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COMMON FIXED POINT IN G-METRIC SPACES VIA GENERALIZED Γ-C_F-SIMULATION FUNCTION

Abstract. We present the generalized Γ-C_F-simulation function and establish the common fixed point result for weak (η_F, g)-contraction in complete G-metric space. The exploration extends to its ramifications on both quasi-metric spaces and metric spaces. The study explores the existence of a solution for a non-linear integral equation as an application of these results.

Key words: *Γ-C_F-simulation functions, G-metric spaces, quasi-metric spaces, weak contraction, common fixed point*

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1. Introduction. Expanding the Banach fixed point theorem to G-metric spaces marks a significant advancement in mathematical analysis. These extensions often entail adjusting the contraction condition to suit the properties of G-metrics. Since Samet et al.'s work [14], it has been recognized that G-metric spaces possess a quasi-metric structure. As a result, many fixed-point theorems established within the domain of G-metric spaces can be inferred from existing results in (quasi-)metric spaces. Specifically, when the contraction condition in a fixed-point theorem for a G-metric space can be simplified to involve only two variables instead of three, it becomes feasible to establish analogous fixed-point results in a metric space.

Further, Khojasteh et al. [9] introduced the notion of simulation functions in order to express different contractivity conditions in a simple, unified, coherent manner. By employing a unified language through simulation functions, researchers can convey and analyze a wide range of contractive mappings using a common set of principles. Later, this principle has been extended in various directions (see [1], [6], [8], [10], [11], [12], [13]).

In this context, we introduce the generalized $\Gamma - C_F$-simulation function by employing the Γ-C-class functions [11]. Additionally, we define weak contraction and establish a common fixed point result applicable to G-metric spaces, along with its implications for quasi-metric spaces and metric spaces. This flexibility is crucial in addressing diverse mathematical problems and adapting to various settings, allowing researchers to tailor contractivity conditions to specific needs. Finally, we apply the derived fixed-point result to solve a specific type of integral equation.

2. Preliminaries. Let us recollect some basic definitions and results for G-metric space.

Definition 1. *[15] Let X be a nonempty set, $G\colon X \times X \times X \to [0, +\infty)$ be a function satisfying the following properties:*

(G_1) $G(x, y, z) = 0$, *if* $x = y = z$,

(G_2) $G(x, x, y) > 0$, *for all* $x, y \in X$ *with* $x \neq y$,

(G_3) $G(x, x, y) \leqslant G(x, y, z)$, *for all* $x, y, z \in X$ *with* $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ *(symmetry in all three variables),*

(G_5) $G(x, y, z) \leqslant G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality).*

The function G is called G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2. [15] A G-metric space (X, G) is said to be symmetric if

$$G(x, y, y) = G(y, x, x), \text{ for all } x, y \in X.$$

Lemma 1. [15] If (X, G) is a G-metric space, then

$$G(x, y, y) \leqslant 2G(y, x, x), \text{ for all } x, y \in X.$$

Proposition 1. [15] Let (X, G) be a G-metric space, $\{x_n\} \subseteq X$ be a sequence, and $x \in X$. Then

(i) $\{x_n\}$ G-converges to $x \iff \lim\limits_{n\to+\infty} G(x_n, x_n, x) = 0 \iff \lim\limits_{n\to+\infty} G(x_n, x, x) = 0$.

(ii) $\{x_n\}$ is G-Cauchy $\iff \lim\limits_{n,m\to+\infty} G(x_n, x_m, x_m) = 0$.

(iii) (X, G) is complete if every G-Cauchy sequence in X is G-convergent in X.

Definition 3. [15] Let (X, G) be a G-metric space. We say that a mapping $T: X \to X$ is G-continuous at $x \in X$ if $\{Tx_m\} \to Tx$ for all sequence $\{x_m\} \subseteq X$ such that $\{x_m\} \to x$.

Definition 4. [2] A sequence $\{x_n\}$ in a G-metric space (X, G) is asymptotically regular if $\lim_{n\to+\infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

Barinde [5] introduced asymptotic regularity for two operators in metric spaces, which can be extended to G-metric spaces as follows:

Definition 5. *Let (X, G) be a G-metric space and $f, g: X \to X$ be two operators. Then the operator g is called f-asymptotically regular in (X, G) if*

$$G(g^n(x), f(g^n(x)), f(g^n(x))) \to 0 \text{ as } n \to +\infty, \text{ for all } x \in X.$$

Let $\Gamma([0, +\infty))$ be the set of all non-decreasing functions $\gamma: [0, +\infty) \to [0, +\infty)$, such that $\gamma(t) = 0$ if and only if $t = 0$.

Definition 6. [11] A function $F: [0, +\infty)^2 \to \mathbb{R}$ is called Γ-C-class function if it is continuous and there exists $\gamma \in \Gamma([0, +\infty))$, such that:

(i) $F(s, t) \leqslant \gamma(s)$;

(ii) $F(s, t) = \gamma(s)$ implies that either $s = 0$ or $t = 0$ for all $s, t \geqslant 0$.

The collection of all Γ-C-class functions is denoted by $\mathcal{C}_\Gamma$. For $\gamma(t) = t$, the Γ-C-class function reduces to C-class function of [3].

Definition 7. [11] A function $F: [0, +\infty)^2 \to \mathbb{R}$ has the property Γ-C_F, if there exists $\gamma \in \Gamma([0, +\infty))$ and $C_F \geqslant 0$, such that:

(F_1) $F(s, t) > C_F$ implies $\gamma(s) > \gamma(t)$, for all $s, t \geqslant 0$;

(F_2) $F(t, t) \leqslant C_F$, for all $t \geqslant 0$.

Example 1. The following functions $F_i: [0, +\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}_\Gamma$ with property Γ-C_F:

(i) $F_1(s, t) = \dfrac{\gamma(s)}{1 + \gamma(t)}$, $C_F = 1, 2$.

(ii) $F_2(s, t) = \dfrac{\gamma(s)}{(1 + \gamma(t))^r}$, $r \in (0, +\infty)$, $C_F = 1$.

3. Main results. In this section, we introduce generalized Γ-C_F-simulation function using Γ-C-class functions. Subsequently, the conditions for the existence and uniqueness of a common fixed point result for weak contractions by using the generalized Γ-C_F-simulation function are established.

Definition 8. *A function $\eta\colon [0,+\infty)\times[0,+\infty)\to\mathbb{R}$ is a (generalized) Γ-C_F-simulation function of type II, if*

(η_1) There exists $C_F \geqslant 0$, such that

$$\text{if } \eta(t,s)\geqslant C_F \text{ then } \eta(t,s)\leqslant F(s,t), \text{ for all } s,t\geqslant 0,$$

where $F\in\mathcal{C}_\Gamma$ with property $\Gamma - C_F$.

(η_2) If $\{t_n\}$ and $\{s_n\}$ are non increasing sequences in $(0,+\infty)$ and $\eta(t_n,s_n)\geqslant C_F$, then

$$\lim_{n\to+\infty}\eta(t_n,s_n)\to C_F \text{ implies } s_n\to 0.$$

We say that η is a (generalized) Γ-C_F-simulation function of type I, if it satisfies (η_1) and the following $(\eta_2)^*$ condition:

$(\eta_2)^*$ If $\{t_n\}$ and $\{s_n\}$ are non increasing sequences in $(0,+\infty)$, such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n > 0$, then $\limsup_{n\to+\infty}\eta(t_n,s_n) < C_F$.

Remark 1.

(i) Every simulation function is a (generalized) Γ-C_F-simulation function of type I.
It follows from definition 8 for $C_F = 0$ and $F(s,t)=\gamma(s)-\gamma(t)$ and $\gamma(t)=t$.

(ii) Also, condition $(\eta_2)^*$ is different from (η_2).

Now, we introduce weak (η_F, g)-contraction for G-metric spaces.

Definition 9. *Let (X,G) be a G-metric space and f,g be self mappings on X. For a function $\eta\colon[0,+\infty)\times[0,+\infty)\to\mathbb{R}$, f is called*

(i) an (η_F, g)-contraction if

$$\eta(G(fx,gy,gy),G(x,y,y))\geqslant C_F, \text{ for all } x,y\in X, \tag{1}$$

$$\eta(G(gx,fy,fy),G(x,y,y))\geqslant C_F, \text{ for all } x,y\in X, \tag{2}$$

(ii) a weak (η_F,g)-contraction if

$$\eta(G(fx,gfx,gfx),G(x,fx,fx))\geqslant C_F, \text{ for all } x\in X, \tag{3}$$

$$\eta(G(gx,fgx,fgx),G(x,gx,gx))\geqslant C_F, \text{ for all } x\in X, \tag{4}$$

(iii) *a generalize weak non-expansive map if*

$$G(fx, gfx, gfx) \leqslant G(x, fx, fx), \text{ for all } x \in X, \quad (5)$$

$$G(gx, fgx, fgx) \leqslant G(x, gx, gx), \text{ for all } x \in X. \quad (6)$$

For $g = f$ *in* (1)–(6), *we get the following contractions:*
A mapping f *is called*

(a) *an* η_F*-contraction if*

$$\eta(G(fx, fy, fy), G(x, y, y)) \geqslant C_F, \text{ for all } x, y \in X, \quad (7)$$

(b) *a weak* η_F*-contraction if*

$$\eta(G(fx, f^2x, f^2x), G(x, fx, fx)) \geqslant C_F, \text{ for all } x \in X, \quad (8)$$

(c) *a weak non-expansive map if*

$$G(fx, f^2x, f^2x) \leqslant G(x, fx, fx), \text{ for all } x \in X. \quad (9)$$

Remark 2. *In definition 9, for* $d_G(x, y) = G(x, y, y)$, *an* (η_F, g)*-contraction for* G*-metric spaces reduced to* (η_F, g)*-contraction for quasi-metric spaces* (X, d_G).

Now, we prove common fixed point result for the pair of mappings in G-metric spaces.

Theorem 1. *Let* (X, G) *be a complete* G*-metric space,* f *and* g *be self mappings on* X *and* $\eta\colon [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ *be a function.*

(i) *Let* f *be an* (η_F, g)*-contraction. If* η *satisfies* (η_1), *then* f *and* g *have at most one common fixed point. Also, if* $\gamma \in \Gamma([0, +\infty))$, *then*

$$G(fx, gy, gy) < G(x, y, y), \text{ for all } x \neq y.$$

(ii) *Let* η *be a* Γ*-*C_F*-simulation function of type II; if* $f(gf)^{n_0}$ *and* $(gf)^{n_0}, n_0 \in \mathbb{N}$ *is a weak* (η_F, g)*-contraction, then* f *is* g*-asymptotically regular. The same result holds if* η *is a* Γ*-*C_F*-simulation function of type I and* f *is a generalized weak non-expansive map.*

(iii) *Let* f *be an* (η_F, g)*-contraction with* f *or* g *continuous, and* η *be a* Γ*-*C_F*-simulation function of type II (or type I with* f *being a generalized weak non-expansive map), then* f *and* g *have a unique common fixed point.*

Proof.

(i) Suppose that $gx = fx = x$, $gy = fy = y$ and $x \neq y$; then $G(x, y, y) = G(fx, gy, gy) = t(say) > 0$.

From (η_1) and (F_2), we get

$$\eta(t,t) \geqslant C_F \implies \eta(t,t) < F(t,t) \leqslant C_F,$$

which is a contradiction. Hence, common fixed point of f and g is unique if exists.

Suppose that $0 < s = G(x, y, y) \leqslant t = G(fx, gy, gy)$, where $x \neq y$. From (1) and (η_1), we have

$$C_F \leqslant \eta(G(fx, gy, gy), G(x, y, y)) < F(G(x, y, y), G(fx, gy, gy)).$$

From (F_1), we get

$$\gamma(G(fx, gy, gy)) < \gamma(G(x, y, y)).$$

Since γ is non-decreasing, $G(fx, gy, gy) < G(x, y, y)$, which is a contradiction. Hence, $G(fx, gy, gy) < G(x, y, y)$.

(ii) For any fixed x_0 in X, construct a sequence $\{x_n\}$ with

$$x_{2n} = (gf)^n(x_0),\ \ x_{2n+1} = f(x_{2n}),\ \text{ for all } n \geqslant 0.$$

Let $t_i = G(x_i, x_{i+1}, x_{i+1})$ for all $i \geqslant 0$. Suppose $t_k = 0$, for some $k \in \mathbb{N}$.
If $x_{2k} = x_{2k+1}$, then x_{2k} is a fixed point of f.
If $x_{2k+1} = x_{2k+2}$, then x_{2k+1} is a fixed point of g.
Thus, at least one mapping of f or g has a fixed point.

Now, assume that $t_k \neq 0$, for all $k \geqslant 0$. Put $x = x_{2n_0+2k} = (gf)^{n_0+k}(x_0)$, $k = 0, 1, \ldots$ in (3) to get

$$\begin{aligned}
C_F &\leqslant \eta\big(G(fx_{2n_0+2k}, gfx_{2n_0+2k}, gfx_{2n_0+2k}), \\
&\quad G(x_{2n_0+2k}, fx_{2n_0+2k}, fx_{2n_0+2k})\big) = \\
&= \eta\big(G(x_{2n_0+2k+1}, x_{2n_0+2k+2}, x_{2n_0+2k+2}), \\
&\quad G(x_{2n_0+2k}, x_{2n_0+2k+1}, x_{2n_0+2k+1})\big) = \\
&= \eta(t_{2n_0+2k+1}, t_{2n_0+2k}) < \\
&< F(t_{2n_0+2k}, t_{2n_0+2k+1}). \qquad (10)
\end{aligned}$$

Put $x = x_{2n_0+2k+1} = f(gf)^{n_0+k}(x_0), k = 0, 1, \ldots$ in (4) to get

$$C_F \leqslant \eta(G(gx_{2n_0+2k+1}, fgx_{2n_0+2k+1}, fgx_{2n_0+2k+1}),$$

$$\begin{aligned}
G(x_{2n_0+2k+1}, gx_{2n_0+2k+1}, gx_{2n_0+2k+1})) = \\
&= \eta(G(x_{2n_0+2k+2}, x_{2n_0+2k+3}, x_{2n_0+2k+3}), \\
G(x_{2n_0+2k+1}, x_{2n_0+2k+2}, x_{2n_0+2k+2})) = \\
&= \eta(t_{2n_0+2k+2}, t_{2n_0+2k+1}) < \\
&< F(t_{2n_0+2k+1}, t_{2n_0+2k+2}). \qquad (11)
\end{aligned}$$

From (10) and (11), we get

$$C_F \leqslant \eta(t_{i+1}, t_i) < F(t_i, t_{i+1}), \text{ for all } i \geqslant n_0. \qquad (12)$$

From (F_1), we get $\gamma(t_{i+1}) < \gamma(t_i)$. Since, γ is non-decreasing $t_{i+1} < t_i$, that is,

$$G(x_{i+1}, x_{i+2}, x_{i+2}) < G(x_i, x_{i+1}, x_{i+1}), \text{ for all } i \geqslant n_0.$$

Hence $\{G(x_i, x_{i+1}, x_{i+1})\}$ is monotonically decreasing sequence of non-negative real numbers. Thus, there exists $r \geqslant 0$, such that $\lim\limits_{i\to+\infty} G(x_i, x_{i+1}, x_{i+1}) = r$.

Let us prove that $r = 0$. Suppose, on the contrary, that $r > 0$. Taking limit as $i \to +\infty$ in (12) and using (F_2), we get

$$C_F \leqslant \lim_{i\to+\infty} \eta(t_{i+1}, t_i) \leqslant F(\lim_{i\to+\infty} t_i, \lim_{i\to+\infty} t_{i+1}) = F(r, r) \leqslant C_F.$$

Hence,

$$\lim_{i\to+\infty} \eta(t_{i+1}, t_i) = C_F. \qquad (13)$$

Type II: From (η_2), we get $r = \lim\limits_{i\to+\infty} t_i = 0$: a contradiction.

Type I: From (5) and (6), we have $t_{i+1} \leqslant t_i$, for all $i \geqslant 0$. Using $(\eta_2)^*$, we get $\limsup\limits_{i\to+\infty} \eta(t_{i+1}, t_i) < C_F$: a contradiction to (13). Hence, $r = 0$. Therefore,

$$\lim_{i\to+\infty} G(x_i, x_{i+1}, x_{i+1}) = 0. \qquad (14)$$

Since $G(x_i, x_i, x_{i+1}) \leqslant 2G(x_i, x_{i+1}, x_{i+1})$, we get

$$\lim_{i\to+\infty} G(x_i, x_i, x_{i+1}) = 0. \qquad (15)$$

(iii) We now show that $\{x_n\}$ is a Cauchy sequence. It is sufficient to show that $\{x_{2n}\}$ is Cauchy in X. On the contrary, assume that $\{x_{2n}\}$ is not Cauchy. Then, from Lemma 4.1.5 in [2], there exists $\varepsilon > 0$ and two subsequences $\{x_{2n(k)}\}$ and $\{x_{2m(k)}\}$ of $\{x_{2n}\}$, such that, for all $k \in \mathbb{N}$, $k \leqslant n(k) < m(k) < n(k+1)$ and for all given $p_1, p_2, p_3 \in \mathbb{Z}$,

$$\lim_{k \to +\infty} G(x_{2n(k)+p_1}, x_{2m(k)+p_2}, x_{2m(k)+p_3}) = \varepsilon. \tag{16}$$

Considering two non-increasing subequences

$$a_l = G(x_{2n(k)(l)}, x_{2m(k)(l)}, x_{2m(k)(l)})$$

and

$$a'_l = G(x_{2n(k)(l)+2}, x_{2m(k)(l)+2}, x_{2m(k)(l)+2})$$

of $G(x_{2n(k)}, x_{2m(k)}, x_{2m(k)})$ and $G(x_{2n(k)+2}, x_{2m(k)+2}, x_{2m(k)+2})$, such that

$$\lim_{l \to +\infty} a_l = \lim_{l \to +\infty} a'_l = \varepsilon. \tag{17}$$

From (1) and (η_1), we have

$$C_F \leqslant \eta(a'_l, a_l) < F(a_l, a'_l).$$

Letting $l \to +\infty$, we get

$$C_F \leqslant \lim_{l \to +\infty} \eta(a'_l, a_l) \leqslant F(\lim_{l \to +\infty} a_l, \lim_{l \to +\infty} a'_l) = F(\varepsilon, \varepsilon) \leqslant C_F.$$

This implies

$$\lim_{l \to +\infty} \eta(a'_l, a_l) = C_F. \tag{18}$$

Type II: From (η_2), $\lim_{l \to +\infty} a_l = 0$: a contradiction to (17).
Type I: From $(\eta_2)^*$, we get $\limsup_{l \to +\infty} \eta(a_l, a'_l) < C_F$: a contradiction to (18).
Thus $\{x_{2n}\}$ is a Cauchy sequence in (X, G). Hence, $\{x_n\}$ is Cauchy in (X, G). Since (X, G) is complete, $x_n \to u \in X$, implies that

$$\lim_{n \to +\infty} x_{2n} = \lim_{n \to +\infty} x_{2n+1} = u.$$

Assume f is continuous; then $\lim_{n \to +\infty} f x_{2n} = \lim_{n \to +\infty} x_{2n+1} = fu$. This implies that $fu = u$. From (1), we have

$$\begin{aligned} C_F &\leqslant \eta(G(fu, gfu, gfu), G(u, fu, fu)) = \\ &= \eta(G(u, gu, gu), G(u, u, u)) < \\ &< F(G(u, u, u), G(u, gu, gu)). \end{aligned}$$

This implies that $0 \leqslant \gamma(G(u, gu, gu)) < \gamma(G(u, u, u)) = \gamma(0) = 0$. Hence, $G(u, gu, gu) = 0$, and so $gu = u$. The uniqueness follows from part (i). □

The following example validates our result.

Example 2. Let $X = [0, 1]$. Define $G\colon X^3 \to [0, +\infty)$ as

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases} \tag{19}$$

Then (X, G) is a complete G-metric space. Define $f, g\colon X \to X$ as $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{x}{2}$, $\forall x \in X$. Also define $\gamma\colon [0, +\infty) \to [0, +\infty)$ by

$$\gamma(t) = \begin{cases} t, & \text{if } 0 \leqslant t < 1, \\ 2t, & \text{if } 1 \leqslant t, \end{cases}$$

and $\eta\colon [0, +\infty)^2 \to \mathbb{R}$ by

$$\eta(t, s) = \frac{\gamma(s)}{1 + \gamma(s)} - \gamma(t), \text{ for all } t, s \in [0, +\infty).$$

Take $F(s, t) = \gamma(s) - \gamma(t)$ with $C_F = 0$, for all $s, t \in [0, +\infty)$. Then η is a (generalized) Γ-C_F-simulation function of type I and all the conditions of Theorem 1 are satisfied, and $x = 0$ is the unique common fixed point of f and g.

4. From G-metric space to quasi-metric space and metric space. We recollect some basic definitions and results for quasi-metric spaces.

Definition 10. *[7] Let X be a non-empty set and let $d\colon X \times X \to [0, +\infty)$ be a function, such that*

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) \leqslant d(x, z) + d(z, y)$, for any points $x, y, z \in X$.

Then d is called a quasi-metric on X and the pair (X, d) is called a quasi-metric space.

Definition 11. *[7] Let (X, d) be a quasi-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is*

- *left-Cauchy if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$, such that $d(x_n, x_m) < \varepsilon$ for all $n \geqslant m > N$.*

- *right-Cauchy if and only if for every* $\varepsilon > 0$*, there exists a positive integer* $N = N(\varepsilon)$*, such that* $d(x_n, x_m) < \varepsilon$ *for all* $m \geqslant n > N$*.*
- *Cauchy if and only if for every* $\varepsilon > 0$*, there exists a positive integer* $N = N(\varepsilon)$*, such that* $d(x_n, x_m) < \varepsilon$ *for all* $m, n > N$*, that is, a sequence* $\{x_n\}$ *in a quasi-metric space is Cauchy if and only if it is both left-Cauchy and right-Cauchy.*

Jleli and Samet [7] gave the following results:

Theorem 2. *Let* (X, G) *be a* G*-metric space. Let* $d_G \colon X \times X \to [0, +\infty)$ *be the function defined by* $d_G(x, y) = G(x, y, y)$*. Then*

(1) (X, d_G) *is a quasi-metric space;*

(2) $\{x_n\} \subset X$ *is* G*-convergent to* $x \in X$ *if and only if* $\{x_n\}$ *is convergent to* x *in* (X, d_G)*;*

(3) $\{x_n\} \subset X$ *is* G*-Cauchy if and only if* $\{x_n\}$ *is Cauchy in* (X, d_G)*;*

(4) (X, G) *is* G*-complete if and only if* (X, d_G) *is complete.*

Definition 12. *[7] Let* (X, d) *be a quasi-metric space. We say that* (X, d) *is complete if and only if each Cauchy sequence in* X *is convergent.*

Asymptotic regularity for two operators for quasi-metric spaces is defined as follows:

Definition 13. *Let* (X, d) *be a quasi-metric space and* $f, g \colon X \to X$ *be two operators. The operator* g *is called* f*-asymptotically regular on* X *if*

$$\lim_{n\to+\infty} d(g^n(x), f(g^n(x))) = 0 = \lim_{n\to+\infty} d(f(g^n(x)), g^n(x)), \text{ for all } x \in X.$$

Theorem 1 in context of quasi-metric spaces is stated as follows. For proving the following result in quasi-metric spaces, we need contractive conditions:

$$\eta(d(fx, gfx), d(x, fx)) \geqslant C_F,$$
$$\eta(d(gx, fgx), d(x, gx)) \geqslant C_F, \quad \text{for all } x \in X,$$

and two more contractive conditions got by changing the order of $d(x, y)$. But here, we can directly derive the result from G-metric space without changing the order of $d(x, y)$ in the contractivity conditions.

Theorem 3. *Let* (X, d) *be a complete metric space,* f *and* g *be self mappings on* X*, and* $\eta \colon [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ *be a function.*

(i) *Let f be an (η_F, g)-contraction. If η satisfies (η_1), then f and g have at most one common fixed point (if any).*
Also, if $\gamma \in \Gamma([0, +\infty))$, then

$$d(fx, gy) < d(x, y), \text{ for all } x \neq y.$$

(ii) *Let η be a $\Gamma - C_F$-simulation function of type II. If $f(gf)^{n_0}$ and $(gf)^{n_0}$, $n_0 \in \mathbb{N}$ is a weak (η_F, g)-contraction, then f is g-asymptotically regular. The same result holds if η is a Γ-C_F-simulation function of type I and f is a generalized weak non-expansive map.*

(iii) *Let f be an (η_F, g)-contraction with f or g continuous, and η be a Γ-C_F-simulation function of type II (or type I, then f is generalized weak non-expansive map). Then f and g have a unique common fixed point.*

Proof. In Theorem 1, take $d_G(x, y) = G(x, y, y)$; then result follows from Theorem 2. □

Theorem 3 is also valid in the context of metric spaces.

Corollary 1. *Let (X, d) be a complete metric space, f be a self mapping on X, and $\zeta\colon [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ be a function.*

(i) *Let f be an ζ-contraction. If ζ satisfies (ζ_1), then f has at most one common fixed point (if any).*
Also, if $\gamma \in \Gamma([0, + \infty))$, then

$$d(fx,fy) < d(x,y), \text{ for all } x \neq y.$$

(ii) *Let ζ be a simulation function of type II; if f^{n_0}, $n_0 \in \mathbb{N}$ is a weak ζ-contraction, then f is asymptotically regular. The same result holds if ζ is a simulation function of type I and f is a weak non-expansive map.*

(iii) *Let f be an ζ-contraction with f continuous, and ζ be a simulation function of type II (or type I, then f is weak non-expansive map). Then f has a unique fixed point.*

Proof. In Theorem 3, take $g = f$, $F(s, t) = \gamma(s) - \gamma(t), \gamma(t) = t$, and $C_F = 0$; then (η_F, g)-contraction reduces to ζ-contraction [9]. □

5. An application. In this section, we present an application of Theorem 1: we guarantee the existence of a solution to an integral equation.

Let $X = C[0,1]$ be the set of all continuous functions defined on [0,1] and let $G\colon X \times X \times X \to \mathbb{R}$ be defined by

$$G(x,y,z) = \sup_{t\in[0,1]} \mid x(t) - y(t) \mid + \sup_{t\in[0,1]} \mid y(t) - z(t) \mid + \sup_{t\in[0,1]} \mid z(t) - x(t) \mid .$$

Then (X,G) is a complete G-metric space. Consider the integral equation:

$$x(t) = \int_0^1 H(t,s)K\big(s, T(x(s))\big)ds, \tag{20}$$

where $H\colon [0,1] \times [0,1] \to \mathbb{R}^+$ and $K\colon [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions and $T\colon X \to X$ is a self mapping on X.

Now we present the following theorem:

Theorem 4. *Suppose the following assumptions hold:*

(1) for all $s \in [0,1]$ and $x, y \in X$, we have

$$\mid K\big(s,x) - K(s,y)\big) \mid \leqslant \mid x - y \mid;$$

(2) for all $t, s \in [0,1]$, we have

$$\sup_{t\in[0,1]} \int_0^1 H(t,s)ds = \frac{1}{4}.$$

Then the integral equation (20) has a solution.

Proof. Let $T\colon X \to X$ be a mapping defined by

$$T(x(t)) = \int_0^1 H(t,s)K\big(s, x(s)\big)ds, \quad t \in [0,1],\ x \in X.$$

From condition (1) and (2), we have

$$G(fx, gfx, gfx) = 2 \sup_{t\in[0,1]} \mid f(x(t)) - gf(x(t)) \mid =$$

$$= 2 \sup_{t\in[0,1]} \mid \int_0^1 H(t,s)K\big(s, x(s)\big)ds - \int_0^1 H(t,s)K\big(s, f(x(s))\big)ds \mid \leqslant$$

$$\leqslant 2 \sup_{t\in[0,1]} \int_0^1 H(t,s) \mid K\big(s,x(s)\big) - K\big(s,f(x(s))\big) \mid ds \leqslant$$

$$\leqslant 2 \sup_{t\in[0,1]} \int_0^1 H(t,s) \mid x(s) - f(x(s)) \mid ds \leqslant$$

$$\leqslant G(x,fx,fx) \sup_{t\in[0,1]} \int_0^1 H(t,s)ds.$$

So, we get

$$G(fx,gfx,gfx) \leqslant \frac{1}{4}G(x,fx,fx). \tag{21}$$

Let $\eta(t,s) = \frac{1}{4}\gamma(s) - \gamma(t)$, $F(s,t) = \gamma(s) - \gamma(t)$ for all $s,t \in [0,+\infty)$, $C_F = 0$, $\gamma(t) = 2t$ for all $t \in [0,+\infty)$.
Now,

$$\eta\big(G(fx,gfx,gfx),G(x,fx,fx)\big) =$$
$$= \frac{1}{4}\gamma\big(G(x,fx,fx)\big) - \gamma\big(G(fx,gfx,gfx)\big) =$$
$$= \frac{1}{4}\big(2G(x,fx,fx)\big) - 2G(fx,gfx,gfx).$$

Then, from (21), we have

$$\eta\Big(G(fx,gfx,gfx),G(x,fx,fx)\Big) \geqslant 0.$$

Thus all the conditions of Theorem 3 are satisfied and hence f and g have a unique common fixed point $x \in X$. Thus x is a solution of the integral equation (20). □

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