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A NORMAL CRITERION CONCERNING SEQUENCE OF FUNCTIONS AND THEIR DIFFERENTIAL POLYNOMIALS

Abstract. In this paper, we study normality of a sequence of meromorphic functions whose differential polynomials satisfy a certain condition. We also give examples to show that the result is sharp.

Key words: *normal families, differential polynomials, meromorphic functions*

2020 Mathematical Subject Classification: *30D30, 30D35, 30D45, 34M05*

1. Introduction. In what follows, $\mathcal{H}(D)$ and $\mathcal{M}(D)$ are the classes of all holomorphic and meromorphic functions in the domain $D \subseteq \mathbb{C}$, respectively. A family $\mathcal{F} \subset \mathcal{M}(D)$ is said to be *normal* in D if every sequence of functions in $\mathcal{F}$ has a subsequence that converges locally uniformly in D with respect to the spherical metric to a limit function, which is either meromorphic in D or the constant ∞. In the case $\mathcal{F} \subset \mathcal{H}(D)$, the Euclidean metric can be substituted for the spherical metric (see [25], [34]). The idea of the normal family is attributed to Paul Montel [22], [23]. Ever since its creation, the theory of normal families has been a cornerstone of complex analysis with far-reaching applications in dynamics of rational as well as transcendental maps, function theory of one and several variables, bicomplex analysis, harmonic mappings, complex projective geometry, functional analysis etc. (see [1], [2], [5], [6], [12], [15], [20], [31], [34]).

The main purpose of this paper is to study the normality of a sequence of non-vanishing meromorphic functions in a domain $D \subseteq \mathbb{C}$, whose differential polynomials have non-exceptional holomorphic functions in D. For $f, g \in \mathcal{M}(D)$, if $f(z) - g(z) \neq 0$ in D, then g is said to be an exceptional function of f in D. On the other hand, if there exist at least one $z \in D$ for which $f(z) - g(z) = 0$, then g is said to be a non-exceptional function of f in D. If g happens to be a constant, say k, then k is said to be an exceptional (respectively, non-exceptional) value of f in D.

Definition 1. *[16] Let $k \in \mathbb{N}$, $f \in \mathcal{M}(D)$ and $n_0, n_1, \ldots, n_k$ be non-negative integers, not all zeros. By a differential monomial of f we mean an expression of the form*

$$M[f] := a \cdot (f)^{n_0} (f')^{n_1} (f'')^{n_2} \cdots (f^{(k)})^{n_k},$$

where $a\ (\not\equiv 0, \infty) \in \mathcal{M}(D)$. If a is taken to be the constant function 1, then we say that the differential monomial $M[f]$ is normalized. Further, the quantities

$$\lambda_M := \sum_{j=0}^{k} n_j \text{ and } \mu_M := \sum_{j=0}^{k} (j+1) n_j$$

are called the degree and weight of the differential monomial $M[f]$, respectively.

For $1 \leqslant i \leqslant m$, let $M_i[f] = \prod_{j=0}^{k} \left(f^{(j)}\right)^{n_{ji}}$ be m differential monomials of f. Then the sum

$$P[f] := \sum_{i=1}^{m} a_i M_i[f]$$

is called a differential polynomial of f and the quantities

$$\lambda_P := \max\{\lambda_{M_i} : 1 \leqslant i \leqslant m\} \text{ and } \mu_P := \max\{\mu_{M_i} : 1 \leqslant i \leqslant m\}$$

are called the degree and weight of the differential polynomial $P[f]$, respectively. If $\lambda_{M_1} = \lambda_{M_2} = \cdots = \lambda_{M_m}$, then $P[f]$ is said to be a homogeneous differential polynomial.

In this work, we are concerned with the homogeneous differential polynomials of the form

$$Q[f] := f^{x_0} (f^{x_1})^{(y_1)} (f^{x_2})^{(y_2)} \cdots (f^{x_k})^{(y_k)}, \tag{1}$$

where x_0, x_1, $\ldots$, x_k, y_1, y_2, $\ldots$, y_k are non-negative integers, such that $x_i \geqslant y_i$ for $i = 1, 2, \ldots, k$.

The differential polynomial (1) first appeared in the literature in [14] and has been used extensively since then, particularly in finding normality criteria of families of meromorphic functions (see [26], [27], [28]).

We set $x' = \sum_{i=1}^{k} x_i$ and $y' = \sum_{i=1}^{k} y_i$. Further, we assume that $x_0 > 0$ and $y' > 0$. Using the generalized Leibniz rule for derivatives, one can easily verify that

$$(f^{x_i})^{(y_i)} = \sum_{n_1+n_2+\cdots+n_{x_i}=y_i} \frac{y_i!}{n_1!n_2!\cdots n_{x_i}!} f^{(n_1)} f^{(n_2)} \cdots f^{(n_{x_i})},$$

where n_i's are non-negative integers. Thus, the degree of $Q[f]$, $\lambda_Q = x_0 + x'$ and the weight of $Q[f]$, $\mu_Q = x_0 + x' + y' = \lambda_Q + y'$.

2. Motivation and main results. In [19, Problem 5.11], Hayman posed the following problem:

Problem A. *Let $\mathcal{F} \subset \mathcal{M}(D)$ and k be a positive integer. Suppose that for each $f \in \mathcal{F}$, $f(z) \neq 0$, $f^{(k)} \neq 1$. Then, what can be said about the normality of $\mathcal{F}$ in D?*

Gu [17] considered Problem A and confirmed that the family $\mathcal{F}$ is indeed normal in D. Subsequently, Yang [29] proved that the exceptional value 1 of $f^{(k)}$ can be replaced by an exceptional holomorphic function. Chang [3] considered the case when $f^{(k)} - 1$ has limited number of zeros and obtained the normality of $\mathcal{F}$. Thin and Oanh [28] replaced $f^{(k)}$ with a differential polynomial of f. Later, Deng et al. [11] established that there is no loss of normality even when $f^{(k)} - h$ has zeros for some $h \in \mathcal{H}(D)$ as long as the number of zeros are bounded by the constant k. Chen et al. [8] took a sequence of exceptional holomorphic functions instead of a single exceptional holomorphic function. Recently, Deng et al. [13] proved the following theorem concerning a sequence of meromorphic functions:

Theorem B. *Let $\{f_j\} \subset \mathcal{M}(D)$ and $\{h_j\} \subset \mathcal{H}(D)$ be sequences of functions in D. Assume that $h_j \to h$ locally uniformly in D, where $h \in \mathcal{H}(D)$ and $h \not\equiv 0$. Let k be a positive integer. If, for each j, $f_j(z) \neq 0$ and $f_j^{(k)} - h_j(z)$ has at most k distinct zeros, ignoring multiplicities, in D, then $\{f_j\}$ is normal in D.*

Following Thin and Oanh [28], a natural question about Theorem B arises:

Question C. *Let $\{f_j\} \subset \mathcal{M}(D)$ and $\{h_j\} \subset \mathcal{H}(D)$ be sequences of functions in D. Is it possible to generalize Theorem B for differential polynomials $Q[f_j]$?*

In this paper, our first objective is to find a complete answer to Question C. Since normality is a local property, one can always restrict the domain to the open unit disk $\mathbb{D}$.

Theorem 1. *Let* $\{f_j\} \subset \mathcal{M}(\mathbb{D})$ *and* $\{h_j\} \subset \mathcal{H}(\mathbb{D})$ *be such that* $h_j \to h$ *locally uniformly in* $\mathbb{D}$*, where* $h \in \mathcal{H}(D)$ *and* $h \not\equiv 0$*. Let* $Q[f_j]$ *be a differential polynomial of* f_j *as defined in* (1)*, having weight* μ_Q*. If, for each* j*,* $f_j(z) \neq 0$ *and* $Q[f_j] - h_j$ *has at most* $\mu_Q - 1$ *zeros, ignoring multiplicities, in* $\mathbb{D}$*, then* $\{f_j\}$ *is normal in* $\mathbb{D}$*.*

Remark 1. *Theorem 1 gives an affirmative answer to Question C.*

Our next objective is to find whether the upper bound for the number of zeros of $Q[f_j] - h_j$ in Theorem 1 can be improved. In view of this, we obtain the following result, which is more general than Theorem 1:

Theorem 2. *Let* $\{f_j\} \subset \mathcal{M}(\mathbb{D})$ *be a sequence, such that, for each* j*,* f_j *has poles of multiplicity at least* m*,* $m \in \mathbb{N}$*. Let* $\{h_j\} \subset \mathcal{H}(\mathbb{D})$ *be such that* $h_j \to h$ *locally uniformly in* $\mathbb{D}$*, where* $h \in \mathcal{H}(\mathbb{D})$ *and* $h \not\equiv 0$*. Let* $Q[f_j]$ *be a differential polynomial of* f_j *as defined in* (1)*, having degree* λ_Q *and weight* μ_Q*. If, for each* j*,* $f_j(z) \neq 0$ *and* $Q[f_j] - h_j$ *has at most* $\mu_Q + \lambda_Q(m-1) - 1$ *zeros, ignoring multiplicities, in* $\mathbb{D}$*, then* $\{f_j\}$ *is normal in* $\mathbb{D}$*.*

Remark 2. *Clearly, if we do not take the multiplicity of poles of* f_j *into account, then Theorem 2 reduces to Theorem 1.*

As a direct consequence of Theorems 1 and 2, we have

Corollary 1. *Let* $\{f_j\} \subset \mathcal{M}(\mathbb{D})$ *and* $\{h_j\} \subset \mathcal{H}(\mathbb{D})$ *be such that* $h_j \to h$ *locally uniformly in* $\mathbb{D}$*, where* $h \in \mathcal{H}(D)$ *and* $h \not\equiv 0$*. If, for each* j*,* $f_j(z) \neq 0$ *and* $Q[f_j](z) \neq h_j(z)$*, then* $\{f_j\}$ *is normal in* $\mathbb{D}$*.*

In the following, we show that the condition '$f_j(z) \neq 0$' in Theorem 2 is essential.

Example 1. Consider a sequence $\{f_j\} \subset \mathcal{M}(\mathbb{D})$ given by $f_j(z) = jz$, $j \in \mathbb{N}$, $j \geqslant 2$. Let $Q[f_j] := f_j f_j'$, so that $\mu_Q = 3$, and let $h_j(z) = z$. Then $h_j \to z \not\equiv 0$ and $Q[f_j](z) - h_j(z)$ has at most one zero in $\mathbb{D}$. However, $\{f_j\}$ is not normal in $\mathbb{D}$.

Taking $h_j(z) = 1/z$ in Example 1, we find that h_j cannot be meromorphic in $\mathbb{D}$. Furthermore, the condition "$h \not\equiv 0$" in Theorem 2 cannot be dropped as demonstrated by the following example:

Example 2. Let $\{f_j\} \subset \mathcal{M}(\mathbb{D})$ be such that $f_j(z) = e^{jz}$, $j \in \mathbb{N}$, and let $h_j \equiv 0$, so that $h_j \to h \equiv 0$. Let $Q[f_j]$ be any differential polynomial of f_j of the form (1). Clearly, $Q[f_j](z) - h(z)$ has no zero in $\mathbb{D}$. But the sequence $\{f_j\}$ is not normal in $\mathbb{D}$.

The following example establishes the sharpness of the condition "$Q[f_j] - h_j$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ distinct zeros in $\mathbb{D}$" in Theorem 2:

Example 3. Let $\{f_j\} \subset \mathcal{M}(\mathbb{D})$ be such that

$$f_j(z) = \frac{1}{jz},\ j \geqslant 3,\ j \in \mathbb{N},$$

and let $Q[f_j] := f_j f_j'$. Then $\lambda_Q = 2$, $\mu_Q = 3$, $m = 1$, and $Q[f_j](z) = -1/j^2 z^3$. Consider $h_j(z) = 1/(z-1)^3$, so that $\{h_j\} \in \mathcal{H}(\mathbb{D})$ and $h_j \to 1/(z-1)^3 \not\equiv 0$. Then, by simple calculations, one can easily see that $Q[f_j](z) - h_j(z)$ has exactly $\mu_Q + \lambda_Q(m-1) = 3$ distinct zeros in $\mathbb{D}$. However, the sequence $\{f_j\}$ is not normal in $\mathbb{D}$.

3. Preliminary results.

What follow are the preparations for the proof of the main result. We assume that the reader is familiar with standard definitions and notations of Nevanlinna's value distribution theory, like $m(r,f), N(r,f), T(r,f), S(r,f)$ (see [18], [30]). Recall that a function $g \in \mathcal{M}(\mathbb{C})$ is said to be a small function of $f \in \mathcal{M}(\mathbb{C})$ if $T(r,g) = S(r,f)$ as $r \to \infty$, possibly outside a set of finite Lebesgue measure.

Notation: By $D_r(a)$, we mean an open disk in $\mathbb{C}$ with center a and radius r. $\mathbb{D} = D_1(0)$ is the open unit disk in $\mathbb{C}$.

The following lemma is an extension of the Zalcman–Pang Lemma due to Chen and Gu [9] (cf. [24, Lemma 2]).

Lemma 1. *(Zalcman–Pang Lemma) Let $\mathcal{F} \subset \mathcal{M}(\mathbb{D})$ be such that each $f \in \mathcal{F}$ has zeros of multiplicity at least m and poles of multiplicity at least p. Let $-p < \alpha < m$. If $\mathcal{F}$ is not normal at $z_0 \in \mathbb{D}$, then there exist sequences $\{f_j\} \subset \mathcal{F}$, $\{z_j\} \subset \mathbb{D}$, satisfying $z_j \to z_0$, and positive numbers ρ_j with $\rho_j \to 0$, such that the sequence $\{g_j\}$ defined by*

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j \zeta) \to g(\zeta)$$

locally uniformly in $\mathbb{C}$ with respect to the spherical metric, where g is a non-constant meromorphic function on $\mathbb{C}$, such that for every $\zeta \in \mathbb{C}$, $g^{\#}(\zeta) \leqslant g^{\#}(0) = 1$.

We remark that if $f(z) \neq 0$ in D for every $f \in \mathcal{F}$, then $\alpha \in (-p, +\infty)$. Likewise, if each $f \in \mathcal{F}$ does not have any pole in D, then $\alpha \in (-\infty, m)$, and if $f(z) \neq 0, \infty$ in D for every $f \in \mathcal{F}$, then $\alpha \in (-\infty, +\infty)$.

Lemma 2. *[10, Lemma 3] Let $\mathcal{F} \subset \mathcal{M}(\mathbb{D})$ and suppose that $h \in \mathcal{H}(\mathbb{D})$ or $h \equiv \infty$. Further, assume that for each $f \in \mathcal{F}$, $f(z) \neq h(z)$ in $\mathbb{D}$. If $\mathcal{F}$ is normal in $\mathbb{D}\backslash\{0\}$ but not normal in $\mathbb{D}$, then there exists a sequence $\{f_j\} \subset \mathcal{F}$, such that $f_j \to h$ in $\mathbb{D}\backslash\{0\}$.*

Proposition 1. *Let $f \in \mathcal{M}(\mathbb{C})$ be a transcendental function and let $Q[f]$ be a differential polynomial of f as defined in (1), having degree λ_Q and weight μ_Q. Assume that ψ ($\not\equiv 0, \infty$) is a small function of f. Then*

$$\lambda_Q T(r,f) \leqslant \overline{N}(r,f) + (1+\mu_Q-\lambda_Q)\overline{N}\Big(r,\frac{1}{f}\Big)\overline{N}\Big(r,\frac{1}{Q[f]-\psi}\Big) + S(r,f).$$

Proof. By definition of $Q[f]$, it is apparent that $Q[f] \not\equiv 0$. Then, from the first fundamental theorem of Nevanlinna, we have

$$\begin{aligned}\lambda_Q T(r,f) &= \lambda_Q\, m\Big(r,\frac{1}{f}\Big) + \lambda_Q N\Big(r,\frac{1}{f}\Big) + O(1) \leqslant \\ &\leqslant m\Big(r,\frac{Q[f]}{f^{\lambda_Q}}\Big) + m\Big(r,\frac{1}{Q[f]}\Big) + \lambda_Q N\Big(r,\frac{1}{f}\Big) + O(1). \qquad (2)\end{aligned}$$

From Nevanlinna's theorem on logarithmic derivative, we find that

$$m\left(r,\frac{Q[f]}{f^{\lambda_Q}}\right) = S(r,f).$$

Thus, from (2), we obtain

$$\begin{aligned}\lambda_Q T(r,f) &\leqslant m\Big(r,\frac{1}{Q[f]}\Big) + \lambda_Q N\Big(r,\frac{1}{f}\Big) + S(r,f) = \\ &= T(r,Q[f]) - N\Big(r,\frac{1}{Q[f]}\Big) + \lambda_Q N\Big(r,\frac{1}{f}\Big) + S(r,f).\end{aligned}$$

Applying the second fundamental theorem of Nevanlinna for small functions to $T(r,Q[f])$, we get

$$\begin{aligned}\lambda_Q T(r,f) \leqslant \lambda_Q N\Big(r,\frac{1}{f}\Big) + \overline{N}(r,Q[f]) + \overline{N}\Big(r,\frac{1}{Q[f]}\Big) + \overline{N}\Big(r,\frac{1}{Q[f]-\psi}\Big) - \\ - N\Big(r,\frac{1}{Q[f]}\Big) + S(r,f) =\end{aligned}$$

$$= \lambda_Q N\Big(r, \frac{1}{f}\Big) + \overline{N}(r,f) + \overline{N}\Big(r, \frac{1}{Q[f]}\Big) + \overline{N}\Big(r, \frac{1}{Q[f]-\psi}\Big) - \tag{3}$$
$$- N\Big(r, \frac{1}{Q[f]}\Big) + S(r,f).$$

Since a zero of f with multiplicity m is also a zero of $Q[f]$ with multiplicity at least $(m+1)\lambda_Q - \mu_Q$,

$$N\Big(r, \frac{1}{Q[f]}\Big) - \overline{N}\Big(r, \frac{1}{Q[f]}\Big) \geqslant [(m+1)\lambda_Q - \mu_Q - 1]\,\overline{N}\Big(r, \frac{1}{f}\Big).$$

Therefore, from (3), we obtain

$$\lambda_Q T(r,f) \leqslant \lambda_Q N\Big(r, \frac{1}{f}\Big) + \overline{N}(r,f) + [1 + \mu_Q - (m+1)\lambda_Q]\,\overline{N}\Big(r, \frac{1}{f}\Big) +$$
$$+ \overline{N}\Big(r, \frac{1}{Q[f]-\psi}\Big) + S(r,f) \leqslant$$
$$\leqslant \overline{N}(r,f) + (1 + \mu_Q - \lambda_Q)\overline{N}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{Q[f]-\psi}\Big) + S(r,f).$$

□

Corollary 2. *Let $f \in \mathcal{M}(\mathbb{C})$ be a transcendental function and let $Q[f]$ be a differential polynomial of f as defined in (1). Assume that ψ $(\not\equiv 0, \infty)$ is a small function of f. If $f \neq 0$, then $Q[f] - \psi$ has infinitely many zeros in $\mathbb{C}$.*

Proof. From Proposition 1, we have

$$\lambda_Q T(r,f) \leqslant \overline{N}(r,f) + (1 + \mu_Q - \lambda_Q)\overline{N}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{Q[f]-\psi}\Big) + S(r,f). \tag{4}$$

Since $f \neq 0$, $\overline{N}(r, 1/f) = 0$. Thus, from (4), we obtain

$$\lambda_Q T(r,f) \leqslant \overline{N}(r,f) + \overline{N}\Big(r, \frac{1}{Q[f]-\psi}\Big) + S(r,f).$$

This implies that

$$(\lambda_Q - 1)T(r,F) \leqslant \overline{N}\Big(r, \frac{1}{Q[F]-\psi}\Big) + S(r,F).$$

Since $\lambda_Q - 1 > 0$, it follows that $Q[F] - \psi$ has infinitely many zeros in $\mathbb{C}$. $\square$

In [3], Chang proved that if f is a non-constant rational function, such that $f \neq 0$, then for $k \geqslant 1$, $f^{(k)} - 1$ has at least $k+1$ distinct zeros in $\mathbb{C}$. Using the method of Chang [3], Deng et al. [11] proved that the constant 1 can be replaced by a polynomial p ($\not\equiv 0$). Recently, Xie and Deng [32] sharpened the lower bound for the distinct zeros of $f^{(k)} - p$ in $\mathbb{C}$ by involving the multiplicity of poles of f. Thin and Oanh [28] extended the result of Chang to differential polynomials by proving that if f ($\neq 0$) is a non-constant rational function, then $Q[f] - 1$ has at least μ_Q distinct zeros in $\mathbb{C}$. We obtain a better result in the following form:

Proposition 2. *Let f be a non-constant rational function, having poles of multiplicity at least m, $m \in \mathbb{N}$, and let p ($\not\equiv 0$) be a polynomial. Let $Q[f]$ be a differential polynomial of f as defined in (1), having degree λ_Q and weight μ_Q. Assume that $f \neq 0$. Then $Q[f] - p$ has at least $\mu_Q + \lambda_Q(m-1)$ distinct zeros in $\mathbb{C}$.*

Proof. Since $f \neq 0$, it follows that f cannot be a polynomial and, so, f has at least one pole. Therefore, we can write

$$f(z) = \frac{C_1}{\prod_{i=1}^{n}(z+\alpha_i)^{n_i+m-1}}. \tag{5}$$

Let

$$p(z) = C_2 \prod_{i=1}^{l}(z+\beta_i)^{l_i}, \tag{6}$$

where C_1, C_2 are non-zero constants; l, n, n_i are positive integers; and l_i are non-negative integers. Also, β_i (when $1 \leqslant i \leqslant l$) are distinct complex numbers and α_i (when $1 \leqslant i \leqslant n$) are distinct complex numbers.

From (5), one can deduce that

$$Q[f](z) = \frac{h_Q(z)}{\prod_{i=1}^{n}(z+\alpha_i)^{\lambda_Q(n_i+m-1)+\mu_Q-\lambda_Q}}, \tag{7}$$

where h_Q is a polynomial of degree $(n-1)(\mu_Q - \lambda_Q)$.

Also, it is easy to see that $Q[f] - p$ has at least one zero in $\mathbb{C}$. Therefore, we can set

$$Q[f](z) = p(z) + \frac{C_3 \prod_{i=1}^{q}(z+\gamma_i)^{q_i}}{\prod_{i=1}^{n}(z+\alpha_i)^{\lambda_Q(n_i+m-1)+\mu_Q-\lambda_Q}}, \tag{8}$$

where $C_3 \in \mathbb{C} \setminus \{0\}$, q_i are positive integers, and γ_i $(1 \leqslant i \leqslant q)$ are distinct complex numbers.

Let $L = \sum\limits_{i=1}^{l} l_i$ and $N = \sum\limits_{i=1}^{n} n_i$. Then from (6), (7) and (8), we have

$$C_2 \prod_{i=1}^{l} (z+\beta_i)^{l_i} \prod_{i=1}^{n} (z+\alpha_i)^{\lambda_Q(n_i+m-1)+\mu_Q-\lambda_Q} + C_3 \prod_{i=1}^{q} (z+\gamma_i)^{q_i} = h_Q(z). \quad (9)$$

From (9), we find that

$$\begin{aligned} \sum_{i=1}^{q} q_i = \sum_{i=1}^{n} [\lambda_Q(n_i+m-1)+\mu_Q-\lambda_Q] + \sum_{i=1}^{l} l_i = \\ = \lambda_Q N + n(m-1)\lambda_Q + n(\mu_Q-\lambda_Q) + L \end{aligned}$$

and $C_3 = -C_2$.

Also, from (9), we get

$$\begin{aligned} &\prod_{i=1}^{l} (1+\beta_i r)^{l_i} \prod_{i=1}^{n} (1+\alpha_i r)^{\lambda_Q(n_i+m-1)+\mu_Q-\lambda_Q} - \prod_{i=1}^{q} (1+\gamma_i r)^{q_i} = \\ &= r^{\mu_Q+\lambda_Q(N+n(m-1)-1)+L} b(r), \end{aligned}$$

where $b(r) := r^{(n-1)(\mu_Q-\lambda_Q)} h_Q(1/r)/C_2$ is a polynomial of degree at most $(n-1)(\mu_Q-\lambda_Q)$. Furthermore, it follows that

$$\begin{aligned} &\frac{\prod\limits_{i=1}^{l} (1+\beta_i r)^{l_i} \prod\limits_{i=1}^{n} (1+\alpha_i r)^{\lambda_Q(n_i+m-1)+\mu_Q-\lambda_Q}}{\prod\limits_{i=1}^{q} (1+\gamma_i r)^{q_i}} = \\ &= 1 + \frac{r^{\mu_Q+\lambda_Q(N+n(m-1)-1)+L}\, b(r)}{\prod\limits_{i=1}^{q} (1+\gamma_i r)^{q_i}} = 1 + O\left(r^{\mu_Q+\lambda_Q(N+n(m-1)-1)+L}\right) \quad (10) \end{aligned}$$

as $r \to 0$. Taking logarithmic derivatives of both sides of (10), we obtain

$$\begin{aligned} &\sum_{i=1}^{l} \frac{l_i \beta_i}{1+\beta_i r} + \sum_{i=1}^{n} \frac{[\lambda_Q(n_i+m-1)+\mu_Q-\lambda_Q]\,\alpha_i}{1+\alpha_i r} - \sum_{i=1}^{q} \frac{q_i \gamma_i}{1+\gamma_i r} = \\ &= O\left(r^{\mu_Q+\lambda_Q(N+n(m-1)-1)+L-1}\right) \text{ as } r \to 0. \quad (11) \end{aligned}$$

Let $S_1 = \{\beta_1, \beta_2, \ldots, \beta_l\} \cap \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $S_2 = \{\beta_1, \beta_2, \ldots, \beta_l\} \cap \{\gamma_1, \gamma_2, \ldots, \gamma_q\}$. Consider the following cases:
Case 1: $S_1 = S_2 = \varnothing$.
Let $\alpha_{n+i} = \beta_i$ when $1 \leqslant i \leqslant l$ and

$$N_i = \begin{cases} \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q, & \text{if } 1 \leqslant i \leqslant n, \\ l_{i-n}, & \text{if } n+1 \leqslant i \leqslant n+l. \end{cases}$$

Then (11) can be written as

$$\sum_{i=1}^{n+l} \frac{N_i \alpha_i}{1 + \alpha_i r} - \sum_{i=1}^{q} \frac{q_i \gamma_i}{1 + \gamma_i r} = O\left(r^{\mu_Q + \lambda_Q(N + n(m-1) - 1) + L - 1}\right) \text{ as } r \to 0. \tag{12}$$

Comparing the coefficients of r^j, $j = 0, 1, \ldots, \mu_Q + \lambda_Q(N + n(m-1) - 1) + L - 2$ in (12), we find that

$$\sum_{i=1}^{n+l} N_i \alpha_i^j - \sum_{i=1}^{q} q_i \gamma_i^j = 0, \text{ for each } j = 1, 2, \ldots, \mu_Q + \lambda_Q(N + n(m-1) - 1) + L - 1. \tag{13}$$

Now, let $\alpha_{n+l+i} = \gamma_i$ for $1 \leqslant i \leqslant q$. Then, from (13) and the fact that $\sum\limits_{i=1}^{n+l} N_i - \sum\limits_{i=1}^{q} q_i = 0$, we deduce that the system of equations

$$\sum_{i=1}^{n+l+q} \alpha_i^j x_i = 0, \; j = 0, 1, \ldots, \mu_Q + \lambda_Q(N + n(m-1) - 1) + L - 1, \tag{14}$$

has a non-zero solution

$$(x_1, \ldots, x_{n+l}, x_{n+l+1}, \ldots, x_{n+l+q}) = (N_1, \ldots, N_{n+l}, -q_1, \ldots, -q_q).$$

This is possible only when the rank of the coefficient matrix of the system (14) is strictly less than $n + l + q$.

Hence, $\mu_Q + \lambda_Q(N + n(m-1) - 1) + L < n + l + q$. Since $N = \sum\limits_{i=1}^{n} n_i \geqslant n$ and $L = \sum\limits_{i=1}^{l} l_i \geqslant l$, it follows that $q \geqslant \mu_Q + \lambda_Q(m-1)$.
Case 2: $S_1 \neq \varnothing$ and $S_2 = \varnothing$.
We may assume, without loss of generality, that $S_1 = \{\beta_1, \beta_2, \ldots, \beta_{s_1}\}$. Then $\beta_i = \alpha_i$ for $1 \leqslant i \leqslant s_1$. Take $s_3 = l - s_1$.

Subcase 2.1: $s_3 \geqslant 1$.
Let $\alpha_{n+i} = \beta_{s_1+i}$ for $1 \leqslant i \leqslant s_3$. If $s_1 < n$, then let

$$N_i = \begin{cases} \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q + l_i, & \text{if } 1 \leqslant i \leqslant s_1, \\ \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q, & \text{if } s_1 + 1 \leqslant i \leqslant n, \\ l_{s_1-n+i}, & \text{if } n + 1 \leqslant i \leqslant n + s_3. \end{cases}$$

If $s_1 = n$, then we take

$$N_i = \begin{cases} \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q + l_i, & \text{if } 1 \leqslant i \leqslant s_1, \\ l_{s_1-n+i}, & \text{if } n + 1 \leqslant i \leqslant n + s_3. \end{cases}$$

Subcase 2.2: $s_3 = 0$.
If $s_1 < n$, then set

$$N_i = \begin{cases} \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q + l_i, & \text{if } 1 \leqslant i \leqslant s_1, \\ \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q, & \text{if } s_1 + 1 \leqslant i \leqslant n \end{cases}$$

and if $s_1 = n$, then set $N_i = \lambda_Q(n_i+m-1)+\mu_Q-\lambda_Q+l_i$, for $1 \leqslant i \leqslant s_1 = n$.

Thus, (11) can be written as:

$$\sum_{i=1}^{n+s_3} \frac{N_i\alpha_i}{1 + \alpha_i r} - \sum_{i=1}^{q} \frac{q_i\gamma_i}{1 + \gamma_i r} = O\left(r^{\mu_Q+\lambda_Q(N+n(m-1)-1)+L-1}\right) \text{ as } r \to 0,$$

where $0 \leqslant s_3 \leqslant l - 1$. Proceeding in the similar fashion as in Case 1, we deduce that $q \geqslant \mu_Q + m - 1$.
Case 3: $S_1 = \varnothing$ and $S_2 \neq \varnothing$.
We may assume, without loss of generality, that $S_2 = \{\beta_1, \beta_2, \dots, \beta_{s_2}\}$. Then $\beta_i = \gamma_i$ for $1 \leqslant i \leqslant s_2$. Take $s_4 = l - s_2$.
Subcase 3.1: $s_4 \geqslant 1$.
Let $\gamma_{q+i} = \beta_{s_2+i}$ for $1 \leqslant i \leqslant s_4$. If $s_2 < q$, then set

$$Q_i = \begin{cases} q_i - l_i, & \text{if } 1 \leqslant i \leqslant s_2, \\ q_i, & \text{if } s_2 + 1 \leqslant i \leqslant q, \\ -l_{s_2-q+i}, & \text{if } q + 1 \leqslant i \leqslant q + s_4. \end{cases}$$

If $s_2 = q$, then set

$$Q_i = \begin{cases} q_i - l_i, & \text{if } 1 \leqslant i \leqslant s_2, \\ -l_{s_2-q+i}, & \text{if } q + 1 \leqslant i \leqslant q + s_4. \end{cases}$$

Subcase 3.2: $s_4 = 0$.
If $s_2 < q$, then set

$$Q_i = \begin{cases} q_i - l_i & \text{if } 1 \leqslant i \leqslant s_2, \\ q_i & \text{if } s_2 + 1 \leqslant i \leqslant q, \end{cases}$$

and if $s_2 = q$, then set $Q_i = q_i - l_i$, for $1 \leqslant i \leqslant s_2 = q$.

Thus, (11) can be written as:

$$\sum_{i=1}^{n} \frac{[\lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q]\,\alpha_i}{1 + \alpha_i r} - \sum_{i=1}^{q+s_4} \frac{Q_i \gamma_i}{1 + \gamma_i r} = \\ = O\left(r^{\mu_Q + \lambda_Q(N + n(m-1) - 1) + L - 1}\right) \text{ as } r \to 0,$$

where $0 \leqslant s_4 \leqslant l - 1$. Proceeding in the similar way as in Case 1, we deduce that $q \geqslant \mu_Q + \lambda_Q(m - 1)$.
Case 4. $S_1 \neq \varnothing$ and $S_2 \neq \varnothing$.
We may assume, without loss of generality, that $S_1 = \{\beta_1, \beta_2, \ldots, \beta_{s_1}\}$, $S_2 = \{\gamma_1, \gamma_2, \ldots, \gamma_{s_2}\}$. Then $\beta_i = \alpha_i$ for $1 \leqslant i \leqslant s_1$ and $\gamma_i = \beta_{s_1+i}$ for $1 \leqslant i \leqslant s_2$. Take $s_5 = l - s_2 - s_1$.
Subcase 4.1: $s_5 \geqslant 1$.
Let $\alpha_{n+i} = u_{s_1+s_2+i}$ for $1 \leqslant i \leqslant s_5$ and if $s_1 < n$, then set

$$N_i = \begin{cases} \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q + l_i, & \text{if } 1 \leqslant i \leqslant s_1, \\ \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q, & \text{if } s_1 + 1 \leqslant i \leqslant n, \\ l_{s_1+s_2-n+i}, & \text{if } n + 1 \leqslant i \leqslant n + s_5. \end{cases}$$

If $s_1 = n$, then set

$$N_i = \begin{cases} \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q + l_i, & \text{if } 1 \leqslant i \leqslant s_1, \\ l_{s_1+s_2-n+i}, & \text{if } n + 1 \leqslant i \leqslant n + s_5. \end{cases}$$

If $s_2 < q$, then set

$$Q_i = \begin{cases} q_i - l_{s_1+i}, & \text{if } 1 \leqslant i \leqslant s_2, \\ q_i, & \text{if } s_2 + 1 \leqslant i \leqslant q, \end{cases}$$

and if $s_2 = q$, then set $Q_i = q_i - l_{s_1+i}$, for $1 \leqslant i \leqslant s_2$.
Subcase 4.2: $s_5 = 0$.
If $s_1 < n$, then set

$$N_i = \begin{cases} \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q + l_i, & \text{if } 1 \leqslant i \leqslant s_1, \\ \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q, & \text{if } s_1 + 1 \leqslant i \leqslant n. \end{cases}$$

If $s_1 = n$, then set $N_i = \lambda_Q(n_i + m - 1) + \mu_Q - \lambda_Q + l_i$ for $1 \leqslant i \leqslant s_1$.

Also, if $s_2 < q$, then set

$$Q_i = \begin{cases} q_i - l_{s_1+i}, & \text{if } 1 \leqslant i \leqslant s_2, \\ q_i, & \text{if } s_2 + 1 \leqslant i \leqslant q, \end{cases}$$

and if $s_2 = q$, then set $Q_i = q_i - l_{s_1+i}$, for $1 \leqslant i \leqslant s_2$.

Thus, in both subcases, (11) can be written as

$$\sum_{i=1}^{n+s_5} \frac{N_i\alpha_i}{1+\alpha_i r} - \sum_{i=1}^{q} \frac{Q_i\gamma_i}{1+\gamma_i r} = O\left(r^{\mu_Q+\lambda_Q(N+n(m-1)-1)+L-1}\right) \text{ as } r \to 0,$$

where $0 \leqslant s_5 \leqslant l-2$. Proceeding in the similar fashion as in Case 1, we deduce that $q \geqslant \mu_Q + \lambda_Q(m-1)$. □

Lemma 3. *Let $\{f_j\} \subset \mathcal{M}(\mathbb{D})$ be a sequence of non-vanishing functions, all of whose poles have multiplicities at least m, $m \in \mathbb{N}$. Let $\{h_j\} \subset \mathcal{H}(\mathbb{D})$ be such that $h_j \to h$ locally uniformly in $\mathbb{D}$, where $h \in \mathcal{H}(\mathbb{D})$ and $h(z) \neq 0$ in $\mathbb{D}$. If, for each j, $Q[f_j] - h_j$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ zeros, ignoring multiplicities, in $\mathbb{D}$, then $\{f_j\}$ is normal in $\mathbb{D}$.*

Proof. Without loss generality, suppose that $\{f_j\}$ is not normal at $0 \in \mathbb{D}$. Then, by Lemma 1, there exists a sequence of points $\{z_j\} \subset \mathbb{D}$ with $z_j \to 0$, a sequence of positive real numbers satisfying $\rho_j \to 0$, and a subsequence of $\{f_j\}$, again denoted by $\{f_j\}$, such that the sequence

$$F_j(\zeta) := \frac{f_j(z_j + \rho_j\zeta)}{\rho_j^{(\mu_Q-\lambda_Q)/\lambda_Q}} \to F(\zeta),$$

spherically locally uniformly in $\mathbb{C}$, where $F \in \mathcal{M}(\mathbb{C})$ is a non-constant and non-vanishing function having poles of multiplicity at least m. Clearly, $Q[F_j] \to Q[F]$ spherically uniformly in every compact subset of $\mathbb{C}$ disjoint from poles of F. Also, one can easily see that $Q[F_j](\zeta) = Q[f_j](z_j + \rho_j\zeta)$. Thus, for every $\zeta \in \mathbb{C}\setminus\{F^{-1}(\infty)\}$,

$$Q[f_j](z_j+\rho_j\zeta) - h_j(z_j+\rho_j\zeta) = Q[F_j](\zeta) - h_j(z_j+\rho_j\zeta) \to Q[F](\zeta) - h(0)$$

spherically locally uniformly. Since F is non-constant and $x_0 > 0$, $x_i \geqslant y_i$ for all $i = 1, 2, \ldots, k$, by a result of Grahl [16, Theorem 7], it follows that $Q[F]$ is non-constant. Next, we claim that $Q[F] - h(0)$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ zeros in $\mathbb{C}$.

Suppose, on the contrary, that $Q[F]-h(0)$ has $\mu_Q+\lambda_Q(m-1)$ distinct zeros in $\mathbb{C}$, say ζ_i, $i=1,2,\dots,\mu_Q+\lambda_Q(m-1)$. Then by Hurwitz's theorem, there exit sequences $\zeta_{j,i}$, $i=1,\ 2,\ \dots,\ \mu_Q+\lambda_Q(m-1)$ with $\zeta_{j,i}\to\zeta_i$, such that for sufficiently large j, $Q[f_j](z_j+\rho_j\zeta_{j,i})-h_j(z_j+\rho_j\zeta_{j,i})=0$ for $i=1,\ 2,\ \dots,\ \mu_Q+\lambda_Q(m-1)$. However, $Q[f_j]-h_j$ has at most $\mu_Q+\lambda_Q(m-1)-1$ distinct zeros in $\mathbb{D}$. This proves the claim. Now, from Corollary 2, it follows that F must be a rational function which contradicts Proposition 2. □

Proposition 3. *Let t be a positive integer. Let $\{f_j\}\subset\mathcal{M}(\mathbb{D})$ be a sequence of non-vanishing functions, all of whose poles have multiplicities at least m, $m\in\mathbb{N}$, and let $\{h_j\}\subset\mathcal{H}(\mathbb{D})$ be such that $h_j\to h$ locally uniformly in $\mathbb{D}$, where $h\in\mathcal{H}(\mathbb{D})$ and $h(z)\neq 0$. If, for every j, $Q[f_j](z)-z^th_j(z)$ has at most $\mu_Q+\lambda_Q(m-1)-1$ zeros in $\mathbb{D}$, then $\{f_j\}$ is normal in $\mathbb{D}$.*

Proof. In view of Lemma 3, it suffices to prove that $\mathcal{F}$ is normal at $z=0$. Since $h(z)\neq 0$ in $\mathbb{D}$, it can be assumed that $h(0)=1$. Now, suppose, on the contrary, that $\{f_j\}$ is not normal at $z=0$. Then, by Lemma 1, there exists a subsequence of $\{f_j\}$, which, for simplicity, is again denoted by $\{f_j\}$, a sequence of points $\{z_j\}\subset\mathbb{D}$ with $z_j\to 0$, and a sequence of positive real numbers satisfying $\rho_j\to 0$, such that the sequence

$$F_j(\zeta):=\frac{f_j(z_j+\rho_j\zeta)}{\rho_j^{(t+\mu_Q-\lambda_Q)/\lambda_Q}}\to F(\zeta)$$

spherically locally uniformly in $\mathbb{C}$, where $F\in\mathcal{M}(\mathbb{C})$ is a non-constant function. Also, since each f_j is non-vanishing, it follows that F is non-vanishing. We now distinguish two cases.

Case 1: Suppose that there exists a subsequence of z_j/ρ_j, again denoted by z_j/ρ_j, such that $z_j/\rho_j\to\infty$ as $j\to\infty$.

Define

$$g_j(\zeta):=z_j^{-(t+\mu_Q-\lambda_Q)/\lambda_Q}f_j(z_j+z_j\zeta).$$

Then an elementary computation shows that

$$Q[g_j](\zeta)=z_j^{-t}Q[f_j](z_j+z_j\zeta),$$

and, hence,

$$Q[f_j](z_j+z_j\zeta)-(z_j+z_j\zeta)^th_j(z_j+z_j\zeta)=$$

$$= z_j^t Q[g_j](\zeta) - (z_j + z_j\zeta)^t h_j(z_j + z_j\zeta) = z_j^t \left[Q[g_j](\zeta) - 1 + \zeta)^t h_j(z_j + z_j\zeta) \right].$$

Since $(1+\zeta)^t h_j(z_j + z_j\zeta) \to (1+\zeta)^t \neq 0$ in $\mathbb{D}$, and $Q[f_j](z_j + z_j\zeta) - z_j^t(1+\zeta)^t h_j(z_j + z_j\zeta)$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ zeros in $\mathbb{D}$, by Lemma 3, it follows that $\{g_j\}$ is normal in $\mathbb{D}$ and, so, there exists a subsequence of $\{g_j\}$, again denoted by $\{g_j\}$, such that $g_j \to g$ spherically locally uniformly in $\mathbb{D}$, where $g \in \mathcal{M}(\mathbb{D})$ or $g \equiv \infty$. If $g \equiv \infty$, then

$$\begin{aligned} F_j(\zeta) &= \rho_j^{-(t+\mu_Q-\lambda_Q)/\lambda_Q} f_j(z_j + \rho_j\zeta) = \\ &= \left(\frac{z_j}{\rho_j}\right)^{(t+\mu_Q-\lambda_Q)/\lambda_Q} z_j^{-(t+\mu_Q-\lambda_Q)/\lambda_Q} f_j(z_j + \rho_j\zeta) = \\ &= \left(\frac{z_j}{\rho_j}\right)^{(t+\mu_Q-\lambda_Q)/\lambda_Q} g_j\left(\frac{\rho_j}{z_j}\zeta\right) \end{aligned}$$

converges spherically locally uniformly to ∞ in $\mathbb{C}$, showing that $F \equiv \infty$: a contradiction to the fact that F is non-constant. Since $g_j(\zeta) \neq 0$, it follows that either $g(\zeta) \neq 0$ or $g \equiv 0$. If $g(\zeta) \neq 0$, then, by the previous argument, we find that $F \equiv \infty$: a contradiction. If $g \equiv 0$, then choose $n \in \mathbb{N}$, such that $n + 1 > (t + \mu_Q - \lambda_Q)/(\lambda_Q)$. Thus, for each $\zeta \in \mathbb{C}$, we have

$$\begin{aligned} F_j^{(n+1)}(\zeta) &= \rho_j^{-(t+\mu_Q-\lambda_Q)/\lambda_Q+n+1} f_j^{(n+1)}(z_j + \rho_j\zeta) = \\ &= \left(\frac{\rho_j}{z_j}\right)^{-(t+\mu_Q-\lambda_Q)/\lambda_Q+n+1} g_j^{(n+1)}\Big)\frac{\rho_j}{z_j}\zeta\Big). \end{aligned}$$

Therefore, $F_j^{(n+1)}(\zeta) \to 0$ spherically uniformly, which implies that F is a polynomial of degree at most n: a contradiction to the fact that F is non-constant and non-vanishing meromorphic function.

Case 2: Suppose that there exists a subsequence of z_j/ρ_j, again denoted by z_j/ρ_j, such that $z_j/\rho_j \to \alpha$ as $j \to \infty$, where $\alpha \in \mathbb{C}$. Then

$$G_j(\zeta) = \rho_j^{-(t+\mu_Q-\lambda_Q)/\lambda_Q} f_j(\rho_j\zeta) = F_j\left(\zeta - \frac{z_j}{\rho_j}\right) \to F(\zeta - \alpha) := G(\zeta),$$

spherically locally uniformly in $\mathbb{C}$. Clearly, $G(\zeta) \neq 0$. Also, it is easy to see that $Q[G_j](\zeta) = \rho_j^{-t} Q[f_j](\rho_j\zeta)$. Thus,

$$Q[G_j](\zeta) - \zeta^t h_j(\rho_j\zeta) = \frac{Q[f_j](\rho_j\zeta) - (\rho_j\zeta)^t h_j(\rho_j\zeta)}{\rho_j^t} \to Q[G](\zeta) - \zeta^t$$

spherically uniformly in every compact subset of $\mathbb{C}$ disjoint from the poles of G. Clearly, $Q[G](\zeta) \not\equiv \zeta^t$, otherwise G has to be a polynomial, which is not possible since $G(\zeta) \neq 0$. Since $Q[f_j](\rho_j\zeta) - (\rho_j\zeta)^t h_j(\rho_j\zeta)$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ distinct zeros in $\mathbb{D}$, it follows that $Q[G](\zeta) - \zeta^m$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ distinct zeros in $\mathbb{C}$ and, hence, by Corollary 2, G must be a rational function. However, this contradicts Proposition 2. Hence $\mathcal{F}$ is normal in $\mathbb{D}$. □

4. Proof of Theorem 2. By virtue of Lemma 3, it is sufficient to prove the normality of $\{f_j\}$ at points $z \in \mathbb{D}$, where $h(z) = 0$. Without loss of generality, assume that $h(z) = z^t a(z)$, where $t \in \mathbb{N}$, $a \in \mathcal{H}(\mathbb{D})$, $a(z) \neq 0$ and $a(0) = 1$. Further, since $h_j \to h$ locally uniformly in $\mathbb{D}$, we can assume that

$$h_j(z) = (z - z_{j,1})^{t_1}(z - z_{j,2})^{t_2} \cdots (z - z_{j,l})^{t_l} a_j(z),$$

where $\sum_{i=1}^{l} t_i = t$, $z_{j,i} \to 0$ for $1 \leqslant i \leqslant l$ and $a_j(z) \to a(z)$ locally uniformly in $\mathbb{D}$. Again, we may assume that $z_{j,1} = 0$, since $\{f_j(z)\}$ is normal in $\mathbb{D}$ if and only if $\{f_j(z + z_{j,1})\}$ is normal in $\mathbb{D}$ (see [25, p. 35]). Now, let us prove the normality of $\{f_j\}$ at $z = 0$ by applying the principle of mathematical induction on t.

Note that if $t = 1$, then $l = 1$ and, so, $h_j(z) = z a_j(z)$. Thus, by Proposition 3, $\{f_j\}$ is normal at $z = 0$. Also, if $l = 1$, then $h_j(z) = z^t a_j(z)$, and, again by Proposition 3, $\{f_j\}$ is normal at $z = 0$. So, let $l \geqslant 2$ and for $n \in \mathbb{N}$ with $1 < t < n$, suppose that $\{f_j\}$ is normal at $z = 0$. In accordance with the principle of mathematical induction, we only need to show that $\{f_j\}$ is normal at $z = 0$ when $n = t$.

Rearranging the zeros of h_j, if necessary, we can assume that $|z_{j,i}| \leqslant |z_{j,l}|$ for $2 \leqslant i \leqslant l$. Let $z_{j,l} = w_j$. Then $w_j \to 0$. Define

$$g_j(z) := \frac{f_j(w_j z)}{w_j^{(t+\mu_Q-\lambda_Q)/\lambda_Q}} \text{ and } v_j(z) := \frac{h_j(w_j z)}{w_j^t},\ z \in D_{r_j}(0),\ r_j \to \infty.$$

Then an easy computation shows that $Q[g_j](z) = w_j^{-t} Q[f_j](w_j z)$ and

$$v_j(z) = z^{t_1}\left(z - \frac{z_{j,2}}{w_j}\right)^{t_2} \cdots \left(z - \frac{z_{j,l-1}}{w_j}\right)^{t_{l-1}} (z-1)^{t_l} a_j(w_j z) \to v(z)$$

locally uniformly in $\mathbb{C}$. Clearly, 0 and 1 are two distinct zeros of v and, hence, all zeros of v have multiplicities at most $t-1$. Since

$$Q[g_j](z) - v_j(z) = \frac{Q[f_j](w_j z) - h_j(w_j z)}{w_j^t} \tag{15}$$

and $Q[f_j](w_j z) - h_j(w_j z)$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ distinct zeros, it follows that $Q[g_j](z) - v_j(z)$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ distinct zeros in $\mathbb{C}$. Thus, by induction hypothesis, we find that $\{g_j\}$ is normal in $\mathbb{C}$. Suppose that $g_j \to g$ spherically locally uniformly in $\mathbb{C}$. Then either $g \in \mathcal{M}(\mathbb{C})$ or $g \equiv \infty$.

Case 1: $g \in \mathcal{M}(\mathbb{C})$.

Since $g_j(z) \neq 0$, it follows that either $g(z) \neq 0$ or $g \equiv 0$. First, suppose that $g(z) \neq 0$. Since $g_j \to g$ spherically locally uniformly in $\mathbb{C}$, it follows that $Q[g_j] \to Q[g]$ in every compact subset of $\mathbb{C}$ disjoint from the poles of g. Then, from (15), we find that $Q[g] - v$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ distinct zeros in $\mathbb{C}$ and, thus, by Corollary 2 and Proposition 2, g has to be a constant.

Next, we claim that $\{f_j\}$ is holomorphic in $D_{\delta/2}(0)$ for some $\delta \in (0,1)$. Suppose, on the contrary, that $\{f_j\}$ is not holomorphic in $D_{\delta/2}(0)$ for any $\delta \in (0,1)$. Then there exists a sequence $\eta_j \in D_{\delta/2}(0)$, such that $\eta_j \to 0$ and $f_j(\eta_j) = \infty$. Assume that η_j has the smallest modulus among the poles of f_j. It is easy to see that $\eta_j / w_j \to \infty$, otherwise

$$f_j(\eta_j) = w_j^{(t+\mu_Q-\lambda_Q)/\lambda_Q} g_j(\eta_j/w_j) \to 0, \text{ a contradiction.}$$

Let

$$\psi_j(z) := \frac{f_j(\eta_j z)}{\eta_j^{(t+\mu_Q-\lambda_Q)/\lambda_Q}} \text{ and } u_j(z) := \frac{h_j(\eta_j z)}{\eta_j^t},\ z \in D_{r_j}(0),\ r_j \to \infty.$$

Then

$$Q[\psi_j](z) - u_j(z) = \frac{Q[f_j](\eta_j z) - h_j(\eta_j z)}{\eta_j^t} \tag{16}$$

and

$$u_j(z) = z^{t_1}\left(z - \frac{z_{j,2}}{\eta_j}\right)^{t_2} \cdots \left(z - \frac{w_j}{\eta_j}\right)^{t_l} a_j(\eta_j z) \to z^t$$

locally uniformly in $\mathbb{C}$. From Lemma 3, it follows that $\{\psi_j\}$ is normal in $\mathbb{C} \setminus \{0\}$. Since $\psi_j(z) \neq 0$ and ψ_j is holomorphic in $\mathbb{D}$, one can easily see that $\{\psi_j\}$ is normal in $\mathbb{D}$ and, hence, in $\mathbb{C}$. Assume that $\psi_j \to \psi$ spherically locally uniformly in $\mathbb{C}$, where $\psi \in \mathcal{M}(\mathbb{C})$ or $\psi \equiv \infty$. Since

$$\psi_j(0) = \frac{f_j(0)}{\eta_j^{(t+\mu_Q-\lambda_Q)/\lambda_Q}} = \left(\frac{w_j}{\eta_j}\right)^{(t+\mu_Q-\lambda_Q)/\lambda_Q} g_j(0) \to 0,$$

therefore, $\psi \not\equiv \infty$. Also, since $\psi_j(z) \neq 0$, we have $\psi(z) \neq 0$ or $\psi \equiv 0$. However, the latter is not possible since $\infty = \psi_j(1) \to \psi(1) = \infty$. Thus, $\psi(z) \neq 0$. Note that $Q[\psi_j](z) - u_j(z) \to Q[\psi](z) - z^t$ spherically uniformly in every compact subset of $\mathbb{C}$ disjoint from the poles of ψ, so, by (16), we conclude that $Q[\psi](z) - z^t$ has at most $\mu_Q + \lambda_Q(m-1) - 1$ distinct zeros in $\mathbb{C}$. By Corollary 2 and Proposition 2, ψ reduces to a constant, which contradicts the fact that $\infty = \psi_j(1) \to \psi(1) = \infty$. Hence, $\{f_j\}$ is holomorphic in $D_{\delta/2}(0)$. Since $f_j(z) \neq 0$, it follows that $\{f_j\}$ is normal at $z = 0$.

Next, suppose that $g \equiv 0$. Then, by the preceding discussion, one can easily see that $\{f_j\}$ is holomorphic in $D_{\delta/2}(0)$ and, hence, $\{f_j\}$ is normal at $z = 0$.

Case 2: $g \equiv \infty$.

Let $\phi_j(z) := f_j(z)/z^{(t+\mu_Q-\lambda_Q)/\lambda_Q}$. Then $1/\phi_j(0) = 0$.

Subcase 2.1: When $\{1/\phi_j\}$ is normal at $z = 0$.

Then $\{\phi_j\}$ is normal at $z = 0$ and, so, there exists $r > 0$ with $D_r(0) \subseteq \mathbb{D}$, such that $\{\phi_j\}$ is normal in $D_r(0)$. Assume that $\phi_j \to \phi$ spherically locally uniformly. Since $\phi_j(0) = \infty$, there exists $\rho > 0$, such that, for sufficiently large j, $|\phi_j(z)| \geqslant 1$ in $D_\rho(0) \subset D_r(0)$. Also, since $f_j(z) \neq 0$ in $D_\rho(0)$, $1/f_j$ is holomorphic in $D_\rho(0)$ and, hence,

$$\left|\frac{1}{f_j(z)}\right| = \left|\frac{1}{\phi_j(z)} \cdot \frac{1}{z^{(t+\mu_Q-\lambda_Q)/\lambda_Q}}\right| \leqslant \left(\frac{2}{\rho}\right)^{(t+\mu_Q-\lambda_Q)/\lambda_Q} \text{ in } \partial D_{\rho/2}(0).$$

Then, by the maximum principle and Montel's theorem [25, p.35], we conclude that $\{f_j\}$ is normal at $z = 0$.

Subcase 2.2: When $\{1/\phi_j\}$ is not normal at $z = 0$.

By Montel's theorem, it follows that, for every $\epsilon > 0$, $\{1/\phi_j(z)\}$ is not locally uniformly bounded in $D_\epsilon(0)$. Therefore, we can find a sequence $\epsilon_j \to 0$, such that $1/\phi_j(\epsilon_j) \to \infty$. Since $|1/\phi_j|$ is continuous, there exists $b_j \to$, such that $|1/\phi_j(b_j)| = 1$.

Let

$$K_j(z) := \frac{f_j(b_j z)}{b_j^{(t+\mu_Q-\lambda_Q)/\lambda_Q}},\ z \in D_{r_j}(0),\ r_j \to \infty \text{ and } q_j(z) := \frac{h_j(b_j z)}{b_j^t}.$$

Then $K_j(z) \neq 0$ and a simple computation shows that

$$Q[K_j](z) = \frac{Q[f_j](b_j z)}{b_j^t} \text{ and } q_j(z) = z^{t_1}\left(z - \frac{z_{j,2}}{b_j}\right)^{t_2} \cdots \left(z - \frac{w_j}{b_j}\right)^{t_l} a_j(b_j z).$$

Note that

$$g_j\left(\frac{b_j}{w_j}\right)=\frac{f_j(b_j)}{w_j^{(t+\mu_Q-\lambda_Q)/\lambda_Q}}=\frac{f_j(b_j)}{b_j^{(t+\mu_Q-\lambda_Q)/\lambda_Q}}\cdot\left(\frac{b_j}{w_j}\right)^{(t+\mu_Q-\lambda_Q)/\lambda_Q}\to\infty.$$

Since $|1/\phi_j(b_j)|=1$ and $(t+\mu_Q-\lambda_Q)/\lambda_Q>0$, it follows that $b_j/w_j\to\infty$, and, hence, $w_j/b_j\to 0$. This implies that $q_j(z)\to z^t$ locally uniformly in $\mathbb{C}$. Further, since $Q[K_j](z)-q_j(z)=(Q[f_j](b_jz)-h_j(b_jz))/b_j^t$, it follows that $Q[K_j]-q_j$ has at most $\mu_Q+\lambda_Q(m-1)-1$ distinct zeros in $\mathbb{C}$ and, hence, by Lemma 3, $\{K_j\}$ is normal in $\mathbb{C}\setminus\{0\}$. We claim that $\{K_j\}$ is normal in $\mathbb{C}$. Suppose otherwise. Then, by Lemma 2, there is a subsequence of $\{K_j\}$, which for the sake of convenience, is again denoted by $\{K_j\}$, such that $K_j(z)\to 0$ in $\mathbb{C}\setminus\{0\}$, which is not possible since $|K_j(1)|=1$. This establishes the claim. Now, suppose that $K_j\to K$ spherically locally uniformly in $\mathbb{C}$. It is evident that $K(z)\neq 0$ in $\mathbb{C}$ and $K\not\equiv\infty$, as $K(1)=1$. Then $Q[K_j]\to Q[K]$ spherically uniformly in every compact subset of $\mathbb{C}$ disjoint from the poles of K. Since $Q[K_j]-q_j$ has at most $\mu_Q+\lambda_Q(m-1)-1$ distinct zeros in $\mathbb{C}$, it follows that $Q[K]-z^t$ has at most $\mu_Q+\lambda_Q(m-1)-1$ distinct zeros in $\mathbb{C}$, and, so, by Corollary 2 and Proposition 2, K reduces to a constant. Using the same arguments as in Case 1, we find that $\{f_j\}$ is normal at $z=0$. This completes the induction process and, hence, the proof. □

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