5].

In [14, Ch. 4, Theorem 85] the following Titchmarsh equivalence theorem is proved:

Theorem 1. *Let $0 < \alpha < 1$ and $f \in L^2(\mathbb{R})$. Then the conditions*

(i) $\|f(\cdot + h) - f(\cdot - h)\|_2 = O(h^\alpha)$, $h > 0$, *and*

(ii) $\int\limits_{|x|\geqslant y} |\widehat{f}(x)|^2\,dx = O(y^{-2\alpha}), \quad y > 0,$

are equivalent.

The norm in $L^2(\mathbb{R})$ is translation-invariant and $\|f(\cdot+h)-f(\cdot-h)\|_2 = =\|f(\cdot + 2h) - f(\cdot)\|_2$, $h > 0$, so the condition (i) may be substituted by $\|f(\cdot + 2h) - f(\cdot)\|_2 = O(h^\alpha)$, $h > 0$.

Lorentz [8] proved

Theorem 2. *If $1 \leqslant p \leqslant 2$, $1 \geqslant \alpha > 1/p-1/2$, and a 2π-periodic function $f \in L^1[0, 2\pi]$ with trigonometrical Fourier coefficients a_n, b_n belongs to Lip(α) (i.e., $|f(x) - f(y)| \leqslant C|x - y|^\alpha$ for all $x, y \in \mathbb{R}$), then*

$$\sum_{k=n}^{\infty}(|a_k|^p + |b_k|^p) \leqslant Cn^{-\alpha p-p/2+1}, \quad n \in \mathbb{N}.$$

Since the proof of Theorem 2 uses the Parseval equality, the condition $f \in Lip(\alpha)$ may be replaced by the condition $f \in Lip(\alpha, 2)$ (i.e., f is 2π-periodic, $f \in L^2[0, 2\pi]$, and $\int\limits_0^{2\pi} |f(x + h) - f(x)|^2\,dx = O(h^{2\alpha})$, $h > 0$.)

The aim of this paper is to obtain analogues and generalizations of Theorems 1 and 2 for the second Hankel–Clifford transform. Note that an analogue of Theorem 1 was obtained for the first Hankel–Clifford transform by El Hamma, Daher, and Mahfoud [4], while estimates of this transform in terms of corresponding differential operator were proved by Lahmadi and El Hamma [7], but there are doubts in the last result. A more elementary estimate for the first Hankel–Clifford transform was obtained by the author [15, Theorem 3]. Some close results and facts about the second Hankel–Clifford transform can be found in [16].

2. Definitions. Let $1 \leqslant p < \infty$, $\mu \geqslant 0$, $\mathbb{R}_+ = [0, +\infty)$, and $L^p_\mu(\mathbb{R}_+)$ be the space of all real-valued measurable functions, such that $\|f\|_{L^p_\mu} = \Big(\int\limits_0^\infty |f(x)|^p x^\mu\,dx\Big)^{1/p} < \infty$. If χ_E is the indicator of a set $E \subset \mathbb{R}_+$ and $f\chi_E \in L^p_\mu(\mathbb{R}_+)$, then $f \in L^p_\mu(E)$.

The Bessel-Clifford function of the first kind of order $\mu \geqslant 0$ (see, e.g., [5]) is defined by

$$c_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu+k+1)}, \quad x \geqslant 0,$$

where by $\Gamma(\alpha)$ we have denoted the Euler gamma function. It is known that $c_\mu(x)$ is a solution of the differential equation $xy'' + (\mu+1)y' + y = 0$.

If $j_\nu(x)$ is the normalized Bessel function of the first kind and order $\nu > -1/2$, given by

$$j_\nu(x) = \Gamma(\nu+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} (x/2)^{2n},$$

then c_μ and j_μ are connected by

$$c_\mu(x) = \Gamma^{-1}(\mu+1) j_\mu(2\sqrt{x}), \quad x \geqslant 0. \tag{2}$$

Hayek [6] introduced the second Hankel–Clifford transform for $f \in L^1_\mu(\mathbb{R}_+)$ by

$$h_{2,\mu}(f)(y) = \int_0^{+\infty} c_\mu(yx) f(x) x^\mu \, dx.$$

By Lemma 2 below and (2), we have $|c_\mu(x)| \leqslant \Gamma^{-1}(\mu+1)$ on $\mathbb{R}_+$. As a corollary, we obtain

$$\|h_{2,\mu}(f)\|_{L^\infty_\mu} \leqslant \Gamma^{-1}(\mu+1) \|f\|_{L^1_\mu}, \quad f \in L^1_\mu(\mathbb{R}_+). \tag{3}$$

For $\mu \geqslant 0$, the transform $h_{2,\mu}$ extends from $L^1_\mu(\mathbb{R}_+) \cap L^2_\mu(\mathbb{R}_+)$ onto $L^2_\mu(\mathbb{R}_+)$ and

$$\|h_{2,\mu}(f)\|_{L^2_\mu} = \|f\|_{L^2_\mu}, \quad f \in L^2_\mu(\mathbb{R}_+). \tag{4}$$

This Plancherel-type equality can be found in [6] or in [9]. Using Riesz–Thorin interpolation theorem (see [2, Ch. 1, Theorem 1.1.1]), we obtain a Hausdorff–Young type inequality

$$\|h_{2,\mu}(f)\|_{L^q_\mu} \leqslant C \|f\|_{L^p_\mu}, \quad f \in L^p_\mu(\mathbb{R}_+), \tag{5}$$

where $1 < p \leqslant 2$ and $q = p/(p-1)$ as in (1).

Let $\Delta(x, y, z) = (p(p-x)(p-y)(p-z))^{1/2}$, where $p = (x+y+z)/2$, be the area of the triangle with sides x, y, z. For $\mu \geqslant 0$, set

$$D_\mu(x,y,z) = \frac{\Delta^{2\mu+1}(x,y,z)}{2^{2\mu}(xyz)^\mu \Gamma(\mu+\frac{1}{2})\sqrt{\pi}},$$

when the triangle with sides x, y, z exists, and $D_\mu(x,y,z) = 0$ in other cases. Then $D_\mu(x,y,z)$ is non-negative and symmetric in x, y, z. In [12], Prasad, Singh, and Dixit suggested the generalized Hankel–Clifford translation of $f \in L^1_\mu(\mathbb{R}_+)$ as follows:

$$T_x(f)(y) = \int_0^{+\infty} f(z) D_\mu(x,y,z) z^\mu dz\ , \quad 0 < x, y < \infty.$$

Using Lemma 1.3 from [12], we have, for $f \in L^1_\mu(\mathbb{R}_+)$:

$$h_{2,\mu}(T_x(f))(y) = c_\mu(xy) h_{2,\mu}(f)(y), \quad y \geqslant 0. \tag{6}$$

By Lemma 2.3 in [16], this result is also valid for $f \in L^p_\mu(\mathbb{R}_+)$, $1 < p \leqslant 2$, a.e. on $\mathbb{R}_+$.

Now we introduce the difference of order $m \in \mathbb{N}$ with step $t > 0$ by

$$\Delta^m_{t,\mu,hc} f(x) = (I - \Gamma(\mu+1)T_t)^m f(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \Gamma^i(\mu+1) T^i_t f(x),$$

where I is the identical operator, and the modulus of smoothness of order m in $L^p_\mu(\mathbb{R}_+)$, $1 \leqslant p < \infty$, by

$$\omega_m(f,\delta)_{p,\mu,hc} = \sup_{0 \leqslant t \leqslant \delta} \|\Delta^m_{t,\mu,hc} f\|_{L^p_\mu}.$$

Due to Lemma 1, for $1 \leqslant p < \infty$, $\mu \geqslant 0$, and $\delta \geqslant 0$, we have $\omega_m(f,\delta)_{p,\mu,hc} \leqslant C\|f\|_{L^p_\mu}$.

Let $S(0,+\infty)$ be the set of all infinitely differentiable functions $\psi(x)$ defined on $(0,+\infty)$, such that

$$\rho_{m.k}(\psi) = \sup_{0 < x < \infty} |x^m \psi^{(k)}(x)| < \infty$$

for all $m, k \in \mathbb{Z}_+$. In [9] it is proved that $h_{2,\mu}$ is an automorphism of $S(0,+\infty)$. Also, in [9, Proposition 6] it is established that for differential operator $B_\mu(\psi) = x\psi'' + (\mu+1)\psi$ and $\psi \in S(0,+\infty)$ the equality

$$h_{2,\mu}(B^i_\mu(\psi))(y) = (-y)^i h_{2,\mu}(\psi)(y), \quad y > 0, \quad i \in \mathbb{N}, \tag{7}$$

holds (see also (1.14) in [12]). For $\mu \geqslant 0$, $1 \leqslant p < \infty$, and $m \in \mathbb{N}$, we define the Sobolev space $W^m_{p,\mu}(\mathbb{R}_+)$ consisting of $f \in L^p_\mu(\mathbb{R}_+)$, such that f, f', …, $f^{(2m-1)}$ are absolutely continuous on each segment from $(0, +\infty)$ and $B^i_\mu(f) \in L^p_\mu(\mathbb{R}_+)$, $i = 1, 2, \dots, m$.

Also, we can consider the space $S_e(\mathbb{R}_+)$ as the space of $\varphi X_{[0,+\infty)}$, where φ are even Schwartz functions. Then $S_e(\mathbb{R}_+) \subset S(0,\infty)$ and $S_e(\mathbb{R}_+)$ is dense in all $L^p_\mu(\mathbb{R}_+)$, $1 \leqslant p < \infty$. Using the usual density arguments, we state that (7) is valid for $\psi \in W^m_{p,\mu}(\mathbb{R}_+)$ and $i = 1, 2, \dots, m$.

Denote by Φ the set of continuous and increasing on $\mathbb{R}_+ = [0,\infty)$ functions ω, such that $\omega(0) = 0$. If $\omega \in \Phi$ and $\int_0^\delta t^{-1}\omega(t)\,dt = O(\omega(\delta))$, $\delta \geqslant 0$, then ω belongs to the Bary class B; if $\omega \in \Phi$ and $\delta^m \int_\delta^\infty t^{-m-1}\omega(t)\,dt = O(\omega(\delta))$ for some $m > 0$ and all $\delta > 0$, then ω belongs to the Bary-Stechkin class B_m (see [1]). We say that $\omega \in \Phi$ satisfies the Δ_2-condition ($\omega \in \Delta_2$), if $\omega(2x) \leqslant C\omega(x)$, $x \in \mathbb{R}_+$.

3. Auxiliary propositions.

Lemma 1. *Let $1 \leqslant p < \infty$, $\mu \geqslant 0$, $f \in L^p_\mu(\mathbb{R}_+)$. Then*

$$\|\Gamma(\mu+1)T_t f\|_{L^p_\mu} \leqslant \|f\|_{L^p_\mu}.$$

The proof of Lemma 1 belongs to Prasad and Singh [13, Lemma 1.1].

Lemma 2. *Let $\mu \geqslant 0$. Then*

(i) $|j_\mu(x)| \leqslant 1$ for $x \geqslant 0$ and $j_\mu(x) < 1$ for $x > 0$;

(ii) $1 - j_\mu(x) \geqslant C > 0$ for $x \geqslant 1$;

(iii) the double inequality $C_1 x^2 \leqslant 1 - j_\mu(x) \leqslant C_2 x^2$ is valid for some $C_2 > C_1 > 0$ and all $x \in [0, 1]$.

Proof. For (i) and (ii), see papers by Platonov [11] and [10, Lemma 3.3]. The assertion of (iii) see, e.g., in [17]. □

From (6), (7), (2), and using induction, we deduce

Lemma 3. *Let $1 \leqslant p \leqslant 2$, $\mu \geqslant 0$, $f \in L^p_\mu(\mathbb{R}_+)$, $m \in \mathbb{N}$, $t \geqslant 0$. Then*

$$h_{2,\mu}(\Delta^m_{t,\mu,hc} f)(y) = (1 - j_\mu(2\sqrt{yt})^m h_{2,\mu}(f)(y) \quad \text{for a.e.} \quad y \in \mathbb{R}_+.$$

For $k \in \mathbb{N}$ and $f \in W^k_{p,\mu}(\mathbb{R}_+)$, we have:

$$h_{2,\mu}(\Delta^m_{t,\mu,hc} B^k_\mu(f))(y) = (1 - j_\mu(2\sqrt{yt})^m (-y)^k h_{2,\mu}(f)(y) \quad \text{for a.e.} \quad y \in \mathbb{R}_+.$$

Lemma 4 can be found in [16].

Lemma 4. *Let $\mu \geqslant 0$, $m > 0$, $\omega \in B_m$, and $G(t)$ be a non-negative measurable function on $\mathbb{R}_+$, such that*

$$\int_y^\infty G(t)t^\mu\,dt = O(\omega(1/y)), \quad y > 0.$$

Then $t^m G(t)$ is integrable on each segment $[a, b] \subset \mathbb{R}_+$ and

$$\int_0^y t^m G(t)t^\mu\,dt = O(y^m \omega(1/y)), \quad y > 0.$$

Lemma 5 is proved in [1].

Lemma 5. *Let $\omega \in \Phi$ and $m \in \mathbb{N}$. Then the conditions (i) $\omega \in B_m$; and (ii) there exists $\alpha \in (0, m)$, such that for all $0 < u \leqslant v < \infty$ the inequality $\omega(v)/v^{m-\alpha} \leqslant C\omega(u)/u^{m-\alpha}$ holds; are equivalent. In particular, if $\omega \in B_m$, then ω satisfies the Δ_2-condition.*

4. Main results. Theorem 3 is an analogue and an extension of Theorem 2.

Theorem 3. *Let $\mu \geqslant 0$, $1 < p \leqslant 2$, $1/p + 1/q = 1$, $m \in \mathbb{N}$, $k \in \mathbb{Z}_+$, $f \in L^p_\mu(\mathbb{R}_+)$ for $k = 0$ or $f \in W^k_{p,\mu}(\mathbb{R}_+)$ for $k \in \mathbb{N}$. If $0 < r \leqslant q$ and $\alpha \in \mathbb{R}$, then for all $N > 0$ we have:*

$$\int_N^\infty y^\alpha |h_{2,\mu}(f)(y)|^r y^\mu\,dy \leqslant C \int_{N/2}^\infty t^{\alpha - kr - (\mu+1)r/q} \omega_m^r(B_\mu^k(f), t^{-1})_{p,\mu,hc} t^\mu\,dt.$$

Proof. By Lemma 3 and Hausdorff–Young inequality (5), we have

$$\int_{\mathbb{R}_+} |h_{2,\mu}(f)(y)|^q y^{kq}(1 - j_\mu(2\sqrt{yt}))^{mq} y^\mu\,dy \leqslant$$

$$\leqslant C_1 \|\Delta^m_{t,\mu,hc} B^k_\mu(f)\|^q_{L^p_\mu} \leqslant C_1 \omega^q_m(B^k_\mu(f), t)_{p,\mu,hc}.$$

Let $N > 0$ and $D_i = [2^i N, 2^{i+1}N)$, $i \in \mathbb{Z}_+$, $t_i = 2^{-i}N^{-1}$. Then, by Lemma 2 (ii), we find that

$$\int_{D_i} |h_{2,\mu}(f)(y)|^q y^\mu\,dy \leqslant C_2 (2^i N)^{-kq} \omega^q_m(B^k_\mu(f), t_i)_{p,\mu,hc}.$$

By the Hölder inequality, for $0 < r < q$ we obtain:

$$\int\limits_{D_i} y^{\alpha}|h_{2,\mu}(f)(y)|^r y^{\mu}\,dy \leqslant$$

$$\Big(\int\limits_{D_i} y^{\alpha q/(q-r)+\mu}\,dy\Big)^{1-r/q}\Big(\int\limits_{D_i} |h_{2,\mu}(f)(y)|^q y^{\mu}\,dy\Big)^{r/q} \leqslant$$

$$\leqslant C_3(2^iN)^{\alpha+(\mu+1)(1-r/q)}(2^iN)^{-kr}\omega_m^r(B_\mu^k(f),2^{-i}N^{-1})_{p,\mu,hc} \leqslant$$

$$\leqslant C_4\int\limits_{2^{i-1}N}^{2^iN}\omega_m^r(f,t^{-1})_{p,\mu,hc}t^{\alpha-kr-(\mu+1)r/q+\mu}\,dt. \tag{8}$$

For $r = q$, we see that

$$\int\limits_{D_i} y^{\alpha}|h_{2,\mu}(f)(y)|^q y^{\mu}\,dy \leqslant C_5(2^iN)^{\alpha-kq}\omega_m^q(B_\mu^k(f),2^{-i}N^{-1})_{p,\mu,hc} \leqslant$$

$$\leqslant C_6\int\limits_{2^{i-1}N}^{2^iN}\omega_m^q(B_\mu^k(f),t^{-1})_{p,\mu,hc}t^{\alpha-kq-1}\,dt. \tag{9}$$

Summing up (9) or (8) over $i = 0, 1, \ldots,$ we obtain

$$\int\limits_{N}^{\infty} y^{\alpha}|h_{2,\mu}(f)(y)|^r y^{\mu}\,dy \leqslant C_7\int\limits_{N/2}^{\infty} t^{\alpha-kr-(\mu+1)r/q}\omega_m^r(B_\mu^k(f),t^{-1})_{p,\mu,hc}t^{\mu}\,dt.$$

□

Corollary 1. *Let* $1 < p \leqslant 2$, $q = p/(p-1)$, $\omega \in \Delta_2$, $m, k \in \mathbb{N}$, $f \in W_{p,\mu}^k(\mathbb{R}_+)$ *and* $\omega_m(B_\mu^k(f),\delta)_{p,\mu,hc} = O(\omega(\delta))$, $\delta \geqslant 0$. *Then*

$$\int\limits_{N}^{\infty} |h_{2,\mu}(f)(y)|^q y^{\mu}\,dy \leqslant C\frac{\omega(N^{-1})}{N^{kq}}, \quad N > 0. \tag{10}$$

Proof. By Theorem 3, we have for $\alpha = 0$ and $r = q$:

$$\int\limits_{N}^{\infty} |h_{2,\mu}(f)(y)|^q y^{\mu}\,dy \leqslant C_1\omega_m^q(B_\mu^k(f),2/N)_{p,\mu,hc}\int\limits_{N/2}^{\infty} t^{-kq-1}\,dt \leqslant$$

$$\leqslant C_2 \frac{\omega(N^{-1})}{N^{kq}}, \quad N > 0,$$

due to the condition $\omega \in \Delta_2$. $\square$

Remark. *It is interesting to compare Corollary 1 with Theorem 2.1 in [7], where a similar to (10) estimate with $h_{1,\mu}$ instead of $h_{2,\mu}$ is obtained. It seems that a factor N^{-2kq} in the analogue of (10) in [7] is not proper.*

Now we can obtain a variant of Theorem 1 (or Titchmarsh equivalence theorem).

Theorem 4. *Let $\mu \geqslant 0$, $f \in L^2_\mu(\mathbb{R})$, $m \in \mathbb{N}$ and $\omega^2 \in B \cap B_{2m}$. Then the conditions* (i) $\omega_m(f,\delta)_{p,\mu,hc} = O(\omega(\delta))$, $\delta \geqslant 0$;

$$\text{(ii)} \quad \int_N^\infty |h_{2,\mu}(f)(y)|^2 y^\mu \, dy = O(\omega^2(N^{-1})), \quad N > 0,$$

and

$$\text{(iii)} \quad \int_N^{2N} |h_{2,\mu}(f)(y)|^2 y^\mu \, dy = O(\omega^2(N^{-1})), \quad N > 0,$$

are equivalent.

Proof. Let (i) be valid. By Theorem 3 in the case $\alpha = k = 0$, $r = p = 2$, we obtain

$$\int_N^\infty |h_{2,\mu}(f)(y)|^2 y^\mu \, dy \leqslant C_1 \int_{N/2}^\infty t^{-\mu-1} \omega^2(t^{-1}) t^\mu \, dt =$$

$$= C_1 \int_0^{2/N} \frac{\omega^2(t)}{t} \, dt \leqslant C_1 \omega^2(2/N) \leqslant C_2 \omega^2(N^{-1}),$$

since $\omega^2 \in B$ and, by Lemma 5, ω^2 satisfies Δ_2-condition. Thus, we prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Conversely, let (iii) be true. Then

$$\int_N^\infty |h_{2,\mu}(f)(y)|^2 y^\mu \, dy = \sum_{i=0}^\infty \int_{2^i N}^{2^{i+1} N} |h_{2,\mu}(f)(y)|^2 y^\mu \, dy \leqslant$$

$$\leqslant C_4\sum_{i=0}^{\infty}\omega^2((2^iN)^{-1}) = C_4\omega^2(N^{-1}) + C_4\sum_{i=1}^{\infty}2^iN\int\limits_{1/(2^iN)}^{1/(2^{i-1}N)}\omega^2(t)\,dt \leqslant$$

$$\leqslant 2C_4\Big(\omega^2(N^{-1}) + \sum_{i=1}^{\infty}\int\limits_{1/(2^iN)}^{1/(2^{i-1}N)}\frac{\omega^2(t)}{t}\,dt\Big) \leqslant C_5\omega^2(N^{-1}), \quad N>0,$$

by the condition $\omega^2 \in B$, i.e., (iii) $\Rightarrow$ (ii).

By Lemma 3 and Plancherel-type equality (4), we have:

$$\|\Delta^m_{t,\mu,hc}f\|^2_{L^2_\mu} = \int\limits_{\mathbb{R}_+}|h_{2,\mu}(f)(y)|^2(1-j_\mu(2\sqrt{yt}))^{2m}y^\mu\,dy =$$

$$= \Big(\int\limits_0^{1/(4t)} + \int\limits_{1/(4t)}^{\infty}\Big)|h_{2,\mu}(f)(y)|^2(1-j_\mu(2\sqrt{yt}))^{2m}y^\mu\,dy = I_1(t)+I_2(t).$$

By Lemma 2 (i), Lemma 5, and condition (ii) of the Theorem, we obtain:

$$I_2(t) \leqslant 2^{2m}\int\limits_{1/(4t)}^{\infty}|h_{2,\mu}(f)(y)|^2y^\mu\,dy \leqslant C_6\omega^2(4t) \leqslant C_7\omega^2(t). \qquad (11)$$

On the other hand, by Lemma 2 (iii):

$$I_1(t) \leqslant C_7\int\limits_0^{1/(4t)}|h_{2,\mu}(f)(y)|^2(yt)^{2m}y^\mu\,dy.$$

But by (11), the condition $\omega^2 \in B_{2m}$, and Lemma 4, we find that

$$\int\limits_0^{1/(4t)}y^{2m}|h_{2,\mu}(f)(y)|^2y^\mu\,dy \leqslant C_8(1/(4t))^{2m}\omega^2(4t)$$

and $I_1(t) \leqslant C_9\omega(t)$, $t>0$, by Lemma 5. From the last inequality and (11), we deduce that $\|\Delta^m_{t,\mu,hc}f\|_{L^2_\mu} \leqslant (C_7+C_9)^{1/2}\omega(t)$, $t>0$, and (ii) $\Rightarrow$ (i) follows. Theorem 4 is proved. □

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