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INTEGRAL MEAN ESTIMATE FOR POLYNOMIALS WITH RESTRICTED ZEROS

Abstract. In this paper, we present certain sharp L^p-inequalities for polynomials with restricted zeros. Our results improve and generalize some known integral inequalities for polynomials in the complex domain.

Key words: *polynomials, inequalities, complex domain*

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1. Introduction and statements of the main results. Let $\mathscr{P}_n$ denote the set of all complex polynomials $P(z) = \sum_{j=0}^{n} b_j z^j$ of degree n. The subset $\mathscr{P}_n^0(\rho)$ consists of polynomials whose zeros all lie within the disk defined by $|z| \leqslant \rho$. Specifically, $\mathscr{P}_n^0 = \mathscr{P}_n^0(1)$ represents polynomials with zeros inside the unit disk. The set $\mathscr{P}_n^\infty$ includes polynomials whose zeros are located in the region $|z| \geqslant 1$.

For a polynomial $P \in \mathscr{P}_n$, the p-norm in the Hardy space is defined as

$$\|P\|_p = \Big(\frac{1}{2\pi}\int\limits_0^{2\pi} |P(e^{i\theta})|^p \, d\theta\Big)^{1/p}, \quad 0 < p < \infty.$$

It is not hard to observe that $\lim\limits_{p\to\infty} \|P\|_p = \max\limits_{|z|=1} |P(z)|$. For this reason, the uniform norm $\max\limits_{|z|=1} |P(z)|$ of $P(z)$ is denoted by $\|P\|_\infty$. On the other hand, $\lim\limits_{p\to 0^+} \|P\|_p = \exp\Big(\frac{1}{2\pi}\int\limits_0^{2\pi} \ln|P(e^{i\theta})| \, d\theta\Big)$ (see [15, p. 139], [21]). This is known as the Mahler measure of $P(z)$ and is denoted by $\|P\|_0$. As an application of Jensen's inequality, the Mahler measure of the n-th degree polynomial $P(z) = b\prod_{v=1}^{n}(z - z_v)$ can be explicitly given by

$$\|P\|_0 = |b|\prod_{v=1}^{n}\max(1, |z_v|). \tag{1}$$

If $P \in \mathscr{P}_n$, then

$$\max_{|z|=1}|P'(z)| \leqslant n\max_{|z|=1}|P(z)| \tag{2}$$

and

$$\max_{|z|=R>1}|P(z)| \leqslant R^n\max_{|z|=1}|P(z)|. \tag{3}$$

Inequality (2) is an immediate consequence of S. Bernstein's Theorem [8] on the derivative of a trigonometric polynomial. Inequality (3) is a simple deduction from the maximum modulus principle. The equality in (2) and (3) holds for $P(z) = az^n$, $a \neq 0$.

If we restrict ourselves to the class of polynomials $P \in \mathscr{P}_n^\infty$, then inequalities (2) and (3) can be, respectively, replaced by

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n}{2}\max_{|z|=1}|P(z)| \tag{4}$$

and

$$\max_{|z|=R>1}|P(z)| \leqslant \frac{R^n+1}{2}\max_{|z|=1}|P(z)|. \tag{5}$$

Inequality (4) was conjectured by P. Erdös and later verified by P. D. Lax [14]. Ankeny and Rivlin [1] used (4) to prove inequality (5). The equality in (4) and (5) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

As an analogue of Bernstein's inequality in the Hardy space norm, Zygmund [23] proved that if $P(z)$ is a polynomial of degree n, then

$$\|P'\|_p \leqslant n\|P\|_p, \quad p \geqslant 1. \tag{6}$$

De Bruijn and Springer [10] and later Mahler [21] proved that this inequality also holds for $p = 0$, but for the case $0 < p < 1$ its validity remained an open question for quite a long time. Finally, Arestov [3] obtained an inequality concerning the Schur-Szegő product of polynomials, which among other things also answered the question.

The Schur-Szegő composition of a polynomial $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n$ with another polynomial $Q(z) = \sum_{j=0}^{n} \binom{n}{j}\gamma_j z^j$ is defined as $P*Q = \sum_{j=0}^{n} \gamma_j b_j z^j$.

For $Q * P$, Arestov [2] (see also [3]) proved the following inequality, which also includes the case $0 \leqslant p < 1$ of (6) as a special case:

$$\|Q * P\|_p \leqslant \|Q\|_0 \|P\|_p, \quad \text{for } p \geqslant 0. \tag{7}$$

Inequality (6) follows at once from (7) by taking $Q(z) = nz(z+1)^{n-1} = \sum_{j=0}^{n} \binom{n}{j} j z^j$. For the class of polynomials $P \in \mathcal{P}_n^{\infty}$, inequality (6) can be sharpened. In fact, in this case inequality (6) can be replaced by

$$\|P'\|_p \leqslant n \frac{\|P\|_p}{\|1+z\|_p}, \quad p \geqslant 0. \tag{8}$$

This inequality is due to N. G. De Bruijn [10] for $p \geqslant 1$, whereas Rahman and Schmeisser [18] extended it for $0 \leqslant p < 1$.

As a generalization of (8) and in the spirit of (7), Arestov [4] also proved that if $P(z) \in \mathcal{P}_n^{\infty}$ and $Q(z) = \sum_{j=0}^{n} \gamma_j z^j \in \mathscr{P}_n^0$, then

$$\|P * Q\|_p \leqslant \frac{\|\gamma_0 + \gamma_n z\|_p}{\|1+z\|_p} \|P\|_p, \quad p \geqslant 0. \tag{9}$$

Inequality (8) follows from (9) by choosing $Q(z) = nz(z+1)^{n-1} = \sum_{j=0}^{n} \binom{n}{j} j z^j$.

For polynomials $P \in \mathscr{P}_n^{\infty}$, Boas and Rahman [9] established an analogue of inequality (5) in the L_p-norm for $p \geqslant 1$:

$$\|P(Rz)\|_p \leqslant \frac{\|z + R^n\|_p}{\|z+1\|_p} \|P(z)\|_p, \quad R > 1. \tag{10}$$

Equality in (10) holds for $P(z) = az^n + b$, with $|a| = |b| \neq 0$. By letting $p \to \infty$ in (10), one can recover inequality (5).

Rahman and Schmeisser [18] (see also [4]) later showed that inequality (9) also holds for $0 \leqslant p < 1$. The above inequality has been generalized in several ways and a good number of papers are available (see, for example, [7], [6], [19], [20]). By applying inequality (10) to the polynomial $z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, we immediately deduce that for $P \in \mathscr{P}_n$ and $0 < r \leqslant 1$, the following inequality holds for each $p > 0$:

$$\|P(rz)\|_p \leqslant \frac{\|r^n z + 1\|_p}{\|z+1\|_p}\|P(z)\|_p. \tag{11}$$

This result is sharp and equality in (11) holds for $P(z) = az^n + b$ with $|a| = |b| \neq 0$.

By letting $p \to \infty$ in (11), we obtain the following sharp inequality under the conditions of (11):

$$\|P(rz)\|_\infty \leqslant \left(\frac{r^n+1}{2}\right)\|P(z)\|_\infty. \tag{12}$$

In this paper, we present the following result, which is a generalization as well as a refinement of inequality (11). More precisely, we prove

Theorem 1. *For any polynomial $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^0$, $0 \leqslant p < \infty$, $0 < r < 1$, and $0 \leqslant t < 1$, we have*

$$\left\| |P(rz)| + tm\frac{\mu_r - r^n}{1+\mu_r} \right\|_p \leqslant \frac{\|r^n + z\|_p}{\|\mu_r + z\|_p}\|P(z)\|_p, \tag{13}$$

where $m = \min_{|z|=1} |p(z)|$ and

$$\mu_r = \left(\frac{r|b_0| + rtm + |b_n|}{|b_0| + tm + r|b_n|}\right).$$

The result is sharp and equality in (13) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

Remark 1. *Since $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^0$,*

$$Q(z) = z^n\overline{P(1/\bar{z})} = \sum_{j=0}^{n} \bar{b}_j z^{n-j} \in \mathscr{P}_n^\infty.$$

By Lemma 4, we have $|b_n| \geqslant |b_0| + m$, where $m = \min_{|z|=1}|Q(z)| = \min_{|z|=1}|P(z)|$. This implies for $0 \leqslant t \leqslant 1$ $|b_n| \geqslant |b_0| + tm$, which gives for $0 < r \leqslant 1$

$$(1-r)|b_n| \geqslant (1-r)|b_0| + tm(1-r),$$

or, equivalently,

$$\frac{|b_n| + r|b_0| + rtm}{|b_0| + r|b_n| + tm} \geqslant 1.$$

That is, $\mu_r \geqslant 1$ *for* $0 < r \leqslant 1$.

Since $\|\mu_r + z\|_p \geqslant \|1+z\|_p$, $p \geqslant 0$, inequality (13) refines inequality (11). For $t = 0$, inequality (13) reduces to the following refinement of inequality (11).

Corollary 1. *If* $P(z) = \sum_{\nu=0}^{n} b_\nu z^\nu \in \mathscr{P}_n^0$, *then for each* $r < 1$, $0 \leqslant p < \infty$:

$$\|P(rz)\|_p \leqslant \frac{\|r^n + z\|_p}{\|\delta_r + z\|_p} \|P(z)\|_p, \tag{14}$$

where

$$\delta_r = \frac{r|b_0| + |b_n|}{|b_0| + r|b_n|}.$$

The result is sharp and equality in (14) *holds for* $P(z) = az^n + b$ *with* $|a| = |b| \neq 0$.

By letting $p \to \infty$ in (13) and noting that $\mu_r \geqslant 1$, we obtain the following refinement of inequality (12):

Corollary 2. *For any polynomial* $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^0$, $0 < r < 1$, *and* $0 \leqslant t < 1$, *we have*

$$\|P(rz)\|_\infty \leqslant \left(\frac{r^n + 1}{\mu_r + 1}\right) \|P(z)\|_\infty - tm\left(\frac{\mu_r - r^n}{1 + \mu_r}\right), \tag{15}$$

where $m = \min_{|z|=1} |P(z)|$ *and* μ_r *is given by* (13). *The inequality is sharp and equality in* (15) *holds for* $P(z) = az^n + b$, $|a| = |b| \neq 0$.

If $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^\infty$, then the polynomial $P^*(z) = z^n\overline{P(1/\bar{z})} \in \mathscr{P}_n^0$. Applying Theorem 1 to the polynomial $P^*(z)$ with $r = \frac{1}{R}$, we obtain the following refinement of inequality (10):

Corollary 3. *If* $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^\infty$, *then for each* $R > 1$, $0 \leqslant p < \infty$, *and* $0 \leqslant t \leqslant 1$:

$$\left\| |P(Rz)| + tm\frac{R^n\delta_R - 1}{1 + \delta_R} \right\|_p \leqslant \frac{\|R^n + z\|_p}{\|\delta_R + z\|_p} \|P(z)\|_p, \tag{16}$$

where $m = \min_{|z|=1} |P(z)|$ *and*

$$\delta_R = \frac{R|b_0| + |b_n| + tm}{|b_0| + R|b_n| + Rtm} \ (\geqslant 1).$$

The bound is sharp and equality in (16) *holds for* $P(z) = az^n + b$, $|a| = |b| \neq 0$. *Since* $\|\delta_R + z\|_p \geqslant \|1 + z\|_p$, $p \geqslant 0$, *inequality* (16) *refines inequality* (10). *For* $t = 0$, *inequality* (16) *also refines inequality* (10).

A. Aziz [5] proved that if $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^0(\rho)$ where $\rho \geqslant 1$, then for $1 \leqslant p < \infty$ and $0 \leqslant t \leqslant 1$:

$$\|P'(z)\|_\infty \geqslant \frac{n}{\|z + \rho^n\|_p} \|P(z)\|_p . \tag{17}$$

Now, we will show that the bound in (17) can be improved by using Corollary 3. More precisely, we prove the following result:

Theorem 2. *If* $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^0(\rho)$ *where* $\rho \geqslant 1$, *then for* $0 \leqslant p < \infty$ *and* $0 \leqslant t \leqslant 1$*:*

$$\|P'(z)\|_\infty \geqslant \frac{n}{\|\rho^n + z\|_p} \frac{\|\phi(\rho) + z\|_p}{\|1 + z\|_p} \left\| |P(z)| + \frac{tm}{\rho^n} \frac{\rho^n \phi(\rho) - 1}{1 + \phi(\rho)} \right\|_p , \tag{18}$$

where

$$\phi(\rho) = \frac{|b_0| + \rho^{n+1}|b_n| + tm}{\rho^n|b_n| + \rho|b_0| + \rho tm} \qquad \text{and} \qquad m = \min_{|z|=\rho} |P(z)|. \tag{19}$$

The result is sharp and equality in (18) *holds for* $P(z) = z^n + \rho^n$.

Remark 2. Since all the zeros of $P(z)$ are in $|z| \leqslant \rho$, $\rho \geqslant 1$, it can be easily seen that $\phi(\rho) \geqslant 1$. In view of this, Theorem 2 is a refinement of the inequality (17).

The following result is obtained by letting $p \to \infty$ in the Theorem 2.

Corollary 4. *If* $P(z) = \sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^0(\rho)$, *where* $\rho \geqslant 1$, *then for* $0 \leqslant p < \infty$ *and* $0 \leqslant t \leqslant 1$

$$\|P'(z)\|_\infty \geqslant \frac{n}{1 + \rho^n} \frac{1 + \phi(\rho)}{2} \left\{ \|P(z)\|_\infty + \frac{tm}{\rho^n} \frac{\rho^n \phi(\rho) - 1}{1 + \phi(\rho)} \right\} , \tag{20}$$

where $\phi(\rho)$ *is given by* (19). *The result is the best possible as shown by* $P(z) = z^n + \rho^n$.

Since $\phi(\rho) \geqslant 1$, inequality (20) improves the result by N. K. Govil [11], which states that if $P(z) \in \mathscr{P}_n^0(\rho)$, $\rho \geqslant 1$, then

$$\|P'\|_\infty \geqslant \frac{n}{1+\rho^n}\|P\|_\infty.$$

2. Lemmas. For the proof of our results, we need the following lemmas. The first lemma is a well-known generalization of the Schwarz lemma by Osserman [16].

Lemma 1. *Let $F(z)$ be analytic in $|z|<1$ with $F(0)=0$, and $|F(z)|<1$ for $|z|<1$; then*

$$|F(z)| \leqslant |z|\frac{|z|+|F'(0)|}{1+|z||F'(0)|}, \quad |z|<1.$$

Lemma 2. *Let a, b be complex numbers independent of α, where α is real. Then for each $p>0$:*

$$\int_0^{2\pi} \left|a+be^{i\alpha}\right|^p d\alpha = \int_0^{2\pi} \left||a|+|b|e^{i\alpha}\right|^p d\alpha.$$

Using periodicity, it is easy to verify the lemma, so we omit the details.

The following Lemma is by Aziz and Rather [6]:

Lemma 3. *If A, B, C are non-negative real numbers and $B+C \leqslant A$, then for every real number α*

$$\left|(A-C)e^{i\alpha}+(B+C)\right| \leqslant \left|Ae^{i\alpha}+B\right|.$$

The next lemma is by Gulzar and Rather [12]:

Lemma 4. *If $P(z)=\sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^0$ and $m=\min_{|z|=1}|P(z)|$, then*

$$|b_n| \geqslant |b_0|+m.$$

The next lemma is a consequence of the result by Arestov [[3], Theorem 4]. Yet, here we deduce it from inequality (7) due to Arestov [2].

Lemma 5. *If $P(z)=\sum_{j=0}^{n} b_j z^j \in \mathscr{P}_n^\infty$, then for every $p>0$, $r<1$ and real β:*

$$\int_0^{2\pi} \left|P(re^{i\theta})+e^{i\beta}r^nP(e^{i\theta}/r)\right|^p d\theta \leqslant \left|r^ne^{i\beta}+1\right|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

Proof. For $0 < r \leqslant 1$ and $|z| > 1$, $|z+r| > |rz+1|$. This gives,

$$|z+r|^n > |rz+1|^n, \quad |z| > 1,$$

which implies that the polynomial $e^{i\beta}(z+r)^n + (rz+1)^n$ has no zeros in $|z| > 1$. Hence, all the zeros of $e^{i\beta}(z+r)^n + (rz+1)^n$ lie in $|z| \leqslant 1$ for $0 < r \leqslant 1$. Setting $Q(z) = e^{i\beta}(z+r)^n + (rz+1)^n$ and noting that by (1), $\|Q\|_0 = |r^n e^{i\beta} + 1|$, we obtain by invoking inequality (7),

$$\|P(rz) + e^{i\beta} r^n P(z/r)\|_p \leqslant |r^n e^{i\beta} + 1| \|P(z)\|_p, \quad p \geqslant 0.$$

That proves Lemma 5. □

Definition 1. [17, pp. 36]. *Let f and g be analytic in $|z| < 1$. We say that the function f is subordinate to g, if there exists a function w, analytic in $|z| < 1$ with $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$, such that*

$$f(z) = g(w(z)) \qquad (|z| < 1).$$

Lemma 6. [17, pp. 36]. *Let f and g be analytic for $|z| \leqslant 1$ and such that f is subordinate to g. In addition, if g is univalent in the same disc, then for each $p > 0$ we have:*

$$\int_0^{2\pi} |f(e^{i\theta})|^p d\theta \leqslant \int_0^{2\pi} |g(e^{i\theta})|^p d\theta.$$

3. Proofs of the theorems.

Proof of Theorem 1. By the assumption, all the zeros of polynomial $P(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ lie in $|z| \leqslant 1$; therefore, the conjugate polynomial $Q(z) = z^n \overline{P(1/\bar{z})} = \sum_{j=0}^{n} \bar{b}_j z^{n-j}$ has all its zeros in $|z| \geqslant 1$ and $m = \min_{|z|=1} |P(z)| = \min_{|z|=1} |Q(z)|$, which implies

$$|z|^n m \leqslant |Q(z)| \quad \text{for} \quad |z| = 1.$$

By the Maximum Modulus principle, we have

$$|z|^n m < |Q(z)| \quad \text{for} \quad |z| < 1. \tag{21}$$

So, for any $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$, the polynomial $G(z) = Q(z) + \alpha m z^n$ does not vanish in $|z| < 1$. Indeed, if $G(z) = Q(z) + \alpha m z^n$ has a zero in $|z| < 1$ at $z = z_0$, then

$$G(z_0) = Q(z_0) + \alpha m z_0^n = 0, \quad |z_0| < 1.$$

This implies

$$|Q(z_0)| = m|\alpha||z_0|^n < m|z_0|^n \quad \text{for} \quad |z_0| < 1,$$

contradicting (21). Hence, we conclude that $\forall \alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$, the polynomial $G(z) = Q(z) + \alpha m z^n = (\bar{b_0} + \alpha m)z^n + \sum_{j=1}^n \bar{b_j} z^{n-j}$ has all its zeros in $|z| \geqslant 1$. Let $H(z) = z^n \overline{G(1/\bar{z})} = P(z) + \bar{\alpha} m$, then the function $F(z) = \dfrac{zH(z)}{G(z)}$ satisfies the assumption of Lemma 1 with $F'(0) = \dfrac{b_0 + \bar{\alpha} m}{\bar{b_n}}$ and, therefore,

$$|F(z)| \leqslant |z| \frac{|z| + \left|\dfrac{b_o + \bar{\alpha} m}{b_n}\right|}{1 + \left|\dfrac{b_o + \bar{\alpha} m}{b_n}\right||z|}.$$

This gives

$$|H(z)| \leqslant \frac{|b_n||z| + |b_0 + \bar{\alpha} m|}{|b_n| + |b_0 + \bar{\alpha} m||z|} |G(z)| \quad \text{for} \quad |z| < 1. \tag{22}$$

Setting $z = re^{i\theta}$ where $0 \leqslant \theta \leqslant 2\pi$ and $r < 1$ in (22), we get

$$|H(re^{i\theta})| \leqslant \frac{r|b_n| + |b_0 + \bar{\alpha} m|}{|b_n| + r|b_0 + \bar{\alpha} m|} |G(re^{i\theta})|. \tag{23}$$

The function $f(x) = \dfrac{r|b_n| + x}{|b_n| + rx}$ is non-decreasing for $x \geqslant 0$. Using the fact that for any $\alpha \in \mathbb{C}$

$$|b_0 + \bar{\alpha} m| \leqslant |b_0| + |\alpha| m,$$

we get from inequality (23) that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leqslant 1$, $r < 1$ and $|z| = 1$:

$$|H(re^{i\theta})| \leqslant \frac{r|b_n| + |b_0| + |\alpha| m}{|b_n| + r|b_0| + r|\alpha| m} |G(re^{i\theta})|.$$

Equivalently,

$$\mu_r|P(re^{i\theta})+\bar{\alpha}m| \leqslant |r^nP(e^{i\theta}/r)+\alpha m r^n|. \tag{24}$$

where $\mu_r = \dfrac{|b_n|+r|b_0|+|\alpha|rm}{r|b_n|+|b_0|+|\alpha|m}$. Choosing the argument of α in the left-hand side of (24) such that

$$|P(re^{i\theta})+\bar{\alpha}m| = |P(re^{i\theta})|+|\alpha|m,$$

we get

$$\mu_r\{|P(re^{i\theta})|+|\alpha|m\} \leqslant |r^nP(e^{i\theta}/r)|+|\alpha|r^nm.$$

This gives

$$\mu_r|P(re^{i\theta})|+|\alpha|m(\mu_r-r^n) \leqslant |r^nP(e^{i\theta}/r)|,$$

equivalently,

$$\mu_r\left\{|P(re^{i\theta})|+|\alpha|m\frac{\mu_r-r^n}{1+\mu_r}\right\} \leqslant |r^nP(e^{i\theta}/r)|-|\alpha|m\frac{\mu_r-r^n}{1+\mu_r}. \tag{25}$$

Since, by Remark 1, $\mu_r \geqslant 1$, therefore, we have

$$|P(re^{i\theta})|+|\alpha|m\frac{\mu_r-r^n}{1+\mu_r} \leqslant |r^nP(e^{i\theta}/r)|-|\alpha|m\frac{\mu_r-r^n}{1+\mu_r}. \tag{26}$$

Taking $A=|r^nP(e^{i\theta}/r)|$, $B=|P(re^{i\theta})|$, and $C=|\alpha|m\dfrac{\mu_r-r^n}{1+\mu_r}$ in Lemma 3 and noting by (26) that $B+C\leqslant A-C\leqslant A$, we get for every real β:

$$\left|\left(|r^nP(e^{i\theta}/r)|-|\alpha|m\frac{\mu_r-r^n}{1+\mu_r}\right)e^{i\beta}+\left(|P(re^{i\theta})|+|\alpha|m\frac{\mu_r-r^n}{1+\mu_r}\right)\right| \leqslant$$
$$\leqslant \left||r^nP(e^{i\theta}/r)|e^{i\beta}+|P(re^{i\theta})|\right|.$$

This yields, for each $p>0$,

$$\int_0^{2\pi}\left|M(\theta)+e^{i\beta}N(\theta)\right|^p d\theta \leqslant \int_0^{2\pi}\left||r^nP(e^{i\theta}/r)|e^{i\beta}+|P(re^{i\theta})|\right|^p d\theta, \tag{27}$$

where

$$M(\theta)=|P(re^{i\theta})|+|\alpha|m\frac{\mu_r-r^n}{1+\mu_r},\quad N(\theta)=|r^nP(e^{i\theta}/r)|-|\alpha|m\frac{\mu_r-r^n}{1+\mu_r}.$$

Integrating both sides of (27) with respect to β from 0 to 2π and using Lemma 2, we get

$$\int_0^{2\pi}\int_0^{2\pi}\left|M(\theta)+e^{i\beta}N(\theta)\right|^p d\theta d\beta\leqslant$$

$$\leqslant\int_0^{2\pi}\int_0^{2\pi}\left||r^nP(e^{i\theta}/r)|e^{i\beta}+|P(re^{i\theta})|\right|^p d\theta d\beta=$$

$$=\int_0^{2\pi}\left\{\int_0^{2\pi}\left||r^nP(e^{i\theta}/r)|e^{i\beta}+|P(re^{i\theta})|\right|^p d\beta\right\}d\theta=$$

$$=\int_0^{2\pi}\int_0^{2\pi}\left|P(re^{i\theta})+r^ne^{i\beta}P(e^{i\theta}/r)\right|^p d\theta d\beta.$$

Combining this with Lemma 5, we have

$$\int_0^{2\pi}\int_0^{2\pi}\left|M(\theta)+e^{i\beta}N(\theta)\right|^p d\theta d\beta\leqslant\int_0^{2\pi}\left|r^ne^{i\beta}+1\right|^p d\beta\int_0^{2\pi}|P(e^{i\theta}|^p d\theta. \tag{28}$$

Now, for every real β and $r_0\geqslant r_1\geqslant 1$, we have

$$\left|r_0+e^{i\beta}\right|\geqslant\left|r_1+e^{i\beta}\right|,$$

which implies, for each $p>0$,

$$\int_0^{2\pi}\left|r_0+e^{i\beta}\right|^p d\alpha\geqslant\int_0^{2\pi}\left|r_1+e^{i\lambda}\right|^p d\alpha.$$

If $|M(\theta)|\neq 0$, we take $r_0=|N(\theta)|/|M(\theta)|$ and $r_1=\mu_r$; then, by (25), $r_0\geqslant r_1\geqslant 1$, and we get, by using Lemma 2:

$$\int_0^{2\pi} \left|M(\theta) + e^{i\beta} r^n N(\theta)\right|^p d\beta = |M(\theta)|^p \int_0^{2\pi} \left|1 + e^{i\beta} \frac{N(\theta)}{M(\theta)}\right|^p d\beta =$$

$$= |M(\theta)|^p \int_0^{2\pi} \left|e^{i\beta} + \frac{N(\theta)}{M(\theta)}\right|^p d\beta = |M(\theta)|^p \int_0^{2\pi} \left|e^{i\beta} + \left|\frac{N(\theta)}{M(\theta)}\right|\right|^p d\beta \geqslant$$

$$\geqslant |M(\theta)|^p \int_0^{2\pi} \left|e^{i\beta} + \mu_r\right|^p d\beta.$$

For $|M(\theta)| = 0$, this inequality is trivially true. Using this inequality in (28), we obtain for each $p > 0$, $r < 1$, and real β:

$$\int_0^{2\pi} \left|e^{i\beta} + \mu_r\right|^p d\beta \int_0^{2\pi} \left||P(re^{i\theta})| + |\alpha| m \frac{\mu_r - r^n}{1 + \mu_r}\right|^p d\theta \leqslant$$

$$\leqslant \int_0^{2\pi} \left|r^n e^{i\beta} + 1\right|^p d\beta \int_0^{2\pi} \left|P(e^{i\theta})\right|^P d\theta =$$

$$= \int_0^{2\pi} \left|r^n + e^{i\beta}\right|^p d\beta \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta.$$

This gives

$$\left\| |P(re^{i\theta})| + |\alpha| m \frac{\mu_r - r^n}{1 + \mu_r} \right\|_p \leqslant \frac{\|r^n + z\|_p}{\|\mu_r + z\|_p} \|P(z)\|_p, \tag{29}$$

which proves the desired result for $p > 0$. To prove the result for $p = 0$, we simply let $p \to 0+$ in (29). □

Proof of Theorem 2. By the assumption, all the zeros of $P(z)$ lie in $|z| \leqslant \rho$, where $\rho \geqslant 1$. Therefore, all the zeros of $T(z) = P(\rho z)$ are in $|z| \leqslant 1$ and, consequently, the zeros of polynomial $R(z) = z^n \overline{T(1/\overline{z})}$ are outside $|z| < 1$. If $z_1, z_2, \ldots, z_n$ are the zeros of $R(z)$, then $|z_j| \geqslant 1$, $j = 1, 2, \ldots, n$, and

$$\frac{zR'(z)}{R(z)} = \sum_{j=1}^{n} \frac{z}{z - z_j}.$$

This gives, for the points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$ with $R(e^{i\theta}) \neq 0$:

$$\mathrm{Re}\frac{e^{i\theta}R'(e^{i\theta})}{R(e^{i\theta})} = \sum_{j=1}^{n} \mathrm{Re}\frac{e^{i\theta}}{e^{i\theta} - z_j} \leqslant \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2}.$$

This implies

$$\left|\frac{e^{i\theta}R'(e^{i\theta})}{nR(e^{i\theta})}\right| \leqslant \left|1 - \frac{e^{i\theta}R'(e^{i\theta})}{nR(e^{i\theta})}\right|, \qquad R(e^{i\theta}) \neq 0.$$

Equivalently,

$$|R'(e^{i\theta})| \leqslant |nR(e^{i\theta}) - e^{i\theta}R'(e^{i\theta})| \tag{30}$$

for the points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are not zeros of $R(z)$. This inequality is also true, even if $e^{i\theta}$ is a zero of $R(z)$. It follows that

$$|R'(z)| \leqslant |nR(z) - zR'(z)| \qquad \text{for} \quad |z| = 1. \tag{31}$$

Since all the zeros of $T(z)$ are in $|z| \leqslant 1$, by the Gauss-Lucas theorem the zeros of $T'(z)$ also lie in $|z| \leqslant 1$. This implies that the polynomial

$$z^{n-1}\overline{T'\left(\frac{1}{\bar{z}}\right)} \equiv nR(z) - zR'(z) \tag{32}$$

does not vanish in $|z| < 1$. Therefore, in view of (30), we conclude that the function

$$f(z) = \frac{zR'(z)}{nR(z) - zR'(z)}$$

is analytic for $|z| \leqslant 1$ and $|f(z)| \leqslant 1$ for $|z| = 1$. Moreover, $f(0) = 0$. Therefore, it follows that the function $1 + f(z)$ is subordinate to the univalent function $1 + z$ for $|z| \leqslant 1$. Hence, by Lemma 6, we obtain

$$\int_0^{2\pi} |1 + f(e^{i\theta})|^p d\theta \leqslant \int_0^{2\pi} |1 + e^{i\theta}|^p d\theta, \qquad p > 0. \tag{33}$$

Now,

$$1 + f(z) = \frac{nR(z)}{nR(z) - zR'(z)}.$$

This gives, for $|z| = 1$, with the help of (22), for each $p > 0$

$$n^p|R(e^{i\theta})|^p = |1 + f(e^{i\theta})|^p|nR(e^{i\theta}) - e^{i\theta}R'(e^{i\theta})|^p = \\ = |1 + f(e^{i\theta})|^p|e^{i(n+1)\theta}\overline{T'(e^{i\theta})}|^p = \\ = |1 + f(e^{i\theta})|^p|T'(e^{i\theta})|^p. \quad (34)$$

inequality (33) in conjunction with (34) gives, for each $p > 0$:

$$n^p \int_0^{2\pi} |R(e^{i\theta})|^p \, d\theta \leqslant \int_0^{2\pi} |1 + e^{i\theta}|^p \, d\theta \left(\max_{|z|=1} |T'(z)| \right)^p \quad (35)$$

Equivalently, for each $p > 0$:

$$n\|R(z)\|_p \leqslant \|1 + z\|_p \max_{|z|=1} |T'(z)|. \quad (36)$$

As the polynomial $R(z)$ does not vanish in $|z| < 1$, we can apply Corollary 3 to $R(z)$ with $R = \rho$ and obtain

$$\||R(\rho z)| + tm^* \frac{\rho^n\phi(\rho) - 1}{1 + \phi(\rho)}\|_p \leqslant \frac{\|\rho^n + z\|_p}{\|\phi(\rho) + z\|_p}\|R(z)\|_p, \quad (37)$$

where

$$\phi(\rho) = \frac{|b_0| + \rho^{n+1}|b_n| + tm^*}{\rho^n|b_n| + \rho|b_0| + t\rho m^*} \quad \text{and} \quad m^* = \min_{|z|=1} |R(z)|.$$

Again, since $R(z) = z^n\overline{T(1/\overline{z})} = z^n\overline{P(\rho/\overline{z})}$, we see that for $0 \leqslant \theta < 2\pi$

$$|R(\rho e^{i\theta})| = \rho^n|P(e^{i\theta})| \quad \text{and} \quad m^* = \min_{|z|=1} |R(z)| = \min_{|z|=1} |T(z)| = \min_{|z|=\rho} |P(z)|.$$

Combining this with (36) and (37), we get:

$$n\|\rho^n|P(z)| + tm\frac{\rho^n\phi(\rho) - 1}{1 + \phi(\rho)}\|_p \leqslant \frac{\|1 + z\|_p}{\|\phi(\rho) + z\|_p}\|\rho^n + z\|_p \max_{|z|=1} |T'(z)|. \quad (38)$$

Applying inequality (2) to the polynomial $T'(z) = \rho P'(\rho z)$ of degree at most $n - 1$, where $\rho \geqslant 1$, we have

$$\max_{|z|=1} |T'(z)| = \rho \max_{|z|=1} |P'(\rho z)| = \rho \max_{|z|=\rho} |P'(z)| \leqslant \rho^n \max_{|z|=1} |P'(z)|. \quad (39)$$

By using inequality (39) in (38), we finally obtain

$$n\left\| |P(z)| + \frac{tm}{\rho^n}\left(\frac{\rho^n\phi(\rho)-1}{1+\phi(\rho)}\right)\right\|_p \leqslant \frac{\|1+z\|_p}{\|\phi(\rho)+z\|_p}\|\rho^n+z\|_p\|P'(z)\|_\infty.$$

□

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