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A NEW CHARACTERIZATION OF q-CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

Abstract. In this work, we introduce the notion of $\mathcal{U}_{(q,\mu)}$-classical orthogonal polynomials, where $\mathcal{U}_{(q,\mu)}$ is the degree raising shift operator defined by $\mathcal{U}_{(q,\mu)} := x(xH_q + q^{-1}I_{\mathcal{P}}) + \mu H_q$, where μ is a nonzero free parameter, $I_{\mathcal{P}}$ represents the identity operator on the space of polynomials $\mathcal{P}$, and H_q is the q-derivative one. We show that the scaled q-Chebychev polynomials of the second kind $\hat{U}_n(x,q), n \geqslant 0$, are the only $\mathcal{U}_{(q,\mu)}$-classical orthogonal polynomials.

Key words: *orthogonal q-polynomials, q-derivative operator, q-Chebyshev polynomials, raising operator*

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1. Introduction. Chebyshev polynomials and their q-analogues are used in many fields in the mathematics as well as in the physical sciences. Note that several contributions have been devoted to the q-extension of the Chebyshev polynomials and their properties [1], [8], [12], [18]. Our objective in this paper is to characterize the scaled q-Chebyshev polynomials of the second kind [18] via a raising operator.

Let $\mathcal{O}$ be a linear operator, acting on the space of polynomials, that sends polynomials of degree n to polynomials of degree $n + n_0$, where n_0 is a fixed integer ($n \geqslant 0$ if $n_0 \geqslant 0$ and $n \geqslant n_0$ if $n_0 < 0$). We call a sequence $\{P_n\}_{n\geqslant 0}$ of orthogonal polynomials $\mathcal{O}$-classical if $\{\mathcal{O}P_n\}_{n\geqslant 0}$ is also orthogonal. An orthogonal polynomial sequence $\{P_n\}_{n\geqslant 0}$ is called classical if $\{P'_n\}_{n\geqslant 0}$ is also orthogonal. This is the Hahn property (see [10]) for the classical orthogonal polynomials. In [11], Hahn gave similar characterization theorems for orthogonal polynomials P_n, such that the polynomials

$D_\omega P_n$ or $H_q P_n (n \geqslant 1)$ are again orthogonal; here D_ω is the divided difference operator and H_q is the q-derivative operator given, respectively, by $D_\omega f(x) = \frac{f(x+\omega)-f(x)}{\omega}$, $\omega \neq 0$ and $H_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}$, $q \neq 1$.

In this paper, we consider the raising operator

$$\mathcal{U}_{(q,\mu)} := x(xH_q + q^{-1} I_{\mathcal{P}}) + \mu H_q,$$

where μ is a nonzero free parameter and $I_{\mathcal{P}}$ represents the identity operator. We show that the scaled q-Chebyshev polynomial sequence of the second kind [18], $\left\{b^{-n}\hat{U}_n(bx)\right\}_{n\geqslant 0}$, where $b^2 = -(q\mu)^{-1}$, is the only $\mathcal{U}_{(q,\mu)}$-classical orthogonal polynomial sequence.

Several authors have been interested in the study of the orthogonal polynomials using the lowering, transfer, and raising operators [2], [3] [5], [4], [6], [14], [17].

The structure of the paper is as follows. In Section 2 , we give some useful results . In Section 3, we solve the problem. In Section 4, a property of the scaled q-Chebychev polynomials of the second kind is given.

2. Preliminaries. We denote by $\mathcal{P}$ the vector space of the polynomials with coefficients in $\mathbb{C}$ and by $\mathcal{P}'$ its dual space. The action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ is denoted as $\langle u, f\rangle$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \geqslant 0$, the moments of u. For instance, for any form u, any polynomial g, and any $(a,c) \in (\mathbb{C}\backslash\{0\}) \times \mathbb{C}$, we let $H_q u$, gu, $h_a u$, Du, $(x-c)^{-1}u$, and δ_c be the forms defined as usually ([15] and [13]) for the images related to the operator H_q

$$\langle H_q u, f\rangle := -\langle u, H_q f\rangle, \ \langle gu, f\rangle := \langle u, gf\rangle, \ \langle h_a u, f\rangle := \langle u, h_a f\rangle,$$

$$\langle Du, f\rangle := -\langle u, f'\rangle, \quad \langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle, \quad \langle \delta_c, f\rangle := f(c),$$

where for all $f \in \mathcal{P}$ and $q \in \widetilde{\mathbb{C}} := \left\{z \in \mathbb{C},\ z \neq 0, z^n \neq 1, n \geqslant 1\right\}$, [13]

$$\begin{cases} H_q(f)(x) = \frac{f(qx)-f(x)}{(q-1)x}, x \neq 0, \\ H_q(f)(0) = f'(0), \end{cases}$$

$$(h_a f)(x) = f(ax)\,, (\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}.$$

In particular, this yields

$$(H_q u)_n = -[n]_q (u)_{n-1}\,, n \geqslant 0\,,$$

where $(u)_{-1} = 0$ and

$$[n]_q := \frac{q^n - 1}{q - 1}, \; n \geqslant 0.$$

Let $\{P_n\}_{n\geqslant 0}$ be a sequence of monic polynomials with $\deg P_n = n, \; n \geqslant 0$, (MPS for short) and let $\{u_n\}_{n\geqslant 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by $\langle u_n, P_m \rangle := \delta_{n,m}, \; n, m \geqslant 0$ [7], [15]. The form u is called regular if we can associate with it a MPS $\{P_n\}_{n\geqslant 0}$, such that ([7], [15]) $\langle u, P_n P_m \rangle = r_n \delta_{n,m}, \; n, m \geqslant 0; \; r_n \neq 0, \; n \geqslant 0$. The sequence $\{P_n\}_{n\geqslant 0}$ is then said to be orthogonal with respect to u (MOPS for short) and is characterized by the following three-term recurrence relation (Favard's theorem) (TTRR for short) [7]:

$$\begin{aligned} &P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ &P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geqslant 0, \end{aligned} \tag{1}$$

where $\beta_n = \dfrac{\langle u, xP_n^2 \rangle}{\langle u, P_n^2 \rangle} \in \mathbb{C}, \; \gamma_{n+1} = \dfrac{\langle u, P_{n+1}^2 \rangle}{\langle u, P_n^2 \rangle} \in \mathbb{C}\backslash\{0\}, \; n \geqslant 0.$

The shifted MOPS $\left\{\widehat{P}_n := a^{-n}\big(h_a P_n\big)\right\}_{n\geqslant 0}$ is then orthogonal with respect to $\widehat{u} = h_{a^{-1}} u$ and satisfies (1) with [15]

$$\widehat{\beta}_n = \frac{\beta_n}{a}, \qquad \widehat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2}, \qquad n \geqslant 0.$$

Moreover, the form u is said to be normalized if $(u)_0 = 1$. In this paper, we suppose that any regular form are normalized. In addition, $\{P_n\}_{n\geqslant 0}$ is a symmetric MOPS if and only if $\beta_n = 0, \; n \geqslant 0$ or, equivalently, $(u)_{2n+1} = 0, \; n \geqslant 0$ [7], [15]. When u is regular, let Φ be a polynomial, such that $\Phi u = 0$, then $\Phi = 0$ [15].

Lemma 1. *[15], [17] Let $\{P_n\}_{n\geqslant 0}$ be a MPS and let $\{u_n\}_{n\geqslant 0}$ be its dual sequence. For any $u \in \mathcal{P}'$ and any integer $m \geqslant 1$, the following statements are equivalent:*

(i) $\langle u, P_{m-1} \rangle \neq 0, \; \langle u, P_n \rangle = 0, \; n \geqslant m$;

(ii) $\exists \lambda_v \in \mathbb{C}, \; 0 \leqslant v \leqslant m-1, \; \lambda_{m-1} \neq 0$ *such that* $u = \sum_{v=0}^{m-1} \lambda_v u_v$.

As a consequence, when the MPS $\{P_n\}_{n\geqslant 0}$ is orthogonal with respect to u, necessarily, $u = u_0$.

Proposition 1. *[15] Let $\{P_n\}_{n\geqslant 0}$ be a MPS with $\deg P_n = n, n \geqslant 0$, and let $\{u_n\}_{n\geqslant 0}$ be its dual sequence. The following statements are equivalent:*

(i) $\{P_n\}_{n\geqslant 0}$ *is orthogonal with respect to* u_0;

(ii) $u_n = \langle u_0, P_n^2\rangle^{-1} P_n u_0,\ n \geqslant 0$;

(iii) $\{P_n\}_{n\geqslant 0}$ *satisfies the three-term recurrence relation* (1).

Let us recall some results in the field of q-theory.

Lemma 2. *[9], [13]*

$$H_q(fg)(x) = (h_q f)(x)(H_q g)(x) + g(x)(H_q f)(x),\ f, g \in \mathcal{P}, \tag{2}$$

$$H_q(fu)(x) = f H_q u + (H_{q^{-1}} f) h_q u,\ f \in \mathcal{P},\ u \in \mathcal{P}', \tag{3}$$

$$h_a(fg)(x) = (h_a f)(x)(h_a g)(x),\ f, g \in \mathcal{P},\ a \in \mathbb{C} - \{0\}, \tag{4}$$

$$h_a(gu) = (h_{a^{-1}} g)(h_a u),\ g \in \mathcal{P}, u \in \mathcal{P}',\ a \in \mathbb{C} - \{0\}, \tag{5}$$

$$H_q \circ h_a = a h_a \circ H_q \quad \textit{in } \mathcal{P}, \tag{6}$$

$$h_{q^{-1}} \circ H_q = H_{q^{-1}} \quad \textit{in } \mathcal{P}. \tag{7}$$

Now, consider a MPS $\{P_n\}_{n\geqslant 0}$ as above and let [13]

$$P_n^{[1]}(x;q) := \frac{1}{[n+1]_q}(H_q P_{n+1})(x), \quad n \geqslant 0.$$

Denote by $\{u_n^{[1]}(q)\}_{n\geqslant 0}$ the dual sequence of $\{P_n^{[1]}(\cdot;q)\}_{n\geqslant 0}$. The following equality holds [13]:

$$H_q\left(u_n^{[1]}(q)\right) = -[n+1]_q u_{n+1}, \quad n \geqslant 0.$$

Definition 1. *[13] The form* u_0 *is said to be* H_q*-classical if it is regular and there exist two polynomials,* Φ *monic,* $\deg\Phi \leqslant 2$*, and* Ψ*,* $\deg\Psi = 1$*, such as:*

$$H_q(\Phi(x)u_0) + \Psi(x)u_0 = 0,$$

where the pair (Φ,Ψ) *is admissible, i.e.,* $\Psi'(0) - \frac{1}{2}\Phi''(0)[n]_q \neq 0,\ n \geqslant 1$. *The corresponding MOPS* $\{P_n\}_{n\geqslant 0}$ *is said to be* H_q*-classical.*

Lemma 3. *[13] When* u_0 *satisfies the equation* $H_q(\Phi u_0) + \Psi u_0 = 0$*, then* $\widehat{u}_0 = h_{a^{-1}} u_0$ *fulfils the equation*

$$H_q\left(\widehat{\Phi}\widehat{u}_0\right) + \widehat{\Psi}\widehat{u}_0 = 0,$$

where $\widehat{\Phi}(x) = a^{-\deg\Phi}\Phi(ax)$, $\widehat{\Psi}(x) = a^{1-\deg\Phi}\Psi(ax)$.

Proposition 2. *[13] For any orthogonal sequence* $\{P_n\}_{n\geqslant 0}$*, the successive assertions are equivalent:*

(i) *The sequence* $\{P_n\}_{n\geqslant 0}$ *is* H_q*-classical.*

(ii) *The sequence* $\left\{P_n^{[1]}\right\}_{n\geqslant 0}$ *is orthogonal.*

(iii) *There exist two polynomials,* Φ *monic,* $\deg\Phi \leqslant 2$, Ψ, $\deg\Psi = 1$, *and a sequence* $\{\lambda_n\}_{n\geqslant 0}$, $\lambda_n \neq 0$, $n \geqslant 0$, *such that*

$$\Phi(x)(H_q\circ H_{q^{-1}}P_{n+1})(x)-\Psi(x)(H_{q^{-1}}P_{n+1})(x)+\lambda_n P_{n+1}(x) = 0, n \geqslant 0. \tag{8}$$

Let us recall the q-Chebyshev MOPS of the first kind: $\{\hat{T}_n(.,q)\}_{n\geqslant 0}$ orthogonal with respect to $\mathcal{T}_q$ and the q-Chebyshev MOPS of the second kind $\{\hat{U}_n(.,q)\}_{n\geqslant 0}$ orthogonal with respect to $\mathcal{U}_q$. We have [18]:

$$\begin{cases} \gamma_1^{\mathcal{T}_q} = \frac{q}{q+1}, \gamma_{n+1}^{\mathcal{T}_q} = \frac{q^{n+1}}{(q^n+1)(q^{n+1}+1)}, & n \geqslant 1, \\ H_q\left((x^2-1)\,\mathcal{T}_q\right) - q^{-1}x\mathcal{T}_q = 0, \end{cases} \tag{9}$$

and

$$\begin{cases} \gamma_{n+1}^{\mathcal{U}_q} = \frac{q^{n+2}}{(q^{n+1}+1)(q^{n+2}+1)}, & n \geqslant 0, \\ H_q\left((x^2-q^{-1})\,\mathcal{U}_q\right) + \frac{1-q^{-3}}{1-q}x\mathcal{U}_q = 0. \end{cases} \tag{10}$$

Denote by $\{\tilde{U}_n(.,q)\}_{n\geqslant 0}$ the MOPS with respect to $\tilde{\mathcal{U}}_q := h_{q^{-\frac{1}{2}}}\mathcal{U}_q$. We have [18]:

$$\tilde{U}_n(x,q) = q^{-\frac{n}{2}}\hat{U}_n\left(q^{\frac{1}{2}}x, q\right), \quad n \geqslant 0, \tag{11}$$

$$\left(x^2-1\right)\mathcal{T}_q = -\frac{1}{q+1}h_{q^{-\frac{1}{2}}}\mathcal{U}_q = -\frac{1}{q+1}\tilde{\mathcal{U}}_q, \tag{12}$$

and

$$H_q\left(\hat{T}_{n+1}(x,q)\right) = \frac{q^{n+1}-1}{q-1}\tilde{U}_n(x,q), \quad n \geqslant 0. \tag{13}$$

Finally, denote by $\{\hat{T}_n\}_{n\geqslant 0}$, $\{\hat{U}_n\}_{n\geqslant 0}$ and $\{\tilde{U}_n\}_{n\geqslant 0}$ respectively, the sequences $\{\hat{T}_n(.,q)\}_{n\geqslant 0}$, $\{\hat{U}_n(.,q)\}_{n\geqslant 0}$ and $\{\tilde{U}_n(.,q)\}_{n\geqslant 0}$.

3. Main results. Let us introduce the operator

$$\begin{aligned} \mathcal{U}_{(q,\mu)} \colon \mathcal{P} &\longrightarrow \mathcal{P} \\ f &\longmapsto \mathcal{U}_{(q,\mu)}(f) = \left(x^2+\mu\right)H_q(f) + q^{-1}xf. \end{aligned} \tag{14}$$

Definition 2. *The MOPS* $\{P_n\}_{n\geqslant 0}$ *is said to be* $\mathcal{U}_{(q,\mu)}$*-classical if* $\{\mathcal{U}_{(q,\mu)}P_n\}_{n\geqslant 0}$ *is also orthogonal.*

For any MPS $\{P_n\}_{n\geqslant 0}$, the MPS $\{Q_n\}_{n\geqslant 0}$ is defined by

$$Q_{n+1}(x) := \frac{\mathcal{U}_{(q,\mu)}P_n}{q^{-1}[n+1]_q}, n \geqslant 0, \tag{15}$$

or, equivalently,

$$q^{-1}[n+1]_q Q_{n+1}(x) := \left(x^2+\mu\right) H_q\left(P_n\right)(x) + q^{-1}xP_n(x), n \geqslant 0, \tag{16}$$

with $Q_0(x) = 1$.

It is clear that the operator $\mathcal{U}_{(q,\mu)}$ raises the degree of any polynomial. Such operator is called raising operator [14]. By transposition of the operator $\mathcal{U}_{(q,\mu)}$, we have:

$$^t\mathcal{U}_{(q,\mu)} = -\mathcal{U}_{(q,\mu)}. \tag{17}$$

Denote by $\{u_n\}_{n\geqslant 0}$ and $\{v_n\}_{n\geqslant 0}$ the dual basis in $\mathcal{P}'$ corresponding to $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, respectively. Then, according to Lemma 1 and (17), we get the relation

$$\left(x^2+\mu\right) H_q\left(v_{n+1}\right) + q^{-1}xv_{n+1} = -q^{-1}[n+1]_q\, u_n, \quad n \geqslant 0. \tag{18}$$

Assume that $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ are MOPSs satisfying

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\ P_{n+2}(x) = \left(x-\beta_{n+1}\right)P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad \gamma_{n+1} \neq 0, n \geqslant 0, \end{cases} \tag{19}$$

$$\begin{cases} Q_0(x) = 1, \quad Q_1(x) = x - \chi_0, \\ Q_{n+2}(x) = \left(x-\chi_{n+1}\right)Q_{n+1}(x) - \theta_{n+1}Q_n(x), \quad \theta_{n+1} \neq 0, n \geqslant 0. \end{cases} \tag{20}$$

Our goal is to describe all the $\mathcal{U}_{(q,\mu)}$-classical orthogonal polynomial sequences. Note that it is necessary that $\mu \neq 0$ to ensure the orthogonality of the sequence $\{Q_n\}_{n\geqslant 0}$. In fact, if we suppose that $\mu = 0$, the relation (16) becomes, for $x = 0$, $Q_{n+1}(0) = 0$, $n \geqslant 0$, which contradicts the orthogonality of $\{Q_n\}_{n\geqslant 0}$. Indeed, from (20) we have $Q_1(x) = x$ and $Q_2(x) = (x-\chi_1)x - \theta_1$. For $x = 0$, we obtain $\theta_1 = 0$, which is impossible.

We are going to establish the connection between the two sequences $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$.

Proposition 3. *The sequences $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ satisfy the following relation:*

$$\left(x^2+\mu\right) h_q P_n(x) = q^n Q_{n+2}(x) + \lambda_n Q_{n+1}(x) + \sigma_n Q_n(x), \, n \geqslant 0,$$

where

$$\lambda_n = q^{-1}[n+1]_q \left(\beta_n - \chi_{n+1}\right), \, n \geqslant 0,$$
$$\sigma_n = q^{-1} \left([n]_q \gamma_n - [n+1]_q \theta_{n+1}\right), \, n \geqslant 0,$$

with $\gamma_0 := 0$.

Proof. By applying the operator H_q to (19) and using (2), we get

$$H_q(P_{n+2})(x) = (qx - \beta_{n+1}) H_q(P_{n+1})(x) - \gamma_{n+1} H_q(P_n)(x) + P_{n+1}(x), \, n \geqslant 0. \tag{21}$$

Multiply equation (21) by $x^2 + \mu$ and relation (20) by x. Then take the sum of these two resulting equations. Next, substituting (16), get

$$\begin{aligned} q^{-1}[n+3]_q \, Q_{n+3}(x) &= \\ = q^{-1}[n+2]_q \, (x - \beta_{n+1}) \, Q_{n+2}(x) &- q^{-1}[n+1]_q \, \gamma_{n+1} Q_{n+1}(x) + \\ &+ \left(x^2 + \mu\right) h_q P_{n+1}(x), \, n \geqslant 0. \end{aligned}$$

On account of the recurrence relation (20), we get

$$\begin{aligned} \left(x^2 + \mu\right) h_q P_{n+1}(x) &= \\ = q^{n+1} Q_{n+3}(x) &+ q^{-1}[n+2]_q \left(\beta_{n+1} - \chi_{n+2}\right) Q_{n+2}(x) + \\ &+ q^{-1} \left([n+1]_q \gamma_{n+1} - [n+2]_q \, \theta_{n+2}\right) Q_{n+1}(x), \; n \geqslant 0. \end{aligned}$$

Now, replacing $n+1$ by n, we have for all $n \geqslant 1$:

$$\begin{aligned} \left(x^2 + \mu\right) h_q P_n(x) = q^n Q_{n+2}(x) &+ q^{-1}[n+1]_q \left(\beta_n - \chi_{n+1}\right) Q_{n+1}(x) + \\ &+ q^{-1} \left([n]_q \gamma_n - [n+1]_q \, \theta_{n+1}\right) Q_n(x), \end{aligned}$$

with the constraint $\gamma_0 := 0$.

For $n = 0$, the Proposition 3 gives

$$Q_2(x) + q^{-1} \left(\beta_0 - \chi_1\right) Q_1(x) = x^2 + \mu + q^{-1}\theta_1,$$

and using the fact that $Q_1(x) = x$, we obtain

$$Q_2(x) = x^2 - \frac{\beta_0}{q+1} x + \frac{q\mu}{q+1}. \tag{22}$$

By comparing (20) and (22) for $n = 0$, we obtain $\chi_1 = \frac{\beta_0}{q+1}$ and $\theta_1 = -\frac{q\mu}{q+1}$. □

In the following lemma, we establish an algebraic relation between the forms u_0 and v_0.

Lemma 4. *The forms u_0 and v_0 satisfy the relation*

$$h_q\left(\left(x^2+\mu\right)v_0\right)=\frac{\mu}{q+1}\,u_0.$$

Proof. By virtue of Proposition 3, we get

$$\left\langle\left(x^2+\mu\right)v_0, h_qP_n\right\rangle=0,\ n\geqslant 1. \tag{23}$$

Moreover, by (22) we have $x^2+\mu=Q_2+\frac{\beta_0}{q+1}Q_1+\frac{\mu}{q+1}$. Since $\{Q_n\}_{n\geqslant 0}$ is orthogonal with respect to the form v_0, and v_0 is supposed to be normalized, we obtain:

$$\left\langle\left(x^2+\mu\right)v_0, P_0\right\rangle=\left\langle v_0, Q_2+\frac{\beta_0}{q+1}Q_1\right\rangle+\frac{\mu}{q+1}=\frac{\mu}{q+1}. \tag{24}$$

On account of Lemma 1, (23), and (24), the desired result holds. □

Using the last lemma, we are going to establish a first-order q-difference equation satisfied by $\{Q_n\}_{n\geqslant 0}$.

Proposition 4. *The following relation holds:*

$$H_q\left(Q_{n+1}\right)(x)=[n+1]_q\,q^{-n}(h_qP_n)(x),\ n\geqslant 0. \tag{25}$$

Proof. Based on Proposition 1, we may write the relation (18) as

$$(x^2+\mu)H_{q^{-1}}(Q_{n+1})(x)\,h_qv_0+q^{-1}xQ_{n+1}(x)\,v_0+(x^2+\mu)Q_{n+1}(x)H_q(v_0)=$$
$$=\lambda_nP_n(x)u_0,\quad n\geqslant 0, \tag{26}$$

where $\lambda_n:=-q^{-1}[n+1]_q\langle v_0,Q_{n+1}^2\rangle\langle u_0,P_n^2\rangle^{-1}$, $n\geqslant 0$.
Making $n=0$ in (26) and using (3), we get:

$$\left(x^2+\mu\right)xH_q\left(v_0\right)=-\left(x^2+\mu\right)h_qv_0-q^{-1}x^2v_0+\lambda_0u_0.$$

Substituting this relation in (26), for $n\geqslant 0$ we obtain:

$$(xH_{q^{-1}}(Q_{n+1})(x)-Q_{n+1}(x))(x^2+\mu)h_qv_0=(\lambda_nxP_n(x)-\lambda_0Q_{n+1}(x))u_0.$$

By virtue of Lemma 4, the fact that $\lambda_0=-\theta_1=\frac{\mu}{q+1}$, and taking into account the regularity of u_0, we finally get

$$\mu\,H_{q^{-1}}\left(Q_{n+1}\right)(x)=(q+1)\lambda_nP_n(x),\ \ n\geqslant 0.$$

The comparison of the degrees in the last equation gives $(q+1)\lambda_n = [n+1]_{q^{-1}}\mu$, $n \geqslant 0$. Therefore,

$$H_{q^{-1}}\left(Q_{n+1}\right)(x) = [n+1]_{q^{-1}} P_n(x),\ n \geqslant 0,$$

which is equivalent to

$$H_q\left(Q_{n+1}\right)(x) = [n+1]_q\, q^{-n} h_q P_n(x),\ n \geqslant 0.$$

□

Now we will show that the scaled q-Chebyshev polynomial sequence $\left\{b^{-n}\hat{U}_n(bx)\right\}_{n\geqslant 0}$, where $b^2 = -(q\mu)^{-1}$, is the only $\mathcal{U}_{(q,\mu)}$-classical orthogonal sequence. In particular, $\left\{\hat{U}_n(x)\right\}_{n\geqslant 0}$ is $\mathcal{U}_{(q,-1)}$-classical orthogonal sequence.

Theorem 1. *For any nonzero complex number μ and any MPS $\{P_n\}_{n\geqslant 0}$, the following statements are equivalent:*

(i) $\{P_n\}_{n\geqslant 0}$ *is $\mathcal{U}_{(q,\mu)}$-classical.*

(ii) There exists $b \in \mathbb{C}$, $b \neq 0$, such that $P_n(x) = b^{-n}\hat{U}_n(bx)$, $n \geqslant 0$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\{P_n\}_{n\geqslant 0}$ is $\mathcal{U}_{(q,\mu)}$-classical. Then there exists a monic orthogonal sequence $\{Q_n\}_{n\geqslant 0}$ satisfying (16). By applying v_0 to (16), we get for $n \geqslant 0$:

$$\langle v_0, q^{-1}[n+1]_q Q_{n+1}(x)\rangle = \langle v_0, (x^2+\mu) H_q\left(P_n\right) + q^{-1} x P_n\rangle = 0.$$

The preceding equation can be written as

$$\langle H_q\left((x^2+\mu)v_0\right) - q^{-1} x v_0,\, P_n\rangle = 0, \quad n \geqslant 0.$$

Equivalently,

$$H_q\left((x^2+\mu)v_0\right) - q^{-1} x v_0 = 0.$$

The choice $a^2 = -\mu^{-1}$ in Lemma 3 gives $v_0 = \mathcal{T}_q$. Then, from (4) and (5),

$$\begin{aligned}\frac{-u_0}{q+1} &= -\mu^{-1} h_q\left((x^2+\mu)v_0\right) = h_q\left(h_a(x^2-1)h_{a^{-1}}\mathcal{T}_q\right) = \\ &= h_q \circ h_{a^{-1}}\left((x^2-1)\mathcal{T}_q\right) = \frac{-1}{q+1} h_{(q^{-1}a)^{-1}}\tilde{U}_q.\end{aligned}$$

Consequently, $u_0 = h_{(q^{-1}a)^{-1}}\tilde{U}_q$. Thus, for $n \geqslant 0$

$$Q_n(x) = a^{-n}\hat{T}_n(ax), \quad P_n(x) = (aq^{-1})^{-n}\tilde{U}_n(aq^{-1}x) = (aq^{\frac{-1}{2}})^{-n}\hat{U}_n(aq^{\frac{-1}{2}}x).$$

The desired result is found by taking $b = aq^{\frac{-1}{2}}$; so, $b^2 = -\left(q\mu\right)^{-1}$.
(ii) $\Rightarrow$ (i). Let b in $\mathbb{C}$, with $b \neq 0$, and let $P_n(x) = b^{-n}\hat{U}_n(bx)$, $n \geqslant 0$. It is clear that $\{P_n\}_{n\geqslant 0}$ is a MOPS. The sequence $\{\hat{T}_n\}_{n\geqslant 0}$ is H_q-classical; then, according to (8), (9), it satisfies the q-diffrence equation

$$(x^2-1)(H_q \circ H_{q^{-1}}\hat{T}_{n+1})(x) - q^{-1}x(H_{q^{-1}}\hat{T}_{n+1})(x) = -\lambda_n \hat{T}_{n+1}(x), \; n \geqslant 0.$$

From (7), we get

$$(x^2-1)H_q\left(h_{q^{-1}}\left(H_q\hat{T}_{n+1}\right)\right)(x) + q^{-1}\,x\,h_{q^{-1}}\left(H_q\hat{T}_{n+1}\right)(x) = \\ = -\lambda_n\hat{T}_{n+1}(x), \quad n \geqslant 0.$$

On account of (13), the last equation becomes

$$[n+1]_q(x^2-1)H_q\left(h_{q^{-1}}\tilde{U}_n\right)(x) + q^{-1}[n+1]_q\,x\,h_{q^{-1}}\tilde{U}_n(x) = \\ = -\lambda_n\hat{T}_{n+1}(x), \quad n \geqslant 0.$$

According to (11), we get

$$q^{\frac{-n}{2}}[n+1]_q(x^2-1)H_q\left(h_{q^{\frac{-1}{2}}}\hat{U}_n\right)(x) + q^{\frac{-n}{2}-1}[n+1]_q\,x\,h_{q^{\frac{-1}{2}}}\hat{U}_n(x) = \\ = -\lambda_n\hat{T}_{n+1}(x), \quad n \geqslant 0.$$

Applying $h_{q^{\frac{1}{2}}}$ to the previous equation and using (6), we get

$$q^{\frac{-n-1}{2}}(qx^2-1)H_q\hat{U}_n(x) + q^{\frac{-n-1}{2}}x\,\hat{U}_n(x) = \frac{-\lambda_n}{[n+1]_q}h_{\frac{1}{2}}\hat{T}_{n+1}(x), \quad n \geqslant 0. \tag{27}$$

Finally, applying h_b to (27) and using (6), we get

$$qb(x^2 - \left(q^{\frac{1}{2}}b\right)^{-2})H_q\left(h_b\hat{U}_n\right)(x) + bx\,h_b\hat{U}_n(x) = \\ = \frac{-q^{\frac{n+1}{2}}\lambda_n}{[n+1]_q}h_{\left(bq^{\frac{1}{2}}\right)}\hat{T}_{n+1}(x), \quad n \geqslant 0. \tag{28}$$

For $\mu = -\left(q^{\frac{1}{2}}b\right)^{-2}$ and multiplying (28) by b^{-n}, we get

$$(x^2+\mu)H_q\left(P_n\right)(x)+q^{-1}x\,P_n(x) = \frac{-\lambda_n\left(bq^{\frac{1}{2}}\right)^{-(n+1)}}{[n+1]_q}h_{\left(bq^{\frac{1}{2}}\right)}\hat{T}_{n+1}(x), \quad n \geqslant 0.$$

Then

$$(\mathcal{U}_{(q,\mu)}P_n)(x) = \frac{-\lambda_n\left(bq^{\frac{1}{2}}\right)^{-(n+1)}}{[n+1]_q}h_{\left(bq^{\frac{1}{2}}\right)}\hat{T}_{n+1}(x), \quad n \geqslant 0.$$

Since $\{\hat{T}_n\}$ an orthogonal polynomials sequence, then $\mathcal{U}_{(q,\mu)}P_n$ is also an orthogonal polynomials sequence. Therefore, $\{P_n\}_{n\geqslant 0}$ is $\mathcal{U}_{(q,\mu)}$-classical. □

4. A property of the scaled q-Chebyshev polynomials.

Lemma 5. *There exists an endomorphism $\mathcal{E}$ of $\mathcal{P}$ into itself, such that the polynomials $P_n(x)$, $n \geqslant 0$, are eigenfunctions. We have:*

$$\mathcal{E}\left(P_n\right) = \lambda_n P_n, \quad n \geqslant 0, \tag{29}$$

with

$$\lambda_n = q^{-(n+1)}([n+1]_q)^2. \tag{30}$$

Moreover,

$$\mathcal{E} := b_1(x)H_{q^{-1}} \circ H_q + b_2(x)H_{q^{-1}} + b_3(x)I_{\mathcal{P}}, \tag{31}$$

where

$$b_1(x) = x^2 + \mu, \quad b_2(x) = \left(q^{-2} + q^{-1} + 1\right)x, \quad b_3(x) = q^{-1}, \tag{32}$$

and $I_{\mathcal{P}}$ represents the identity operator on the space of polynomials $\mathcal{P}$.

Proof. By applying the operator H_q to (15) and using (24), we obtain

$$H_q \circ \mathcal{U}_{(q,\mu)}\left(P_n\right) = q^{-(n+1)}\left([n+1]_q\right)^2\left(h_q P_n\right), \quad n \geqslant 0. \tag{33}$$

Then, applying the operator $h_{q^{-1}}$ to (33) and using (7), we get

$$H_{q^{-1}} \circ \mathcal{U}_{(q,\mu)}\left(P_n\right) = q^{-(n+1)}\left([n+1]_q\right)^2 P_n, \quad n \geqslant 0. \tag{34}$$

Consequently, from (2), (7), (15), and (34), we deduce (31)–(32). In addition, we have:

$$\mathcal{E}\left(X^n\right) = \lambda_n X^n + \mu_n X^{n-2}, n \geqslant 0,$$

with

$$\mu_n = q^{-(n-2)}[n]_q[n-1]_q\mu, n \geqslant 0.$$

Thus, the matrix of the endomorphism $\mathcal{E}$ in the canonical basis $\{X^n\}_{n\geqslant 0}$ of $\mathcal{P}$ is given by

$$\mathbf{M}_{\mathcal{E}} = \begin{pmatrix} \lambda_0 & 0 & \mu_2 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \ddots & \ddots & \vdots \\ & & \lambda_2 & \ddots & \mu_n & 0 \\ & & & \ddots & 0 & \ddots \\ & & & & \lambda_n & \ddots \\ 0 & & & & & \ddots \end{pmatrix}.$$

Using the relation (29), the matrix of $\mathcal{E}$ in the basis $\{P_n\}_{n\geqslant 0}$ is as follows:

$$\mathbf{E} = \begin{pmatrix} \lambda_0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda_n & 0 \\ 0 & \cdots & \cdots & 0 & \ddots \end{pmatrix}.$$

□

Remark 1.

1. *When $q \to 1$ in Proposition 3, Lemma 4, Proposition 4* and *Theorem 1, we recover the results, as well as the characterization of Chebyshev polynomials of the second kind in [3].*
2. *When $q \to 1$ in Lemma 5, we find the property described in [19] with $\xi_1 = 0$ for the Chebyshev polynomials of the second kind.*

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