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FIXED POINTS OF FUZZY MULTIVALUED F-CONTRACTIVE MAPPINGS WITH A DIRECTED GRAPH IN PARAMETRIC METRIC SPACES

Abstract. In the structure of a parametric metric space accompanied by directed graph, we introduce the idea of fuzzy multivalued F-contractive mappings. Results related to the existence of common fuzzy fixed points are introduced. The proved results are supported by an example. Our results bring together, sum up, and supplement different familiar related results in the literature. We hope that the acclaimed results in our work will encourage new analysis aspects in fixed-point theory and parallel hybrid models in the literature of fuzzy mathematics supplemented with a graph.

Key words: *F-contraction, fuzzy set, fuzzy multivalued mapping, fuzzy fixed point*

2020 Mathematical Subject Classification: *46S40, 47H10, 54H25*

1. Introduction A trouble in mathematical modeling of workable incidents is the indefiniteness persuaded by our inability to classify events with ample precision. The crisp set theory cannot manage imprecisions. As an effort to deal with problems of insufficient data, fuzzy sets [31] were considered, which gave birth to fuzzy set theory. It provides suitable mathematical tools for handling information with no statistical uncertainty. As a result, fuzzy set theory has gained much regard because of its applications in multiple domains like management sciences, engineering, environmental sciences, medical sciences, and other fields. The mean concepts of fuzzy sets have been modified and upgraded in different aspects; for example, see [7], [13], [18]. In 1973, Markin [23] commenced the investigation of fixed points for multivalued contractions and nonexpansive maps utilizing the Pompeiu-Hausdorff metric.

In 1981, Heilpern [14] commenced the study of fuzzy multivalued maps and obtained a fuzzy correspondent of Nadler's fixed-point theorem [24]. Later, many authors reviewed the existence of fixed points of fuzzy multivalued maps, for instance, Al-Mazrooei et al. [6], Azam et al. [8], [9], Bose and Sahani [12], Mohammed [16], Mohammed and Azam [17], Qiu and Shu [26], and so on. In 2003, Rus [28] presented the idea of R-multivalued mappings by upgrading the idea of K-multivalued mapping presented by Latif and Beg [22] in 1997. In 2009, Abbas and Rhoades [4] worked on R-multivalued mappings and introduced the idea of a generalized R-multivalued mappings to establish some common fixed-point consequences for such mappings.

In 2004, Ran and Reurings [27] first examined the existence of fixed points in partially ordered metric spaces and afterwards by Nieto and Lopez [25]. In 2007, Jachymski [19] established some generalized fixed-point results in metric fixed-point theory by using graph structure on metric spaces instead of order structure.

In 2012, Wardowski [30] generalized the Banach contraction principle results by introducing an advance contraction named as F-contraction. In 2013, Abbas et al. [1] used the concept of F-contraction mapping with respect to a self mapping on a metric space and acquired some common fixed-point results. In 2013, Sgroi and Vetro [29] got some fixed-point results for F-contraction multivalued maps in metric spaces (see also [5]).

In 2013, Abbas and Nazir [3] acquired some fixed-point consequences for power graphic contraction pair on a metric space equipped with a graph. In the structure of parametric metric space enriched with directed graph, this work aims to validate some common fuzzy fixed-point consequences for fuzzy multivalued generalized graphic F-contraction mappings. It is important to note that our consequences have been proved without Hausdorff metric, in addition our consequences extend and bind together different parallel results in the existing literature ([21], [22], [28], and [29]).

2. Preliminaries. Denote the set of natural numbers, the set of positive real numbers, and the set of real numbers by the symbols $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$, respectively. Let us describe some definitions and results needed for our main result.

Definition 1. [15] *Let Υ be a non-empty set. A mapping $f\colon \Upsilon \times \Upsilon \times (0,\infty) \to [0,\infty)$ is called a parametric metric if the following conditions are satisfied:*

1) $f(\varkappa_1, \varkappa_2, \theta) = 0$, *for all* $\theta > 0$ *if and only if* $\varkappa_1 = \varkappa_2$;
2) $f(\varkappa_1, \varkappa_2, \theta) = f(\varkappa_2, \varkappa_1, \theta)$ *for all* $\varkappa_1, \varkappa_2 \in \Upsilon$ *and* $\theta > 0$;
3) $f(\varkappa_1, \varkappa_3, \theta) \leqslant f(\varkappa_1, \varkappa_2, \theta) + f(\varkappa_2, \varkappa_3, \theta)$ *for all* $\varkappa_1, \varkappa_2, \varkappa_3 \in \Upsilon$ *and* $\theta > 0$;

then the pair $(\Upsilon,\ f)$ *is called a parametric metric space.*

Example 1. [15] Let the set of all functions $g, h : (0, \infty) \to \mathbb{R}$ be denoted by X. Define $f : \Upsilon \times \Upsilon \times (0, \infty) \to [0, \infty)$ by $f(h,g,\theta) = |h(\theta) - g(\theta)|$ for all $h, g \in X$ and all $\theta > 0$. Then f is a parametric metric on Υ and the pair (f, Υ) is a parametric metric space.

Definition 2. [15] *Consider a parametric metric space* (Υ, f) *having a sequence* $\{\varkappa_n\}$*; the following definitions are needed for our results:*

1) *if* $\lim_{n\to\infty} f(\varkappa_n, \varkappa, \theta) = 0$, *then* $\{\varkappa_n\}$ *is called convergent to* $\varkappa \in X$, *written as* $\lim_{n\to\infty} \varkappa_n = \varkappa$, *for all* $\theta > 0$;
2) *if for all* $\theta > 0$ *we have* $\lim_{n,m\to\infty} f(\varkappa_n, \varkappa_m, \theta) = 0$, *then* $\{\varkappa_n\}$ *is called Cauchy sequence in* Υ;
3) *a parametric space* (Υ, f) *is called complete if every Cauchy sequence converges in it.*

Definition 3. [15] *Let* $(\Upsilon,\ f)$ *be a parametric metric space and* $T : \Upsilon \to \Upsilon$ *be a mapping.* T *is said to be a continuous mapping at* a *in* Υ, *if for any sequence* $\{a_n\}$ *in* Υ *such that* $\lim_{n\to\infty} a_n = a$, *then* $\lim_{n\to\infty} Ta_n = Ta$.

Definition 4. *A graph* G *consists of two sets* $V = \{v_1, v_2, v_3, \dots\}$ *and* $E = \{e_1, e_2, e_3, \dots\}$*; the elements of* V *are called vertices while the elements of* E *are called edges. In the literature on the graph theory, the set* $V(G)$ *represents the vertex set of* G *and* $E(G)$ *represents the edge set. Let* $\{v_1, v_2\}$ *be an edge of graph* G, *as* $\{v_1, v_2\}$ *is a 2-objects set, so we may write* $\{v_2, v_1\}$ *instead of* $\{v_1, v_2\}$.

Following Jachymski [19], consider a parametric metric space (Υ, f) and a directed graph G with vertices $V(G)$ coinciding with Υ. $E(G)$ represents the set having all edges and loops also $E(G) \supseteq \Delta$, where Δ represents the diagonal of $\Upsilon \times\ \Upsilon$. Furthermore, $E^*(G)$ represents a set having only edges of the graph G. Also, it is supposed that there are no multiple edges in the graph G and the pair $(V(G), E(G))$ can perceive the graph G.

Definition 5. [19] *In a parametric space (Υ, f), an operator $\eta : \Upsilon \to \Upsilon$ is said to be a Banach G-contraction if*

1. *for every $a, b \in X$ with $(a, b) \in E(G)$, we have $(\eta(a), \eta(b)) \in E(G)$, that is, η preserves edges.*
2. *There exists $\gamma \in (0, 1)$, such that for all $a, b \in X$ with $(a, b) \in E(G)$ we have $f(\eta(a), \eta(b)) \leqslant \gamma f(a, b)$, that is, η decreases weights of edges of G.*

A (directed) path in a graph G of length $l \in \mathbb{N}$ between the vertices $\varkappa_1$ and $\varkappa_2$ is a finite sequence $\{\varkappa_n\}$ (where $n \in \{0, 1, 2, \ldots, l\}$) of vertices, such that $\varkappa_0 = \varkappa_1$, $\varkappa_l = \varkappa_2$ and $(\varkappa_{j-1}, \varkappa_j) \in E(G)$ for $j \in \{1, 2, \ldots, l\}$.

Note that a graph G is connected if there is a (directed) path between every pair of vertices, and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ represents the undirected graph obtained from G by neglecting the direction of edges. The graph obtained from G by reversing the direction of edges is denoted by G^{-1}. Thus,

$$E\left(G^{-1}\right) = \{(\varkappa_1,\, \varkappa_2) \in \Upsilon \times \Upsilon : (\varkappa_2, \varkappa_1) \in E\left(G\right)\}.$$

It is important to note that $\widetilde{G}$ is such a directed graph that the set of its edges is symmetric, hence we can write:

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If the set of edges of a graph G is symmetric, then, for $\varkappa \in V(G)$, $[\varkappa]_G$ represents the class of equivalence of the relation R defined on $V(G)$ by the rule:

$$\varkappa_2 R \varkappa_3 \text{ if there is a directed path in } G \text{ from } \varkappa_2 \text{ to } \varkappa_3\ .$$

If $\phi : \Upsilon \to \Upsilon$ is an operator, set:

$$\Upsilon_\phi := \{\varkappa \in \Upsilon : (\varkappa, \phi(\varkappa)) \in E(G)\}.$$

Jachymski [20] obtained the following property:

A graph G has a property

(P) : for every sequence $\{\varkappa_n\}$ in Υ, if $\varkappa_n \to \varkappa$, such as $n \to \infty$ and $(\varkappa_n, \varkappa_{n+1}) \in E(G)$, then $(\varkappa_n, \varkappa) \in E(G)$.

Theorem 1. [20] *Let (Υ, f) be a complete parametric metric space and let G be a directed graph, such that $V(G) = \Upsilon$. Let $E(G)$ and the triplet (Υ, f, G) have property (P) and $\eta : \Upsilon \to \Upsilon$ be a G-contraction. Then the following statements hold:*

(i) $\Upsilon_\eta \neq \varnothing$ *if and only if* η *has a fixed point;*

(ii) *if* $\Upsilon_\eta \neq \varnothing$ *and* G *is weakly connected, then* η *is a Picard operator;*

(iii) *for any* $\varkappa \in \Upsilon_\eta$, $\eta\,|_{[\varkappa]_{\tilde{G}}}$ *is a Picard operator;*

(iv) *if* $\Upsilon_\eta \subseteq E(G)$, *then* η *is a weakly Picard operator.*

See Berinde [10, 11], for further details of Picard operators.
[30] Denote by Γ the sets of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) For all $\varkappa_1, \varkappa_2 \in \mathbb{R}^+$, such that $\varkappa_1 < \varkappa$ implies that $F(\varkappa_1) < F(\varkappa_2)$, implies F to be strictly increasing.

(F_2) For any sequence $\{\varkappa_n\}$ of positive real numbers, $\lim\limits_{n\to\infty} \varkappa_n = 0$ and $\lim\limits_{n\to\infty} F(\varkappa_n) = -\infty$ are equivalent.

(F_3) There exists $h \in (0,1)$, such that $\lim\limits_{\varkappa\to 0^+} \varkappa^h F(\varkappa) = 0$.

Recall that an ordinary subset B of Υ is determined by its characteristic function χ_B, where $\chi_B : B \longrightarrow \{0,1\}$ is defined as

$$\chi_B(b) = \begin{cases} 1, & \text{if } b \in B, \\ 0, & \text{if } b \notin B. \end{cases}$$

The value $\chi_B(b)$ defines whether an element belongs to B or not. This suggestion is employed to establish fuzzy sets by permitting an element $x \in A$ to correspond to any value in the interval [0,1]. Thus, a fuzzy set in Υ is a function with domain Υ and values in $[0,1] = I$. The set of all fuzzy sets in Υ is denoted by I^Υ. If B is a fuzzy set in Υ, then the function value $B(\varkappa)$ is called the grade of membership of $\varkappa$ in B. The α-level set of a fuzzy set B is denoted by $[B]_\alpha$ and is defined as follows:

$$[B]_\alpha = \begin{cases} \overline{\{\varkappa \in \Upsilon : B(\varkappa) > 0\}}, & \text{if } \alpha = 0, \\ \{\varkappa \in \Upsilon : B(\varkappa) \geqslant \alpha\}, & \text{if } \alpha \in (0,1], \end{cases}$$

where $\overline{X}$ represents the closure of set X.

Example 2. Let Υ be the set of all individuals in a certain town, and

$$A = \{\varkappa \in \Upsilon \mid \varkappa \text{ is an old person}\}.$$

Then, it is more appropriate to identify an individual to be an old person by a membership function A on Υ, because the term «old» is not well defined.

Example 3. Let $\Upsilon = \{1, 2, 3, 4\}$ be equipped with the usual metric. Let $T : \Upsilon \to I^{\Upsilon}$ be a fuzzy multivalued map, that is, for each $\varkappa \in \Upsilon$, $T(\varkappa) : \Upsilon \to [0, 1]$ is a fuzzy set. For instance, for some $\alpha \in (0, 1]$, we may define one of the fuzzy sets $T(1)$ by

$$T(1)(\kappa) = \begin{cases} \alpha, & \text{if } \kappa = 1, \\ \frac{\alpha}{3}, & \text{if } \kappa = 2, \\ \frac{\alpha}{7}, & \text{if } \kappa = 3, \\ \frac{\alpha}{9}, & \text{if } \kappa = 4. \end{cases}$$

Let (Υ, f) be a parametric metric space. The family of all nonempty closed subsets of Υ is denoted by $P_{cl}(\Upsilon)$.

Define

$$F_{P_{cl}}(\Upsilon) = \{A \in I^{\Upsilon} \colon [A]_\alpha \in P_{cl}(\Upsilon)\} \text{ for some } \alpha \in (0, 1].$$

A point $\varkappa$ in Υ is a fuzzy fixed point of a fuzzy mapping $T \colon \Upsilon \to I^{\Upsilon}$ iff $\varkappa \in [T\varkappa]_\alpha$. $Fuz(T)$ denotes the set of all fuzzy fixed points of a fuzzy mapping T.

Suppose T_1, $T_2 \colon \to F_{P_{cl}}(\Upsilon)$ be fuzzy mappings. Set

$$X_{T_1, T_2} := \{\varkappa \in \Upsilon \colon (\varkappa, v_\varkappa) \in E(G) \text{ where } v_\varkappa \in [T_1(\varkappa)]_\alpha \cap [T_2(\varkappa)]_\beta\},$$

for some $\alpha, \beta \in (0, 1]$, where α and β need not be equal.

In the further discussion, we will take $\alpha, \beta \in (0, 1]$, where α and β need not be equal, so in the rest of the paper it will not be minded if α and β are not mentioned to be in $(0, 1]$.

Now we give the following definition in the setup of parametric metric space:

Definition 6. *Let $T_1, T_2 \colon \Upsilon \to F_{P_{cl}}(\Upsilon)$ be two fuzzy multivalued mappings in parametric metric space (Υ, f). Assume that for every vertex $\varkappa$ in G and for any $v_\varkappa \in [T_i(\varkappa)]_\alpha$, for $i \in \{1, 2\}$ we have $(\varkappa, v_\varkappa) \in E(G)$. A pair (T_1, T_2) is said to form:*

(i) a fuzzy graphic F_1-contraction, if for every $\varkappa_1, \varkappa_2 \in \Upsilon$ with $(\varkappa_1, \varkappa_2) \in E(G)$ and $v_{\varkappa_1} \in [T_i(\varkappa_1)]_\alpha$, there exists $v_{\varkappa_2} \in [T_j(\varkappa_2)]_\beta$

for $i, j \in \{1, 2\}$ with $i \neq j$, such that $(v_{\varkappa_1}, v_{\varkappa_2}) \in E^*(G)$ and

$$\tau + F\left(f(v_{\varkappa_1}, v_{\varkappa_2}, \theta)\right) \leqslant F(M_1(\varkappa_1, \varkappa_2; v_{\varkappa_1}, v_{\varkappa_2}, \theta)), \tag{1}$$

holds, where τ denotes a positive real number and

$$M_1(\varkappa_1, \varkappa_2; v_{\varkappa_1}, v_{\varkappa_2}, \theta) = \max\Big\{ f(\varkappa_1, \varkappa_2, \theta), f(\varkappa_1, v_{\varkappa_1}, \theta), f(\varkappa_2, v_{\varkappa_2}, \theta), \frac{f(\varkappa_1, v_{\varkappa_2}, \theta) + f(\varkappa_2, v_{\varkappa_1}, \theta)}{2} \Big\}.$$

(ii) a fuzzy graphic F_2-contraction, if for $\varkappa_1$ and $\varkappa_2 \in \Upsilon$ with $(\varkappa_1, \varkappa_2) \in E(G)$ and $u_{\varkappa_1} \in [T_i(\varkappa_1)]_\alpha$ there exists $u_{\varkappa_2} \in [T_j(\varkappa_2)]_\beta$ and $i, j \in \{1, 2\}$ with $i \neq j$, such that $(u_{\varkappa_1}, u_{\varkappa_2}) \in E^*(G)$ and we have

$$\tau + F\left(f(u_{\varkappa_1}, u_{\varkappa_2}, \theta)\right) \leqslant F(M_2(\varkappa_1, \varkappa_2; u_{\varkappa_1}, u_{\varkappa_2}, \theta)), \tag{2}$$

where τ is a positive real number and

$$M_2(\varkappa_1, \varkappa_2; u_{\varkappa_1}, u_{\varkappa_2}, \theta) = \alpha_1 f(\varkappa_1, \varkappa_2, \theta) + \beta_1 f(\varkappa_1, u_{\varkappa_1}, \theta) + \\ + \gamma_1 f(\varkappa_2, u_{\varkappa_2}, \theta) + \delta_1 f\left(\varkappa_1, u_{\varkappa_2}, \theta\right) + \delta_2 f\left(\varkappa_2, u_{\varkappa_1}, \theta\right),$$

and $\alpha_1, \beta_1, \gamma_1, \delta_1, \delta_2 \geqslant 0, \delta_1 \leqslant \delta_2$ with $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 + \delta_2 \leqslant 1$.

It is important to note that for different selections of mappings F, one may obtain different conductivity conditions.

Remember that a fuzzy map $T\colon \Upsilon \to F_{P_{cl}}(\Upsilon)$ is called an upper semi-continuous, if for $\varkappa_n \in \Upsilon$ and $\varkappa_n^* \in [T\varkappa_n]_\alpha$ with $\varkappa_n \to \varkappa_0$ and $\varkappa_n^* \to \varkappa_0^*$, we have $\varkappa_0^* \in [T\varkappa_0]_\beta$.

A subset W of V, that is, $W \subset V$, is named a clique of a graph G if for every two vertices belonging to W, there exists an edge connecting them. This is similar to the condition that the subgraph induced by W is complete, that is, for every $\varkappa$, $\varkappa^* \in W(G)$, we have $(\varkappa, \varkappa^*) \in E(G)$.

3. Main results. We present common fuzzy fixed point results for two fuzzy multivalued mappings on a parametric metric space enriched with a directed graph. We start with the following result:

Theorem 2. *Suppose (Υ, f) is a complete parametric metric space enriched with a directed graph G, such that $V(G) = \Upsilon$ and $E(G) \supseteq \Delta$. If fuzzy mappings T_1, $T_2\colon \Upsilon \to F_{P_{cl}}(\Upsilon)$ form a fuzzy graphic F_1-contraction pair, then the following statements hold:*

(i) *$Fuz\,(T_1) = Fuz\,(T_2) \neq \varnothing$ if and only if $Fuz(T_i) \neq \varnothing$ for any $i \in \{1, 2\}$.*

(ii) *$\Upsilon_{T_1,T_2} \neq \varnothing$ given that $Fuz\,(T_1) \cap Fuz\,(T_2) \neq \varnothing$.*

(iii) *The graph G is weakly connected and $\Upsilon_{T_1,T_2} \neq \varnothing$; then $Fuz\,(T_1) = Fuz\,(T_2) \neq \varnothing$ given that either (a), or T_1 or T_2 are upper semicontinuous, or (b) F is continuous, either T_1 or T_2 are bounded, and G has property (P).*

(iv) *$Fuz\,(T_1) \cap Fuz\,(T_2)$ is a clique of graph $\widetilde{G}$ if and only if $Fuz\,(T_1) \cap Fuz\,(T_2)$ is a singleton set.*

Proof. To validate (i), let $\varkappa^*$ be any point of Υ and suppose that $\varkappa^* \in [T_1(\varkappa^*)]_\alpha$, such that $\varkappa^* \notin [T_2\,(\varkappa^*)]_\beta$. As it is given that (T_1, T_2) form a graphic fuzzy F_1-contraction pair, this implies that there exists a $\varkappa \in [T_2\,(\varkappa^*)]_\beta$ with $(\varkappa^*, \varkappa) \in E^*\,(G)$, such that

$$\tau + F\,(f(\varkappa^*, \varkappa, \theta)) \leqslant F(M_1(\varkappa^*, \varkappa^*; \varkappa^*, \varkappa, \theta)),$$

where

$$M_1(\varkappa^*, \varkappa^*; \varkappa^*, \varkappa, \theta) = \max\Big\{f(\varkappa^*, \varkappa^*, \theta), f(\varkappa^*, \varkappa^*, \theta), f(\varkappa, \varkappa^*, \theta), \frac{f(\varkappa^*, \varkappa, \theta) + f(\varkappa^*, \varkappa^*, \theta)}{2}\Big\} = f(\varkappa, \varkappa^*, \theta).$$

Thus we have

$$\tau + F\,(f(\varkappa^*, \varkappa, \theta)) \leqslant F(f(\varkappa^*, \varkappa, \theta)),$$

a contradiction as $\tau > 0$. Hence, $\varkappa^* \in [T_2(\varkappa^*)]_\beta$ and so $Fuz(T_1) \subseteq Fuz(T_2)$. Similarly, $Fuz(T_2) \subseteq Fuz(T_1)$ and, therefore, $Fuz(T_1) = Fuz(T_2)$. Also, if $\varkappa^* \in [T_2(\varkappa^*)]_\beta$, then we have $\varkappa^* \in [T_1(\varkappa^*)]_\alpha$. The converse statement can be proved easily by simple steps.

To prove (ii), let $Fuz\,(T_1) \cap Fuz\,(T_2) \neq \varnothing$. Then there exists $\varkappa \in \Upsilon$, such that $\varkappa \in [T_1(\varkappa)]_\alpha \cap [T_2(\varkappa)]_\beta$. As $\Delta \subseteq E(G)$, we deduce that $X_{T_1,T_2} \neq \varnothing$.

To validate (iii), assume that $\varkappa_0$ is an arbitrary point of Υ. If $\varkappa_0 \in [T_1\,(\varkappa_0)]_\alpha$ or $\varkappa_0 \in [T_2\,(\varkappa_0)]_\beta$, then the proof is completed. So, assume that $\varkappa_0 \notin [T_1\,(\varkappa_0)]_\alpha$ and $\varkappa_0 \notin [T_2\,(\varkappa_0)]_\beta$. Now, for $i, j \in \{1, 2\}$ with $i \neq j$, if $\varkappa_1 \in [T_i(\varkappa_0)]_\alpha$, then there exists $\varkappa_2 \in [T_j(\varkappa_1)]_\beta$ with $(\varkappa_1, \varkappa_2) \in E^*(G)$, such that

$$\tau + F\,(f(\varkappa_1,\varkappa_2,\theta)) \leqslant F(M_1(\varkappa_0,\varkappa_1; \varkappa_1,\varkappa_2,\theta)),$$

where

$$M_1(\varkappa_0, \varkappa_1; \varkappa_1, \varkappa_2, \theta) =$$
$$= \max\left\{f(\varkappa_0, \varkappa_1, \theta), f(\varkappa_0, \varkappa_1, \theta), f(\varkappa_1, \varkappa_2, \theta), \frac{f(\varkappa_0, \varkappa_2, \theta) + f(\varkappa_1, \varkappa_1, \theta)}{2}\right\} =$$
$$= \max\left\{f(\varkappa_0, \varkappa_1, \theta), f(\varkappa_1, \varkappa_2, \theta), \frac{f(\varkappa_0, \varkappa_2, \theta)}{2}\right\} \leqslant$$
$$\leqslant \max\left\{f(\varkappa_0, \varkappa_1, \theta), f(\varkappa_1, \varkappa_2, \theta), \frac{f(\varkappa_0, \varkappa_1, \theta), f(\varkappa_1, \varkappa_2, \theta)}{2}\right\} =$$
$$= \max\{f(\varkappa_0, \varkappa_1, \theta), f(\varkappa_1, \varkappa_2, \theta)\}.$$

If $M_1(\varkappa_0, \varkappa_1; \varkappa_1, \varkappa_2, \theta) = f(\varkappa_1, \varkappa_2, \theta)$, then

$$\tau + F\left(f(\varkappa_1, \varkappa_2, \theta)\right) \leqslant F(f(\varkappa_1, \varkappa_2, \theta)),$$

gives a contradiction as $\tau > 0$. Therefore, $M_1(\varkappa_0, \varkappa_1; \varkappa_1, \varkappa_2, \theta) = f(\varkappa_0, \varkappa_1, \theta)$ and we have

$$\tau + F\left(f(\varkappa_1, \varkappa_2, \theta)\right) \leqslant F\left(f(\varkappa_0, \varkappa_1, \theta)\right).$$

Correspondingly, for the point $\varkappa_2$ in $[T_j(\varkappa_1)]_\alpha$, there exists $\varkappa_3 \in [T_i(\varkappa_2)]_\beta$ with $(\varkappa_2, \varkappa_3) \in E^*(G)$ such that

$$\tau + F\left(f(\varkappa_2, \varkappa_3, \theta)\right) \leqslant F(M_1(\varkappa_1, \varkappa_2; \varkappa_2, \varkappa_3, \theta)),$$

where

$$M_1(\varkappa_1, \varkappa_2; \varkappa_2, \varkappa_3, \theta) =$$
$$= \max\left\{f(\varkappa_1, \varkappa_2, \theta), f(\varkappa_1, \varkappa_2, \theta), f(\varkappa_2, \varkappa_3, \theta), \frac{f(\varkappa_1, \varkappa_3, \theta) + f(\varkappa_2, \varkappa_2, \theta)}{2}\right\} =$$
$$= \max\{f(\varkappa_1, \varkappa_2, \theta), f(\varkappa_2, \varkappa_3, \theta)\}.$$

In case $M_1(\varkappa_1, \varkappa_2; \varkappa_2, \varkappa_3, \theta) = f(\varkappa_2, \varkappa_3, \theta)$:

$$\tau + F\left(f(\varkappa_2, \varkappa_3, \theta)\right) \leqslant F(f(\varkappa_2, \varkappa_3, \theta)),$$

gives a contradiction as $\tau > 0$. Therefore, $M_1(\varkappa_1, \varkappa_2; \varkappa_2, \varkappa_3, \theta) = f(\varkappa_1, \varkappa_2, \theta)$ and we have:

$$\tau + F\left(f(\varkappa_2, \varkappa_3, \theta)\right) \leqslant F\left(f(\varkappa_1, \varkappa_2, \theta)\right).$$

Enduring this way, for $\varkappa_{2n} \in [T_j(\varkappa_{2n-1})]_\alpha$, there exist $\varkappa_{2n+1} \in [T_i(\varkappa_{2n})]_\beta$ with $(\varkappa_{2n}, \varkappa_{2n+1}) \in E^*(G)$, such that

$$\tau + F(f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)) \leqslant F(M_1(\varkappa_{2n-1}, \varkappa_{2n}; \varkappa_{2n}, \varkappa_{2n+1}, \theta)),$$

that is,

$$\tau + F\left(f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)\right) \leqslant F\left(f(\varkappa_{2n-1}, \varkappa_{2n})\right).$$

In the similar manner, for $\varkappa_{2n+1} \in [T_j(\varkappa_{2n})]_\alpha$, there exist $\varkappa_{2n+2} \in [T_i(\varkappa_{2n+1})]_\beta$, such that for $(\varkappa_{2n+1}, \varkappa_{2n+2}) \in E^*(G)$ implies

$$\tau + F\left(f(\varkappa_{2n+1}, \varkappa_{2n+2}, \theta)\right) \leqslant F\left(f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)\right).$$

Hence, we got a sequence $\{\varkappa_n\}$ in Υ, such that for $\varkappa_n \in [T_j(\varkappa_{n-1})]_\alpha$, there exist $\varkappa_{n+1} \in [T_i(\varkappa_n)]_\beta$ with $(\varkappa_n, \varkappa_{n+1}) \in E^*(G)$ and it satisfies

$$\tau + F\left(f(\varkappa_n, \varkappa_{n+1}, \theta)\right) \leqslant F\left(f(\varkappa_{n-1}, \varkappa_n, \theta)\right).$$

Therefore,

$$F\left(f(\varkappa_n, \varkappa_{n+1}, \theta)\right) \leqslant F(f(\varkappa_{n-1}, \varkappa_n, \theta)) - \tau \leqslant F(f(\varkappa_{n-2}, \varkappa_{n-1}, \theta)) - 2\tau \leqslant \\ \leqslant \ldots \leqslant F\left(f(\varkappa_0, \varkappa_1, \theta)\right) - n\tau. \quad (3)$$

From (3), we have got $\lim_{n\to\infty} F\left(f(\varkappa_n, \varkappa_{n+1}, \theta)\right) = -\infty$ that together with (F_2) yields $\lim_{n\to\infty} f(\varkappa_n, \varkappa_{n+1}, \theta) = 0$.

Now, by (F_3), there exists $h \in (0,1)$, such that

$$\lim_{n\to\infty} [f(\varkappa_n, \varkappa_{n+1}, \theta)]^h F\left(f(\varkappa_n, \varkappa_{n+1}, \theta)\right) = 0.$$

From (3), we have

$$[f(\varkappa_n, \varkappa_{n+1}, \theta)]^h F\left(f(\varkappa_n, \varkappa_{n+1}, \theta)\right) - [f(\varkappa_n, \varkappa_{n+1}, \theta)]^h F\left(f(\varkappa_0, \varkappa_{n+1}, \theta)\right) \leqslant \\ \leqslant -n\tau[f(\varkappa_n, \varkappa_{n+1}, \theta)]^h \leqslant 0.$$

On taking limit as $n \to \infty$, we get $\lim_{n\to\infty} n[f(\varkappa_n, \varkappa_{n+1}, \theta)]^h = 0$.

So, $\lim_{n\to\infty} n^{\frac{1}{h}} f(\varkappa_n, \varkappa_{n+1}, \theta) = 0$ and there exists $n_1 \in \mathbb{N}$, such that $n^{\frac{1}{h}} f(\varkappa_n, \varkappa_{n+1}, \theta) \leqslant 1$ for all $n \geqslant n_1$. So, we get

$$f(\varkappa_n, \varkappa_{n+1}, \theta) \leqslant \frac{1}{n^{1/h}}$$

for all $n \geqslant n_1$. Now consider $m,n \in \mathbb{N}$, such that $m > n \geqslant n_1$; we get:

$$f(\varkappa_n, \varkappa_m, \theta) \leqslant$$
$$\leqslant f(\varkappa_n, \varkappa_{n+1}, \theta) + f(\varkappa_{n+1}, \varkappa_{n+2}, \theta) + \ldots + f(\varkappa_{m-1}, \varkappa_m, \theta) \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}.$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$, we get $f(\varkappa_n, \varkappa_m, \theta) \to 0$ as $n, m \to \infty$. Therefore, $\{\varkappa_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $\varkappa^* \in \Upsilon$, such that $\varkappa_n \to \varkappa^*$ as $n \to \infty$.

Now, if T_i is upper semicontinuous, then, as $\varkappa_{2n} \in X$, $\varkappa_{2n+1} \in [T_i(\varkappa_{2n})]_\alpha$ with $\varkappa_{2n} \to \varkappa^*$ and $\varkappa_{2n+1} \to \varkappa^*$ as $n \to \infty$ implies that $\varkappa^* \in [T_i(\varkappa^*)]_\beta$. Using (i), we get $\varkappa^* \in [T_i(\varkappa^*)]_\alpha = [T_j(\varkappa^*)]_\beta$. In the same fashion, the result holds when T_j is upper semicontinuous.

Assume that F is continuous. Since $\varkappa_{2n}$ converges to $\varkappa^*$ as $n \to \infty$ and $(\varkappa_{2n}, \varkappa_{2n+1}) \in E(G)$, we have $(\varkappa_{2n}, \varkappa^*) \in E(G)$. For $\varkappa_{2n} \in [T_j(\varkappa_{2n-1})]_\alpha$, there exists $v_n \in [T_i(\varkappa^*)]_\beta$, such that $(\varkappa_{2n}, v_n) \in E^*(G)$. As $\{v_n\}$ is bounded, $\limsup\limits_{n\to\infty} v_n = v^*$, and $\liminf\limits_{n\to\infty} v_n = v^*$ both exist. Assume that $v^* \neq x^*$. Since (T_1, T_2) form a graphic F_1-contraction,

$$\tau + F(f(\varkappa_{2n}, v_n, \theta)) \leqslant F(M_1(\varkappa_{2n-1}, \varkappa^*; \varkappa_{2n}, v_n, \theta)),$$

where

$$M_1(\varkappa_{2n-1}, \varkappa^*; \varkappa_{2n}, v_n, \theta) = \max\Big\{ f(\varkappa_{2n-1}, \varkappa^*, \theta), f(\varkappa_{2n-1}, \varkappa_{2n}, \theta),$$
$$f(\varkappa^*, v_n, \theta), \frac{f(\varkappa_{2n-1}, v_n, \theta) + f(\varkappa^*, \varkappa_{2n}, \theta)}{2} \Big\}.$$

Taking $\limsup$ implies

$$\tau + F(f(\varkappa^*, v^*, \theta)) \leqslant F(f(\varkappa^*, v^*, \theta)),$$

a contradiction. Hence, $v^* = \varkappa^*$. In the same way, taking the $\liminf$ yields $v^* = \varkappa^*$. Since $v_n \in [T_i(\varkappa^*)]_\alpha$ for all $n \geqslant 1$ and $[T_i(\varkappa^*)]_\alpha$ is a closed set, it follows that $\varkappa^* \in [T_i(\varkappa^*)]_\alpha$. Now, from (i), we get $\varkappa^* \in [T_i(\varkappa^*)]_\beta$ and, hence, $Fuz(T_1) = Fuz(T_2)$.

Finally, to verify (iv), suppose that the set $Fuz(T_1) \cap Fuz(T_2)$ is a clique set of $\widetilde{G}$. We need to verify that $Fuz(T_1) \cap Fuz(T_2)$ is singleton. Assume the contrary: that there exists v and u, such that

$v, u \in Fuz\,(T_1) \cap Fuz\,(T_2)$, but $u \neq v$. As $(v, u) \in E^*(G)$ and T_1 and T_2 form a graphic F_1-contraction, so for $(v_x, u_y) \in E^*\,(G)$ implies

$$\begin{aligned}
&\tau + F\,(f(v, u, \theta)) \leqslant F(M_1(v, u; v, u, \theta)) = \\
&= F\Big(\Big\{\max\{f(v, u, \theta), f(v, v, \theta), f(u, u, \theta), \frac{f(v, u, \theta) + f(u, v, \theta)}{2}\Big\}\Big) = \\
&\qquad = F\,(f\,(v, u, \theta))\,,
\end{aligned}$$

a contradiction as $\tau > 0$. Hence, $v = u$. Conversely, if $Fuz(T_1) \cap Fuz(T_2)$ is singleton, then it follows that $Fuz(T_1) \cap Fuz(T_2)$ is a clique set of $\widetilde{G}$. □

Example 4. Let $\Upsilon = \{0\} \cup \{a_n = \dfrac{n(n+1)}{2} : n \in \mathbb{N}\} = V\,(G)$,

$$\begin{aligned}
E\,(G) &= \{(\varkappa, \varkappa^*) \colon \varkappa \leqslant \varkappa^* \text{ where } \varkappa, \varkappa^* \in V\,(G)\} \text{ and} \\
E^*\,(G) &= \{(\varkappa, \varkappa^*) \colon \varkappa < \varkappa^* \text{ where } \varkappa, \varkappa^* \in V\,(G)\}.
\end{aligned}$$

Let $V\,(G)$ be endowed with the parametric metric defined as

$$f\,(\varkappa, \varkappa^*, \theta) = \begin{cases} \theta \max\{\varkappa, \varkappa^*\}, & \varkappa \neq \varkappa^*, \\ 0, & \varkappa = \varkappa^*. \end{cases}$$

Define $T_1 \colon \Upsilon \to F_{P_{cl}}(\Upsilon)$ as follows:

$$T_1(\varkappa)(t) = \begin{cases} \frac{1}{2}, & \text{if } \ t = \varkappa_1, \\ \frac{1}{3}, & \text{elsewhere.} \end{cases}$$

Now, for $\alpha = \frac{1}{2}$, we have

$$[T_1(\varkappa)]_{\frac{1}{2}} = \Big\{t \colon [T_1(\varkappa)(t)]_\alpha \geqslant \frac{1}{2}\Big\} = \{\varkappa_1\}$$

and $T_2 \colon \Upsilon \to F_{P_{cl}}(\Upsilon)$ as in the following cases:
1) For $\varkappa = \varkappa_1$

$$T_2(\varkappa)(t) = \begin{cases} \frac{1}{2}, & \text{if } \ t = \varkappa_1, \\ \frac{1}{3}, & \text{elsewhere.} \end{cases}$$

2) For $\varkappa = \varkappa_n$ with $n > 1$

$$T_2(\varkappa)(t) = \begin{cases} \frac{1}{2}, & \text{if } t \in \{\varkappa_1, \varkappa_{n-1}\}, \\ \\ \frac{1}{3}, & \text{elsewhere.} \end{cases}$$

Now, for $\beta = \dfrac{1}{2}$,

$$\begin{aligned} [T_2(\varkappa)]_{\frac{1}{2}} &= \Big\{t : [T(\varkappa)(t)]_\beta \geqslant \frac{1}{2}\Big\} = \\ &= \begin{cases} \{\varkappa_1\}, & \text{if } \varkappa = \varkappa_1, \\ \{\varkappa_1, \varkappa_{n-1}\}, & \text{if } \varkappa = \varkappa_n \text{ with } n > 1. \end{cases} \end{aligned}$$

Take $F(\omega) = \ln \omega + \omega$, $\omega > 0$, and $\tau = 1$. For $(u_\varkappa, u_{\varkappa^*}) \in E^*(G)$, and $\alpha = \beta = \dfrac{1}{2}$, we observe the following cases:

(i) If $\varkappa = \varkappa_1, \varkappa^* = \varkappa_m$, for $m > 1$, then for $v_\varkappa = \varkappa_1 \in [T_1(\varkappa)]_{\frac{1}{2}}$, there exists $v_{\varkappa^*} = \varkappa_{m-1} \in [T_2(\varkappa^*)]_{\frac{1}{2}}$, such that

$$\begin{aligned} &f(v_\varkappa, v_{\varkappa^*}, \theta) e^{f(v_\varkappa, v_{\varkappa^*}, \theta) - M(\varkappa, \varkappa^*; v_\varkappa, v_{\varkappa^*}, \theta)} \leqslant \\ &\qquad \leqslant f(v_\varkappa, v_{\varkappa^*}, \theta) e^{f(v_\varkappa, v_{\varkappa^*}, \theta) - f(\varkappa \varkappa^*, \theta)} = \theta \frac{m^2 - m}{2} e^{-m\theta} < \\ &\qquad < \theta \frac{m^2 + m}{2} e^{-\theta} = e^{-\theta} f(\varkappa, \varkappa^*, \theta) \leqslant e^{-\theta} M_1(\varkappa, \varkappa^*; v_\varkappa, v_{\varkappa *}, \theta). \end{aligned}$$

(ii) If $\varkappa = \varkappa_n$, $\varkappa^* = \varkappa_{n+1}$ with $n > 1$, then for $v_\varkappa = \varkappa_1 \in [T_1(\varkappa)]_{\frac{1}{2}}$ there exists $v_{\varkappa^*} = \varkappa_{n-1} \in [T_2(\varkappa^*)]_{\frac{1}{2}}$, such that:

$$\begin{aligned} &f(v_\varkappa, v_{\varkappa^*}, \theta) e^{f(v_\varkappa, v_{\varkappa^*}, \theta) - M(\varkappa, \varkappa^*; v_\varkappa, v_{\varkappa^*}, \theta)} \leqslant \\ &\leqslant f(v_\varkappa, v_{\varkappa^*}, \theta) e^{f(v_\varkappa, v_{\varkappa^*}, \theta) - \frac{f\left(\varkappa, v_{\varkappa^*}, \theta\right) + f\left(\varkappa^*, v_{\varkappa^*}, \theta\right)}{2}} = \theta \frac{n^2 - n}{2} e^{\theta \frac{-3n-1}{2}} < \\ &\qquad < \theta \frac{n^2 + 2n + 1}{2} e^{-\theta} = e^{-\theta} \Big[\frac{f(\varkappa, u_{\varkappa^*}, \theta) + f(\varkappa^*, u_\varkappa, \theta)}{2}\Big] \leqslant \\ &\qquad\qquad \leqslant e^{-\theta} M_1(\varkappa, \varkappa^*; v_\varkappa, v_{\varkappa^*}, \theta). \end{aligned}$$

(iii) When $\varkappa = \varkappa_n$, $\varkappa^* = \varkappa_m$ with $m > n > 1$, then for $v_\varkappa = \varkappa_1 \in [T_1(\varkappa)]_{\frac{1}{2}}$ there exists $v_{\varkappa^*} = \varkappa_{n-1} \in [T_2(\varkappa^*)]_{\frac{1}{2}}$, such that

$$
\begin{aligned}
&f(v_\varkappa, v_{\varkappa^*}, \theta) e^{f(v_\varkappa, v_{\varkappa^*}, \theta) - M(\varkappa, \varkappa^*; v_\varkappa, v_{\varkappa *}, \theta)} \leqslant \\
&\quad \leqslant f(v_\varkappa, v_{\varkappa^*}, \theta) e^{f(v_\varkappa, v_{\varkappa^*}, \theta) - f(\varkappa, v_\varkappa, \theta)} = \theta \frac{n^2 - n}{2} e^{-n\theta} < \theta \frac{n^2 + n}{2} e^{-\theta} = \\
&\qquad = e^{-\theta} f(\varkappa, v_\varkappa, \theta) \leqslant e^{-\theta} M_1(\varkappa, \varkappa^*; v_\varkappa, v_{\varkappa^*}, \theta).
\end{aligned}
$$

Now we show that for $\varkappa, \varkappa^* \in \Upsilon$, $v_\varkappa \in [T_2(\varkappa)]_{\frac{1}{2}}$; there exists $v_{\varkappa^*} \in [T_1(\varkappa^*)]_{\frac{1}{2}}$, such that $(v_\varkappa, v_{\varkappa^*}) \in E^*(G)$ and (1) is satisfied. For this, we consider the following cases:

(I) If $\varkappa = \varkappa_n$, $\varkappa^* = \varkappa_1$ with $n > 1$, we have for $v_\varkappa = \varkappa_{n-1} \in [T_2(\varkappa)]_{\frac{1}{2}}$, there exists $v_{\varkappa^*} = \varkappa_1 \in [T_1(\varkappa^*)]_{\frac{1}{2}}$, such that:

$$
\begin{aligned}
&f(v_\varkappa, v_{\varkappa^*}, \theta) e^{f(v_\varkappa, v_{\varkappa^*}, \theta) - M(\varkappa, \varkappa^* v_\varkappa, v_{\varkappa^*}, \theta)} \leqslant \\
&\quad \leqslant f(v_\varkappa, v_{\varkappa^*}, \theta) e^{f(v_\varkappa, v_{\varkappa^*}, \theta) - f(\varkappa, \varkappa^* \theta)} = \theta \frac{n^2 - n}{2} e^{-n\theta} < \theta \frac{n^2 + n}{2} e^{-\theta} = \\
&\qquad = e^{-\theta} f(\varkappa, \varkappa^*, \theta) \leqslant e^{-\theta} M_1(\varkappa, \varkappa^*; v_\varkappa, v_{\varkappa^*}, \theta).
\end{aligned}
$$

(II) In case $\varkappa = \varkappa_n, \varkappa^* = \varkappa_m$ with $m > n > 1$, for $v_\varkappa = \varkappa_{n-1} \in [T_2(\varkappa)]_{\frac{1}{2}}$, there exists $v_{\varkappa^*} = \varkappa_1 \in [T_2(\varkappa^*)]_{\frac{1}{2}}$, such that

$$
\begin{aligned}
&f(u_\varkappa, u_{\varkappa^*}, \theta) e^{f(u_\varkappa, u_{\varkappa^*}, \theta) - M(\varkappa, \varkappa^* u_\varkappa, u_{\varkappa^*}, \theta)} \leqslant \\
&\quad \leqslant f(u_\varkappa, u_{\varkappa^*}, \theta) e^{f(u_\varkappa, u_{\varkappa^*}, \theta) - f(\varkappa, u_{\varkappa^*}, \theta)} = \theta \frac{n^2 - n}{2} e^{\theta(n^2 - n - m^2 - m)} < \\
&\qquad < \theta \frac{m^2 + m}{2} e^{-\theta} = e^{-\theta} f(\varkappa^*, v_{\varkappa^*}, \theta) \leqslant e^{-\theta} M_1(\varkappa, \varkappa^*; v_\varkappa, v_{\varkappa^*}, \theta).
\end{aligned}
$$

Henceforth, for all $\varkappa$, $\varkappa^*$ in $V(G)$, condition (1) is satisfied. Hence, all the requirements of Theorem 2 are satisfied. Moreover, $\{1\}$ is the common fuzzy fixed point of T_1 and T_2 with $Fuz(T_1) = Fuz(T_2)$.

The following result generalizes Theorem 3.4 in [28].

Theorem 3. *Let (Υ, f) be a complete parametric metric space endowed with a directed graph G, such that $V(G) = \Upsilon$ and $E(G) \supseteq \Delta$. If $T_1, T_2 : \Upsilon \to F_{P_{cl}}(\Upsilon)$ form a fuzzy graphic F_2-contraction pair, then the following statements hold:*

(i) *$Fuz(T_1) \neq \varnothing$ or $Fuz(T_2) \neq \varnothing$ if and only if $Fuz(T_1) = Fuz(T_2) \neq \varnothing$.*

(ii) *$\Upsilon_{T_1, T_2} \neq \varnothing$ provided that $Fuz(T_1) \cap Fuz(T_2) \neq \varnothing$.*

(iii) *If* $\Upsilon_{T_1,T_2} \neq \varnothing$ *and* G *is weakly connected, then* $Fuz(T_1) = Fuz(T_2) \neq \varnothing$, *provided that either (a), or* T_1 *or* T_2 *is upper semicontinuous, or (b)* F *is continuous, either* T_1 *or* T_2 *is bounded, and* G *has property (P).*

(iv) $Fuz(T_1) \cap Fuz(T_2)$ *is a clique set of* $\widetilde{G}$ *if and only if* $Fuz\,(T_1) \cap Fuz\,(T_2)$ *is a singleton set.*

Proof. To validate (i), let $\varkappa^* \in [T_1(\varkappa^*)]_\alpha$. Assume $\varkappa^* \notin [T_2\,(\varkappa^*)]_\beta$; then, since (T_1, T_2) form a fuzzy graphic F_2-contraction pair, there exists a $\varkappa \in [T_2\,(\varkappa^*)]_\beta$ with $(\varkappa^*, \varkappa) \in E^*\,(G)$, such that

$$\tau + F\,(f(\varkappa^*, \varkappa, \theta)) \leqslant F(M_2(\varkappa^*, \varkappa^*; \varkappa^*, \varkappa, \theta)),$$

where

$$\begin{aligned} M_2(\varkappa^*, \varkappa^*; \varkappa^*, \varkappa, \theta) &= \alpha_1 f(\varkappa^*, \varkappa^*, \theta) + \beta_1 f(\varkappa^*, \varkappa^*, \theta) + \gamma_1 f(\varkappa, \varkappa^*, \theta) + \\ &+ \delta_1 f(\varkappa^*, \varkappa, \theta) + \delta_2 f(\varkappa^*, \varkappa^*, \theta) = (\gamma_1 + \delta_1) f(\varkappa, \varkappa^*, \theta). \end{aligned}$$

Thus, we have

$$\tau + F\,(f(\varkappa^*, \varkappa, \theta)) \leqslant F((\gamma_1 + \delta_1) f(\varkappa^*, \varkappa, \theta)) \leqslant F(f(\varkappa^*, \varkappa, \theta)),$$

a contradiction as $\tau > 0$. Hence, $\varkappa^* \in [T_2\,(\varkappa^*)]_\beta$ and, so, $Fuz(T_1) \subseteq Fuz(T_2)$. Similarly, $Fuz(T_2) \subseteq Fuz(T_1)$ and, therefore, $Fuz(T_1) = Fuz(T_2)$. Also, if $\varkappa^* \in [T_2(\varkappa^*)]_\beta$, then we have $\varkappa^* \in [T_1(\varkappa^*)]_\alpha$. Converse can be proved by straightforward steps.

To validate (ii), let $Fuz\,(T_1) \cap Fuz\,(T_2) \neq \varnothing$. Then there exists $\varkappa \in \Upsilon$, such that $\varkappa \in [T_1(\varkappa)]_\alpha \cap [T_2(\varkappa)]_\beta$. Since $\Delta \subseteq E(G)$, we conclude that $\Upsilon_{T_1,T_2} \neq \varnothing$.

To validate (iii), suppose that $\varkappa_0$ is an arbitrary point of Υ. For $i, j \in \{1, 2\}$, with $i \neq j$, take $\varkappa_1 \in [T_i(\varkappa_0)]_\alpha$; there exists $\varkappa_2 \in [T_j(\varkappa_1)]_\beta$ with $(\varkappa_1, \varkappa_2) \in E^*\,(G)$, such that

$$\tau + F\,(f(\varkappa_1, \varkappa_2, \theta)) \leqslant F(M_2(\varkappa_0, \varkappa_1; \varkappa_1, \varkappa_2, \theta)),$$

where

$$\begin{aligned} M_2(\varkappa_0, \varkappa_1; \varkappa_1, \varkappa_2, \theta) &= \alpha_1 f(\varkappa_0, \varkappa_1, \theta) + \beta_1 f(\varkappa_0, \varkappa_1, \theta) + \\ &+ \gamma_1 f(\varkappa_1, \varkappa_2, \theta) + +\delta_1 f(\varkappa_0, \varkappa_2, \theta) + \delta_2 f(\varkappa_1, \varkappa_1, \theta) \leqslant \\ &\leqslant (\alpha_1 + \beta_1 + \delta_1) f(\varkappa_0, \varkappa_1, \theta) + (\gamma_1 + \delta_1) f(\varkappa_1, \varkappa_2, \theta). \end{aligned}$$

If $f(\varkappa_0, \varkappa_1, \theta) \leqslant f(\varkappa_1, \varkappa_2, \theta)$, then we have

$$\tau + F(f(\varkappa_1, \varkappa_2, \theta)) \leqslant F((\alpha_1 + \beta_1 + \gamma_1 + 2\delta_1) f(\varkappa_1, \varkappa_2, \theta)) \leqslant F(f(\varkappa_1, \varkappa_2, \theta)),$$

gives a contradiction as $\tau > 0$. Therefore,

$$\tau + F\left(f(\varkappa_1, \varkappa_2, \theta)\right) \leqslant F\left(f(\varkappa_0, \varkappa_1, \theta)\right).$$

Continuing this process, for $\varkappa_{2n} \in [T_j(\varkappa_{2n-1})]_\alpha$, we see that there exists $\varkappa_{2n+1} \in [T_i \varkappa_{2n}]_\beta$, such that for $(\varkappa_{2n}, \varkappa_{2n+1}) \in E^*(G)$, we have

$$\tau + F\left(f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)\right) \leqslant F\left(M_2(\varkappa_{2n-1}, \varkappa_{2n}; \varkappa_{2n}, \varkappa_{2n+1}, \theta)\right),$$

where

$$\begin{aligned} & M_2(\varkappa_{2n-1}, \varkappa_{2n}; \varkappa_{2n}, \varkappa_{2n+1}, \theta) = \\ & \quad = \alpha_1 f(\varkappa_{2n-1}, \varkappa_{2n}, \theta) + \beta_1 f(\varkappa_{2n-1}, \varkappa_{2n}, \theta) + \gamma_1 f(\varkappa_{2n}, \varkappa_{2n+1}, \theta) + \\ & \qquad + \delta_1 f(\varkappa_{2n-1}, \varkappa_{2n+1}, \theta) + \delta_2 f(\varkappa_{2n}, \varkappa_{2n}, \theta) \leqslant \\ & \qquad \leqslant (\alpha_1 + \beta_1 + \delta_1) f(\varkappa_{2n-1}, \varkappa_{2n}, \theta) + (\gamma_1 + \delta_1) f(\varkappa_{2n}, \varkappa_{2n+1}, \theta). \end{aligned}$$

If $f(\varkappa_{2n-1}, \varkappa_{2n}, \theta) \leqslant f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)$, then

$$\begin{aligned} \tau + F\left(f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)\right) & \leqslant F\left((\alpha_1 + \beta_1 + \gamma_1 + 2\delta_1, \theta) f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)\right) \leqslant \\ & \leqslant F\left(f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)\right), \end{aligned}$$

gives a contradiction as $\tau > 0$. Therefore,

$$\tau + F\left(f(x_{2n}, x_{2n+1}, \theta)\right) \leqslant F\left(f(x_{2n-1}, x_{2n}, \theta)\right).$$

In a similar way, for $\varkappa_{2n+1} \in [T_j(\varkappa_{2n})]_\alpha$, there exists $\varkappa_{2n+2} \in [T_i\left(\varkappa_{2n+1}\right)]_\beta$ with $(\varkappa_{2n+1}, \varkappa_{2n+2}) \in E^*(G)$, such that

$$\tau + F\left(f(\varkappa_{2n+1}, \varkappa_{2n+2}, \theta)\right) \leqslant F\left(f(\varkappa_{2n}, \varkappa_{2n+1}, \theta)\right).$$

Hence, we obtain a sequence $\{\varkappa_n\}$ in Υ, such that for $\varkappa_n \in [T_j(\varkappa_{n-1})]_\alpha$, there exists $\varkappa_{n+1} \in [T_i\left(\varkappa_n\right)]_\beta$ with $(\varkappa_n, \varkappa_{n+1}) \in E^*(G)$, such that

$$\tau + F\left(f(\varkappa_n, \varkappa_{n+1}, \theta)\right) \leqslant F\left(f(\varkappa_{n-1}, \varkappa_n, \theta)\right).$$

Therefore,

$$F\left(f(\varkappa_n,\varkappa_{n+1},\theta)\right) \leqslant F\left(f(\varkappa_{n-1},\varkappa_n,\theta)\right)-\tau \leqslant F\left(f(\varkappa_{n-2},\varkappa_{n-1},\theta)\right)-2\tau \leqslant \\ \leqslant \ldots \leqslant F\left(f(\varkappa_0,\varkappa_1,\theta)\right)-n\tau.$$

Consequently, $\lim\limits_{n\to\infty} F\left(f(\varkappa_n,\varkappa_{n+1},\theta)\right) = -\infty$ together with (F_2) gives $\lim\limits_{n\to\infty} f(\varkappa_n,\varkappa_{n+1},\theta) = 0$. Following arguments similar to those in proof of Theorem 2, $\{\varkappa_n\}$ is a Cauchy sequence in Υ. Since Υ is complete, there exists an element $\varkappa^* \in \Upsilon$, such that $\varkappa_n \to \varkappa^*$ as $n \to \infty$.

Now, if T_i is upper semicontinuous, then, as $\varkappa_{2n}\in\Upsilon$, $\varkappa_{2n+1}\in[T_i(\varkappa_{2n})]_\alpha$ with $\varkappa_{2n} \to \varkappa^*$ and $\varkappa_{2n+1} \to \varkappa^*$ as $n \to \infty$ implies that $\varkappa^* \in [T_i(\varkappa^*)]_\beta$. Using (i), we get $\varkappa^* \in [T_i(\varkappa^*)]_\alpha = [T_j(\varkappa^*)]_\beta$. In the same manner, the result holds when T_j is upper semicontinuous.

Assume that F is continuous. Since $\varkappa_{2n}$ converges to $\varkappa^*$ as $n \to \infty$ and $(\varkappa_{2n},\varkappa_{2n+1}) \in E(G)$, we have $(\varkappa_{2n},\varkappa^*) \in E(G)$. For $\varkappa_{2n} \in [T_j(\varkappa_{2n-1})]_\alpha$, there exists $v_n \in [T_i(\varkappa^*)]_\beta$, such that $(\varkappa_{2n}, v_n) \in E^*(G)$. As $\{v_n\}$ is bounded, $\limsup\limits_{n\to\infty} v_n = v^*$, and $\liminf\limits_{n\to\infty} v_n = v^*$ both exist. Assume that $v^* \neq \varkappa^*$. Since (T_1,T_2) form a fuzzy graphic F_2-contraction,

$$\tau + F\left(f(\varkappa_{2n},v_n,\theta)\right) \leqslant F(M_2(\varkappa_{2n-1},\varkappa^*;\varkappa_{2n},v_n,\theta)),$$

where

$$M_2(\varkappa_{2n-1},\varkappa^*;\varkappa_{2n},v_n,\theta) = \alpha f(\varkappa_{2n-1},\varkappa^*,\theta) + \beta\rho(\varkappa_{2n-1},\varkappa_{2n},\theta)+ \\ + \gamma f(\varkappa^*,v_n,\theta) + \delta_1 f(\varkappa_{2n-1},v_n,\theta) + \delta_2 f(\varkappa^*,\varkappa_{2n},\theta).$$

Taking $\limsup$ implies

$$\tau + F\left(f(\varkappa^*,v^*,\theta)\right) \leqslant F((\gamma+\delta_1)f(\varkappa^*,v^*,\theta)) \leqslant F(f(\varkappa^*,v^*,\theta)),$$

a contradiction. Hence, $v^* = \varkappa^*$. Similarly, taking the $\liminf$ gives $v^* = \varkappa^*$. Since $v_n \in T_i(\varkappa^*)$ for all $n \geqslant 1$ and $[T_i(\varkappa^*)]_\alpha$ is a closed set, it follows that $\varkappa^* \in [T_i(\varkappa^*)]_\alpha$. Now, from (i) we get $\varkappa^* \in [T_i(\varkappa^*)]_\alpha$ and, hence, $Fuz(T_1) = Fuz(T_2)$.

Finally, to validate (iv), assume that the set $Fuz(T_1) \cap Fuz(T_2)$ is a clique of $\widetilde{G}$. We are to prove that $Fuz(T_1)\cap Fuz(T_2)$ is singleton. Suppose the contrary: that there exist v and u, such that $u, v \in Fuz(T_1) \cap Fuz(T_2)$ but $u \neq v$. As $(v,u) \in E^*(G)$, and T_1, and T_2 form a graphic F_2- contraction, so, for $(v_a,u_b) \in E^*(G)$, we have:

$$\begin{aligned}\tau + F\left(f(v,u,\theta)\right) \leqslant F(M_2(v,u;v,u,\theta)) = \\ = F(\alpha_1 f(v,u,\theta) + \beta_1 f(v,v,\theta) + \gamma_1 f(u,u,\theta) + \\ + \delta_1 f\left(v,u,\theta\right) + \delta_2 f\left(u,v,\theta\right)) = \\ = F\left((\alpha_1+\delta_1+\delta_2) f(v,u,\theta)\right) \leqslant F\left(f\left(v,u,\theta\right)\right),\end{aligned}$$

a contradiction as $\tau > 0$. Hence, $v = u$. Conversely, if $Fuz(T_1) \cap Fuz(T_2)$ is singleton, then it follows that $Fuz(T_1) \cap Fuz(T_2)$ is a clique of $\widetilde{G}$. □

Remark 1. *Let (Υ, f) be a complete parametric metric space enriched with a directed graph G. If we replace (2) by either of the following three conditions:*

$$\tau + F\left(f(v_a, v_b, \theta)\right) \leqslant F(\alpha_1 f(a,b,\theta) + \beta_1 f(a, v_b, \theta) + \gamma_1 f(b, v_b, \theta)), \quad (4)$$

where $\alpha_1, \beta_1, \gamma_1 \geqslant 0$ and $\alpha_1 + \beta_1 + \gamma_1 \leqslant 1$, or

$$\tau + F\left(f(v_a, v_b, \theta)\right) \leqslant F(h[f(a, v_b, \theta) + f(b, v_b, \theta)]), \quad (5)$$

where $h \in [0, \frac{1}{2}]$, or

$$\tau + F\left(f(v_a, v_b, \theta)\right) \leqslant F(f(a,b,\theta)). \quad (6)$$

Then the deductions acquired in Theorem 3 remain true.

Remark 2.

1) *If $E(G) := \Upsilon \times \Upsilon$, then, clearly, G is connected and our Theorem 2 improves and generalizes (i) Theorem 2.1 in [2], (ii) Theorem 1.9 in [4], (iii) Theorem 4.1 in [22], (iv) Theorem 3.4 of [28], and (v) Theorem 3.1 of [29].*

2) *If $E(G) := \Upsilon \times \Upsilon$, then Theorem 3 improves and extends Theorem 3.4 in [28], and Theorem 3.4 in [29].*

3) *If $E(G) := \Upsilon \times \Upsilon$, then our Remark 1 extends and generalizes (i) Theorem 3.4 in [28] and (ii) Theorem 4.1 of [22].*

4) *If $E(G) := \Upsilon \times, \Upsilon$ then our Remark 1 improves and generalizes Theorem 4.1 in [22].*

5) *If we take $T_1 = T_2$ in graphic F_1-contraction pair and graphic F_2-contraction pair, then we obtain the fixed point results for graphic F_1-contraction and graphic F_2-contraction of a single map.*

4. Problem Statement. The results of this paper expand the common fuzzy fixed point theory of multivalued mappings by incorporating fuzzy graphic F-contractions. The concepts discussed within the context of parametric metric spaces are foundational. Therefore, these ideas can be enhanced when applied to more generalized metric structures, such as b-metric spaces, controlled metric spaces, semi-metric spaces, and quasi-metric spaces. Additionally, the component of the fuzzy multivalued map can be extended to include L-fuzzy mappings, intuitionistic fuzzy mappings, soft multivalued maps, and others.

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