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ON $\mathcal{I}^{\mathcal{K}}$-CONVERGENCE OF SEQUENCES OF BI-COMPLEX NUMBERS

Abstract. We propose the concept of $\mathcal{I}^{\mathcal{K}}$-convergence of sequences of bi-complex numbers. We explore the fundamental properties of this newly introduced notion and its relationships with other convergence methods.

Key words: *bi-complex number, ideal, filter, $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}^{\mathcal{K}}$-convergence.*

2020 Mathematical Subject Classification: *40A35, 40G15.*

1. Introduction. Ideal convergence, originally proposed by Kostyrko et al. [7] in 2001 as an extension of statistical convergence, has garnered significant attention from researchers in subsequent years. Starting from this foundational work, researchers (e.g.) Debnath and Rakshit [6], Choudhury and Debnath [3], Savas and Das [13], among others, have conducted extensive research in this area, exploring its applications and properties. Through their investigations, they have highlighted ideal convergence as a generalized form encompassing various established convergence concepts, contributing to the advancement of mathematical analysis and its applications. For an extensive study on Ideal convergence, one may refer to [9], [10], [13].

In 2011, Macaj and Sleziak [8] proposed the concept of $\mathcal{I}^{\mathcal{K}}$-convergence, which extends the idea of $\mathcal{I}^*$-convergence by incorporating two ideals, $\mathcal{I}$ and $\mathcal{K}$. Unlike the traditional convergence, where convergence is assessed along a single set, $\mathcal{I}^{\mathcal{K}}$-convergence considers convergence along a large set with respect to another ideal, $\mathcal{K}$. This extension presents an intriguing analogy and offers avenues for further exploration. For an extensive study on $\mathcal{I}^{\mathcal{K}}$-convergence, one may refer to [4].

The notion of $\mathcal{I}^{\mathcal{K}}$-convergence being a generalization of $\mathcal{I}^*$-convergence suggests potential for deeper investigation and application. Recent advancements in this direction, particularly from a topological perspective,

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have been made by Debnath et al. [5] and other researchers. Their works shed light on the topological aspects of $\mathcal{I}^{\mathcal{K}}$-convergence, contributing to a broader understanding of this generalized convergence concept and its implications in various mathematical contexts.

The exploration of convergence is a fundamental aspect of analysis, playing a crucial role in various mathematical investigations. However, the study of convergence of sequences of bi-complex numbers remains relatively underdeveloped and has not yet received substantial attention. Despite its nascent stage, recent research indicates a notable analogy in the convergence behavior of sequences of bi-complex numbers.

Recently, Bera and Tripathy [1] made significant strides by introducing the concept of statistical convergence for sequences of bi-complex numbers. Their work marks a pivotal advancement in the study of convergence in this domain, as they explored the different properties from both algebraic and topological perspectives.

Given this recent progress, it is indeed a natural progression to explore the $\mathcal{I}^{\mathcal{K}}$-convergence of sequences of bi-complex numbers. Building upon the foundation laid by Bera and Tripathy, investigating $\mathcal{I}^{\mathcal{K}}$-convergence offers the opportunity to deepen our understanding of the convergence behavior of bi-complex sequences in a more generalized setting. This exploration may unveil new insights into the convergence properties of bi-complex numbers, contributing to the broader landscape of mathematical analysis.

Throughout the paper, $\mathbb{C}_2$ represent the set of all bi-complex numbers.

2. Definitions and Preliminaries. Segre [15] defined a bi-complex number as $\xi = z_1 + i_2 z_2 = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$, where $z_1 = x_1 + i_1 x_2$, $z_2 = x_3 + i_2 x_4 \in \mathbb{C}$ (set of complex numbers) and $x_1, x_2, x_3, x_4 \in \mathbb{R}$ (set of real numbers) and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. Denote the set of all bi-complex numbers by $\mathbb{C}_2$; it is defined as: $\mathbb{C}_2 = \{\xi \colon \xi = z_1 + i_2 z_2 \colon z_1, z_2 \in \mathbb{C}\}$.

In the realm of bi-complex numbers, a number $\xi = x_1 + i_1 x_2 + i_2 x_3 + {} + i_1 i_2 x_4$ is classified as a hyperbolic number if $x_2 = 0$ and $x_3 = 0$. These hyperbolic numbers are collectively denoted as $\mathcal{H}$, and the set comprising them is referred to as the $\mathcal{H}$-plane.

Equipped with coordinate-wise addition, real scalar multiplication, and term-by-term multiplication, the set $\mathbb{C}_2$ becomes a commutative algebra with the identity $1 = 1 + i_1 \cdot 0 + i_2 \cdot 0 + i_1 i_2 \cdot 0$.

Within $\mathbb{C}_2$, there exist four idempotent elements, specifically 0, 1,

$e_1 = \dfrac{1+i_1i_2}{2}$, and $e_2 = \dfrac{1-i_1i_2}{2}$.
It is obvious that $e_1+e_2 = 1$ and $e_1e_2 = e_2e_1 = 0$. Every bi-complex number $\xi = z_1 + i_2z_2$ has a unique idempotent representation as $\xi = T_1e_1 + T_2e_2$, where $T_1 = z_1 - i_1z_2$ and $T_2 = z_1 + i_1z_2$ are called the idempotent components of ξ. The Euclidean norm $\|\cdot\|$ on $\mathbb{C}_2$ is defined as

$$\|\xi\|_{\mathbb{C}_2} = \sqrt{x_1^2+x_2^2+x_3^2+x_4^2} = \sqrt{|z_1|^2+|z_2|^2} = \sqrt{\frac{|T_1|^2+|T_2|^2}{2}},$$

where $\xi = z_1 + i_2z_2 = x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 = T_1e_1 + T_2e_2$ with this norm $\mathbb{C}_2$ is a Banach space. For an extensive view, [11], [14], [16] can be addressed where many more references can be found.

Definition 1. [2] *In the context of bi-complex numbers, several conjugates are defined as follows:*

The i_1-conjugate of a bi-complex number $\xi = z_1+i_2z_2$ is denoted by ξ^* and is defined as $\xi^* = \bar{z}_1 + i_2\bar{z}_2$ for all $z_1, z_2 \in \mathbb{C}$. Here, $\bar{z}_1$ and $\bar{z}_2$ represent the complex conjugates of z_1 and z_2, respectively, and $i_1^2 = i_2^2 = -1$.

The i_2-conjugate of a bi-complex number $\xi = z_1 + i_2z_2$ is denoted by $\bar{\xi}$ and is defined as $\bar{\xi} = z_1 - i_2z_2$ for all $z_1, z_2 \in \mathbb{C}$. Again, $i_1^2 = i_2^2 = -1$.

The i_1i_2-conjugate of a bi-complex number $\xi = z_1 + i_2z_2$ is denoted by ξ' and defined as $\xi' = \bar{z}_1 - i_2\bar{z}_2$ for all $z_1, z_2 \in \mathbb{C}$, where $\bar{z}_1$ and $\bar{z}_2$ are the complex conjugates of z_1 and z_2, respectively, and $i_1^2 = i_2^2 = -1$.

Properties of i_1-conjugation. Some of the properties of i_1-conjugation, which were obtained by Rochon and Shapiro [12] are listed as follows:

(i) $(\xi+\eta)^* = \xi^* + \eta^*$;

(ii) $(\alpha\xi)^* = \alpha\xi^*$;

(iii) $(\xi^*)^* = \xi$;

(iv) $(\xi\eta)^* = \xi^*\eta^*$;

(v) $(\xi^{-1})^* = (\xi^*)^{-1}$if ξ^{-1} exists;

(vi) $(\frac{\xi}{\eta})^* = \frac{\xi^*}{\eta^*}$.

They are obtained similarly to the properties of i_2 -conjugation.

Definition 2. [2] *A sequence of bi-complex number (ξ_k) is considered statistically convergent to $\xi \in \mathbb{C}_2$ if, for every $\varepsilon > 0$,*

$$\delta(\{k \in \mathbb{N} \colon \|\xi_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\}) = 0.$$

Symbolically, we write stat-$\lim \xi_k = \xi$.

Definition 3. [7] *Consider a nonempty set X. A family of subsets $\mathcal{I} \subset \mathcal{P}(X)$ is called an ideal on X if it satisfies the following conditions:*

(1) For every X_1,$X_2 \in \mathcal{I}$, the union $X_1 \cup X_2$ belongs to $\mathcal{I}$.

(2) For every $X_1 \in \mathcal{I}$ and every subset X_2 of X_1, X_2 is also in $\mathcal{I}$.

Further $\mathcal{I}$ is said to be admissible if $\forall x \in X$,{x} $\in \mathcal{I}$, and it is said to be nontrivial if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$.

Example 1. Here are some standard examples of ideals:

(1) The collection of all finite subsets of $\mathbb{N}$ constitutes a nontrivial admissible ideal on $\mathbb{N}$, denoted as $\mathcal{I}_f$.

(2) The set comprising all subsets of $\mathbb{N}$ with natural density zero forms a nontrivial admissible ideal on $\mathbb{N}$. This particular ideal is denoted as $\mathcal{I}_\delta$.

(3) Let $\mathcal{I}_c = \{A \subseteq \mathbb{N}\colon \sum\limits_{a \in A} a^{-1} < \infty\} \subset \mathbb{N}$. Then $\mathcal{I}_c$ forms an ideal in $\mathbb{N}$.

(4) Consider a partitioning of the natural numbers $\mathbb{N}$ into disjoint sets $D_1, D_2, D_3, \ldots$, such that $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ and $D_a \cap D_b = \varnothing$ for $a \neq b$. The set $\mathcal{I}$, comprising all subsets of $\mathbb{N}$ that have finite intersections with the sets D_p, constitutes an ideal in $\mathbb{N}$.

Definition 4. [3] *A family $\mathcal{F} \subset 2^X$ of subsets of a nonempty set X is called a filter in X if it satisfies the following conditions:*

(1) The empty set ϕ does not belong to $\mathcal{F}$.

(2) For all $X_1, X_2 \in \mathcal{F}$, the intersection $X_1 \cap X_2$ is also in $\mathcal{F}$.

(3) For every $X_1 \in \mathcal{F}$ and every superset X_2 of X_1 containing X_1, X_2 is also in $\mathcal{F}$.

Definition 5. [3] *If $\mathcal{I}$ is a proper nontrivial ideal in Y, then $\mathcal{F}(\mathcal{I}) = \{A \subset Y : \exists B \in \mathcal{I} : A = Y - B\}$ constitutes a filter in Y. This filter is commonly referred to as the filter associated with the ideal $\mathcal{I}$.*

Definition 6. [7] *Let $\mathcal{I} \subset P(\mathbb{N})$ denote a nontrivial ideal over $\mathbb{N}$. We define an $\mathcal{I}$-convergence for a real-valued sequence (ξ_n) towards l as follows: for every $\varepsilon > 0$, the set $H(\varepsilon) = \{n \in \mathbb{N} : |\xi_n - l| \geqslant \varepsilon\}$ must be an element of $\mathcal{I}$. Here, l is called the $\mathcal{I}$-limit of the sequence (ξ_n) and is denoted by $\mathcal{I} - \lim\limits_{k} \xi_k = l$.*

Definition 7. [4] *Consider an admissible ideal $\mathcal{I}$ in $\mathbb{N}$. Define $\mathcal{I}^*$-convergence for a real-valued sequence (ξ_k) towards l as follows: there exists a* set $T = \{t_1 < t_2 < \ldots < t_k < \ldots\}$ in the associated filter $\mathcal{F}(\mathcal{I})$ such that $\lim_{k \in T} \xi_k = l$. Symbolically, $\mathcal{I}^* - \lim_k \xi_k = l$.

Definition 8. [8] *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. A real-valued sequence (ξ_k) is said to be $\mathcal{I}^{\mathcal{K}}$-convergent to l if there exists $M \in \mathcal{F}(\mathcal{I})$, such that the sequence (η_k) defined by $\eta_k = \begin{cases} \xi_k, & k \in M, \\ l, & k \notin M, \end{cases}$ is $\mathcal{K}$-convergent to l.*

Definition 9. [8] *Let $\mathcal{K}$ be an ideal on $\mathbb{N}$. Then $P \subset_{\mathcal{K}} Q$ denotes the property $P \backslash Q \in \mathcal{K}$. Also, $P \subset_{\mathcal{K}} Q$ and $Q \subset_{\mathcal{K}} P$ together imply $P \sim_{\mathcal{K}} Q$. Thus, $P \sim_{\mathcal{K}} Q$ if and only if $P \Delta Q \in \mathcal{K}$. A set P is said to be $\mathcal{K}$-pseudo intersection of a system $\{P_i : i \in \mathbb{N}\}$ if for every $i \in \mathbb{N}, P \subset_{\mathcal{K}} P_i$ holds.*

Definition 10. [8] *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals on $\mathbb{N}$. Then $\mathcal{I}$ is said to have the additive property with respect to $\mathcal{K}$ (or the condition $AP(\mathcal{I}, \mathcal{K})$ holds), if every sequence $(F_n)_{n \in \mathbb{N}}$ of sets from $\mathcal{F}(\mathcal{I})$ has $\mathcal{K}$-pseudo intersection in $\mathcal{F}(\mathcal{I})$.*

Definition 11. *With respect to the Euclidean norm on $\mathbb{C}_2$, a sequence of bi-complex numbers (ξ_k) is called $\mathcal{I}$-convergent to $t \in \mathbb{C}_2$ if, for each $\varepsilon_1 > 0$, the set*

$$F(\varepsilon_1) = \{k \in \mathbb{N} : \|\xi_k - t\|_{\mathbb{C}_2} \geqslant \varepsilon_1\} \in \mathcal{I}.$$

Symbolically, we write, $\xi_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathbb{C}_2}} t$.

Definition 12. *Let $\mathcal{I}$ be an admissible ideal. A sequence (ξ_k) of $\mathbb{C}_2$ is called $\mathcal{I}^*$-convergent to $\xi \in \mathbb{C}_2$ with respect to the Euclidean norm on $\mathbb{C}_2$ if $\exists$ a set $T = \{t_1 < t_2 < \ldots < t_k < \ldots\}$ in the associated filter $\mathcal{F}(\mathcal{I})$, such that the sub-sequence (ξ_{t_k}) converges to ξ. Symbolically, we write, $\xi_k \xrightarrow{\mathcal{I}^*-\|\cdot\|_{\mathbb{C}_2}} \xi$.*

3. Main Results.

Definition 13. *Let $\mathcal{I}$ and $\mathcal{K}$ be two admissible ideals in $\mathbb{N}$ and (ξ_k) be a sequence of bi-complex numbers. Then (ξ_k) is considered $\mathcal{I}^{\mathcal{K}}$-convergent to $l \in \mathbb{C}_2$ with respect to the Euclidean norm on $\mathbb{C}_2$ if there exists $M \in \mathcal{F}(\mathcal{I})$, such that the sequence (η_k) defined by $\eta_k = \begin{cases} \xi_k, & k \in M, \\ l, & k \notin M, \end{cases}$ is $\mathcal{K}$-convergent to l. Symbolically, we write, $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-||\cdot||_{\mathbb{C}_2}} l$.*

Example 2. Let $D_p = \{2^{p-1}k \colon k \text{ is an odd number}\}$. Then, $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ is a decomposition of $\mathbb{N}$. Let $\mathcal{I} = \mathcal{I}_D$, those subsets of $\mathbb{N}$ that intersect with finitely many D_p's. Consider the sequence (ξ_k) defined by $\xi_k = \dfrac{e_1 - e_2}{p}$, $\forall k \in D_p$'s. Then the sequence is $\mathcal{I}^{\mathcal{I}}$-convergent to 0.

Justification. Let $M = \mathbb{N} \backslash D_1$. Then $M \in \mathcal{F}(\mathcal{I})$. Now, consider the sequence (η_k) defined by $\eta_k = \begin{cases} \xi_k, & k \in M, \\ 0, & k \notin M. \end{cases}$

Now, the sequence $(\eta_k) = (0, \dfrac{e_1 - e_2}{2}, 0, \dfrac{e_1 - e_2}{3}, 0, \ldots)$ is $\mathcal{I}$-convergent to 0.

If $\varepsilon > \frac{1}{2}$, then $A(\varepsilon) = \{k \in \mathbb{N} \colon \|\eta_k - 0\|_{\mathbb{C}_2} \geqslant \varepsilon\} = \varnothing \in \mathcal{I}$, as $\varnothing$ intersects none of D_p's. If $\varepsilon = \frac{1}{2}$, then $A(\frac{1}{2}) = \{k \in \mathbb{N} \colon \|\eta_k - 0\|_{\mathbb{C}_2} \geqslant \frac{1}{2}\} = D_2 \in \mathcal{I}$, as it intersects only D_2. If $\varepsilon = \frac{1}{3}$, then $A(\frac{1}{3}) = \{k \in \mathbb{N} \colon \|\eta_k - 0\|_{\mathbb{C}_2} \geqslant \frac{1}{3}\} = D_2 \cup D_3 \in \mathcal{I}$, as it intersects only D_2 and D_3. Proceeding like this, we can say that if $\varepsilon = \frac{1}{p}$, then $A(\frac{1}{p}) = \{k \in \mathbb{N} \colon \|\eta_k - 0\|_{\mathbb{C}_2} \geqslant \frac{1}{p}\}$ intersects $(p-1)$ of sets namely $D_2, D_3, D_4, \ldots D_p$ and, hence, $A(\frac{1}{p}) \in \mathcal{I}$. Now, by the Archimedean property, we can say that there exists $\frac{1}{p} < \varepsilon$ that implies $A(\varepsilon) \subset A(\frac{1}{p}) \in \mathcal{I}$. Hence, $\mathcal{I} - \lim \eta_k = 0$. Thus, our claim is established.

Theorem 1. *Let (ξ_k) be a sequence in $\mathbb{C}_2$, such that $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}} - \|\cdot\|_{\mathbb{C}_2}} \xi$. Then ξ is uniquely determined.*

Proof. If possible, suppose: $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}} - \|\cdot\|_{\mathbb{C}_2}} \xi$ and $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}} - \|\cdot\|_{\mathbb{C}_2}} \eta$ hold for some $\xi \neq \eta$ in $\mathbb{C}_2$. Choose $\varepsilon > 0$, such that $\|\xi - \eta\|_{\mathbb{C}_2} = 2\varepsilon$. Then, by Definition 13, there exists $M_1, M_2 \in \mathcal{F}(\mathcal{I})$, such that the sequences (x_k) and (y_k) defined by $x_k = \begin{cases} \xi_k, & k \in M_1, \\ \xi, & k \notin M_1, \end{cases}$ and $y_k = \begin{cases} \xi_k, & k \in M_2, \\ \eta, & k \notin M_2, \end{cases}$ satisfy the following properties:

- $\forall \varepsilon > 0$, $\{k \in \mathbb{N} \colon \|x_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}$,
- $\forall \varepsilon > 0$, $\{k \in \mathbb{N} \colon \|y_k - \eta\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}$.

Then, $\forall \varepsilon > 0$, $\{k \in \mathbb{N} \colon \|(x_k - y_k) - (\xi - \eta)\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}$. Now, as the inclusion, $M_1 \cap M_2 \subseteq \{k \in \mathbb{N} \colon \|(x_k - y_k) - (\xi - \eta)\|_{\mathbb{C}_2} \geqslant \varepsilon\}$ holds, so, by hereditary of $\mathcal{K}$, $M_1 \cap M_2 \in \mathcal{K}$, which implies $\mathbb{N} \backslash (M_1 \cap M_2) \in \mathcal{F}(\mathcal{K})$. Again, as $M_1, M_2 \in \mathcal{F}(\mathcal{I})$, so $M_1 \cap M_2 \in \mathcal{F}(\mathcal{I})$. Now, $\mathbb{N} \backslash (M_1 \cap M_2) \in \mathcal{F}(\mathcal{K})$

and $M_1 \cap M_2 \in \mathcal{F}(\mathcal{I})$ implies $\mathbb{N}\backslash(M_1 \cap M_2) \cap (M_1 \cap M_2) \in \mathcal{F}(\mathcal{I} \vee \mathcal{K})$ i.e $\varnothing \in \mathcal{F}(\mathcal{I} \vee \mathcal{K})$, a contradiction. □

Theorem 2. *Let (ξ_k) and (η_k) be two sequences in $\mathbb{C}_2$, such that $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$ and $\eta_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \eta$ hold. Then*

(i) $\xi_k + \eta_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi + \eta$,

(ii) $c\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} c\xi$ *where* $c \in \mathbb{R}$.

Proof. (i) Suppose $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$ and $\eta_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \eta$. Then there exists $M_1, M_2 \in \mathcal{F}(\mathcal{I})$, such that the sequences (x_k) and (y_k) defined by $x_k = \begin{cases} \xi_k, & k \in M_1, \\ \xi, & k \notin M_1, \end{cases}$ and $y_k = \begin{cases} \eta_k, & k \in M_2, \\ \eta, & k \notin M_2, \end{cases}$ are $\mathcal{K}$-convergent to ξ and η, respectively.

In other words, $\forall \varepsilon > 0$, we have $D_1(\varepsilon) = \{k \in \mathbb{N} \colon \|x_k - \xi\|_{\mathbb{C}_2} \geqslant \frac{\varepsilon}{2}\} \in \mathcal{K}$ and $D_2(\varepsilon) = \{k \in \mathbb{N} \colon \|y_k - \eta\|_{\mathbb{C}_2} \geqslant \frac{\varepsilon}{2}\} \in \mathcal{K}$. Now, as the inclusion $(\mathbb{N}\backslash D_1) \cap (\mathbb{N}\backslash D_2) \subseteq \{k \in \mathbb{N} \colon \|x_k + y_k - \xi - \eta\|_{\mathbb{C}_2} < \varepsilon\}$ holds, we must have $\{k \in \mathbb{N} \colon \|x_k + y_k - \xi - \eta\|_{\mathbb{C}_2} \geqslant \varepsilon\} \subseteq D_1 \cup D_2 \in \mathcal{K}$.

$$\text{Proving } x_k + y_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi + \eta. \tag{1}$$

Now we take $M = M_1 \cap M_2 \in \mathcal{F}(\mathcal{I})$ and define the sequence (S_k) as $S_k = \begin{cases} \xi_k + \eta_k, & k \in M, \\ \xi + \eta, & k \notin M. \end{cases}$ Then, by virtue of (i), we have $S_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi + \eta$.

This implies that $\xi_k + \eta_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi + \eta$.

(ii) Proof of this part is easy and so is omitted. □

Theorem 3. *Let (ξ_k) be any sequence in $\mathbb{C}_2$, such that $\xi_k \xrightarrow{\mathcal{I}^*-\|\cdot\|_{\mathbb{C}_2}} \xi$. Then $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$, for any admissible ideal $\mathcal{K}$.*

Proof. Let $\xi_k \xrightarrow{\mathcal{I}^*-\|\cdot\|_{\mathbb{C}_2}} \xi$. Then, by Definition 8, there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that the subsequence (ξ_{m_k}) is convergent to ξ. This implies that for any $\varepsilon > 0$, the set $A = \{k \in \mathbb{N} \colon \|\eta_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\}$ contains finite number of elements, where the sequence (η_k) is given by $\eta_k = \begin{cases} \xi_k, & k \in M, \\ \xi, & k \notin M. \end{cases}$

Since $\mathcal{K}$ is admissible, so the above set $A \in \mathcal{K}$ and the result follows. □

Remark 1. *The converse of the theorem mentioned earlier may not be valid.*

Proof. The result is illustrated by the following example. □

Example 3. Consider Example 2. The sequence (ξ_k) is $\mathcal{I}^{\mathcal{K}}$-convergent to 0 for $\mathcal{K} = \mathcal{I}$. But (ξ_k) is not $\mathcal{I}^*$-convergent to 0.

Justification. If $\xi_k \xrightarrow{\mathcal{I}^* - \|\cdot\|_{\mathbb{C}_2}} 0$ holds, then, by Definition 8, there exists a set $T = \{t_1 < t_2 < \ldots < t_k < \ldots\} \in \mathcal{F}(\mathcal{I})$, such that

$$\text{for any } \varepsilon > 0, \text{there exists } N = N_\varepsilon \in \mathbb{N}\colon \|\xi_{t_k} - 0\|_{\mathbb{C}_2} < \varepsilon \text{ for all } k > N. \quad (2)$$

Now, as $T \in \mathcal{F}(\mathcal{I})$, so $T = \mathbb{N}\backslash B$ for some $B \in \mathcal{I}$. By definition of $\mathcal{I}$, there exists some $j \in \mathbb{N}$, such that $B \subset \bigcup\limits_{p=1}^{j} D_p$, which as a consequence implies $D_{j+1} \subset T$. But then we have, $\xi_{t_k} = \frac{i_2}{j+1}$ for infinitely many $k's$, which is a contradiction to (2).

Theorem 4. *Assuming (ξ_k) is an arbitrary sequence in $\mathbb{C}_2$, if (ξ_k) converges with respect to $\mathcal{K}$ to $\xi \in \mathbb{C}_2$, then the sequence (ξ_k) is also $\mathcal{I}^{\mathcal{K}}$-convergent to ξ.*

Proof. Since $\xi_k \xrightarrow{\mathcal{K} - \|\cdot\|_{\mathbb{C}_2}} \xi$, so,

$$\text{for any } \varepsilon > 0, \{k \in \mathbb{N}\colon \|\xi_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}. \quad (3)$$

Choose $M = \mathbb{N}$ from $\mathcal{F}(\mathcal{I})$. Define the sequence η_k as $\eta_k = \xi_k, k \in \mathcal{M}$. Then, using (3), we get for any $\varepsilon > 0$, $\{k \in \mathbb{N}\colon \|\eta_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}$, i.e., η_k is $\mathcal{K}-$convergent to ξ. Hence, $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}} - \|\cdot\|_{\mathbb{C}_2}} \xi$. □

Remark 2. *The reverse of the previously mentioned theorem may not necessarily be valid.*

Proof. The result is illustrated by the following example. □

Example 4. Define $Q_1 = q_1\mathbb{N} \cup \{1\}$ and $Q_p = q_p\mathbb{N}\backslash \bigcup\limits_{i=1}^{p-1} D_i$, $p > 1$, where q_p is the p^{th} prime number. Then $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} Q_p$ is a decomposition of $\mathbb{N}$. Consider the ideal $\mathcal{I}$ that consists of those subsets of $\mathbb{N}$ that intersect with finitely many Q_p's, and another ideal $\mathcal{I}_\delta$. Let (ξ_k) be any sequence in $\mathbb{C}_2$ defined by $\xi_k = \begin{cases} e_1 + e_2, & k \text{ is prime}, \\ e_1 e_2, & k \text{ is not prime}. \end{cases}$

Then, (ξ_k) is ${\mathcal{I}_\delta}^{\mathcal{I}}$-convergent to 0 but not $\mathcal{I}$- convergent to 0.

Justification. Let P be the set of all prime numbers. Then $\delta(P) = 0$, so $P \in \mathcal{I}_\delta$. Let $M = \mathbb{N}\backslash P$. Then $M \in \mathcal{F}(\mathcal{I}_\delta)$. Now the sequence (η_k) defined as $\eta_k = \begin{cases} \xi_k, & k \in M, \\ 0, & k \notin M, \end{cases}$ is the null sequence and, therefore, (η_k) is $\mathcal{I}$-convergent to 0. Hence, the sequence (ξ_k) is ${\mathcal{I}_\delta}^{\mathcal{I}}$-convergent to 0. But (ξ_k) is not $\mathcal{I}$-convergent to 0. Since for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} \colon \|\xi_k - 0\|_{\mathbb{C}_2} \geqslant \varepsilon\} \notin \mathcal{I}$.

Theorem 5. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$, such that $\mathcal{I} \subseteq \mathcal{K}$. Consider a sequence (ξ_k) in $\mathbb{C}_2$, such that $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$. Then $\xi_k \xrightarrow{\mathcal{K}-\|\cdot\|_{\mathbb{C}_2}} \xi$.*

Proof. Let $\mathcal{I} \subseteq \mathcal{K}$ hold, and suppose $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$. So, by definition, there exists $M \in \mathcal{F}(\mathcal{I})$, such that the sequence (η_k) defined as $\eta_k = \begin{cases} \xi_K, & k \in M, \\ \xi, & k \notin M, \end{cases}$ is $\mathcal{K}$-convergent to ξ, which as a consequence implies

$$\forall \varepsilon > 0 \; \{k \in M \colon \|\xi_k - \xi|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}. \tag{4}$$

Thus, $\{k \in \mathbb{N} \colon \|\xi_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\} \subseteq \{k \in M \colon \|\xi_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\} \cup (\mathbb{N}\backslash M) \in \mathcal{K}$, by equation (4) and our assumption $\mathcal{I} \subseteq \mathcal{K}$. Hence, $\xi_k \xrightarrow{\mathcal{K}-\|\cdot\|_{\mathbb{C}_2}} \xi$. □

Remark 3. *If a sequence is $\mathcal{I}^{\mathcal{K}}$-convergent, then it may not be $\mathcal{I}$-convergent to the same limit.*

Example 5. Let $D_p = \{2^{p-1}k \colon k \text{ be an odd number}\}$. Then $\mathbb{N} = \bigcup_{p=1}^{\infty} D_p$ is a decomposition of $\mathbb{N}$. Let $\mathcal{I} = \mathcal{I}_D$, those subsets of $\mathbb{N}$ that intersect finitely many D_p's. Consider the sequence (ξ_k) defined by $\xi_k = \frac{e_1 - e_2}{p}$, $\forall k \in D_p$'s. Then the sequence is $\mathcal{I}$-convergent to 0. Let $\mathcal{I}_f$ consist of all finite subsets of $\mathbb{N}$. Then ξ_k is not $\mathcal{I}^{\mathcal{I}_f}$-convergent to 0.

Theorem 6. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. Let (ξ_k) be a sequence in $\mathbb{C}_2$. Then $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$ implies $\xi_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathbb{C}_2}} \xi$ if and only if $\mathcal{K} \subseteq \mathcal{I}$.*

Proof. Let $\mathcal{K} \subseteq \mathcal{I}$ and suppose $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$. Then the result follows directly from the following inclusion:

$$\{k \in \mathbb{N} \colon \|\xi_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\} \subseteq \{k \in M \colon \|\xi_k - \xi\|_{\mathbb{C}_2} \geqslant \varepsilon\} \cup (\mathbb{N}\backslash M).$$

For the converse part, let (ξ_k) be a sequence in $\mathbb{C}_2$, such that $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$ implies $\xi_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathbb{C}_2}} \xi$. To prove that, $\mathcal{K} \subseteq \mathcal{I}$. Let us assume the contrary. Then, there exists a set, say $A \in \mathcal{K}\backslash\mathcal{I}$. Choose $\xi_1, \xi_2 \in \mathbb{C}_2$ such that $\xi_1 \neq \xi_2$.

Define a sequence (ξ_k) as $\xi_k = \begin{cases} \xi_1, & k \in A, \\ \xi_2, & k \in (\mathbb{N}\backslash A). \end{cases}$

Let $\varepsilon > 0$ be arbitrary. Then, clearly, $\{k \in \mathbb{N} \colon \|\xi_k - \xi_2\|_{\mathbb{C}_2} \geqslant \varepsilon\} \subseteq A \in \mathcal{K}$; this implies (ξ_k) is $\mathcal{K}$-convergent to ξ_2. Thus, (ξ_k) is $\mathcal{I}^{\mathcal{K}}$-convergent to ξ_2. For $\varepsilon = \|\xi_1 - \xi_2\|_{\mathbb{C}_2}$, $\{k \in \mathbb{N} \colon \|\xi_k - \xi_2\|_{\mathbb{C}_2} \geqslant \varepsilon\} = A \in \mathcal{I}$: a contradiction. Hence, we must have $\mathcal{K} \subseteq \mathcal{I}$. □

Theorem 7. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$, such that the condition $AP(\mathcal{I}, \mathcal{K})$ holds. Then for a sequence (ξ_k) in $\mathbb{C}_2$, $\xi_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathbb{C}_2}} \xi$ implies $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$.*

Proof. Let $\xi_k \xrightarrow{\mathcal{I}-\|\cdot\|_{\mathbb{C}_2}} \xi$. Choose a sequence of rationals $(\varepsilon_i)_{i\in\mathbb{N}}$.

Then, for all i, $M_i = \{k \in \mathbb{N} : ||\xi_k - \xi||_{\mathbb{C}_2} < \varepsilon_i\} \in \mathcal{F}(\mathcal{I})$. Then, by Definition 11, there exists a set $M \in \mathcal{F}(\mathcal{I})$, such that for any $i \in \mathbb{N}$, $(M\backslash M_i) \in K$. Consider the sequence (η_k) defined by $\eta_k = \begin{cases} \xi_k, & k \in M, \\ \xi, & k \notin M. \end{cases}$

To complete the proof, it is sufficient to show that the sequence (η_k) is $\mathcal{K}$-convergent to $\xi \in \mathbb{C}_2$. Now,

$$\{k \in \mathbb{N} \colon \|\eta_k - \xi\|_{\mathbb{C}_2} < \varepsilon_i\} =$$
$$= (\mathbb{N}\backslash M) \cup \{k \in M \colon \|\xi_k - \xi\|_{\mathbb{C}_2} < \varepsilon_i\} = (\mathbb{N}\backslash M) \cup (M_i \cap M) = \mathbb{N}\backslash(M\backslash M_i).$$

Now, as $(M\backslash M_i) \in \mathcal{K}$, so $\mathbb{N}\backslash(M\backslash M_i) \in \mathcal{F}(\mathcal{K})$ and, consequently, we have $\{k \in \mathbb{N} \colon \|\eta_k - \xi\|_{\mathbb{C}_2} < \varepsilon_i\} \in \mathcal{F}(\mathcal{K})$, i.e., $\eta_k \xrightarrow{\mathcal{K}-\|\cdot\|_{\mathbb{C}_2}} \xi$.

Hence, $\xi_k \xrightarrow{\mathcal{I}^{\mathcal{K}}-\|\cdot\|_{\mathbb{C}_2}} \xi$. □

Theorem 8. *Let $\mathcal{I}, \mathcal{K}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{K}_1, \mathcal{K}_2$ be ideals in $\mathbb{N}$ and (ξ_k) be any sequence in $\mathbb{C}_2$. Then:*

(i) *If* $\mathcal{I}^{\mathcal{K}_1} - \lim \xi_k = l = \mathcal{I}^{\mathcal{K}_2} - \lim \xi_k$ *then* $\mathcal{I}^{\mathcal{K}_1 \vee \mathcal{K}_2} - \lim \xi_k = l$.

(ii) *If* $\mathcal{I}_1^{\mathcal{K}} - \lim \xi_k = l = \mathcal{I}_2^{\mathcal{K}} - \lim \xi_k$ *then* $(\mathcal{I}_1 \vee \mathcal{I}_2)^{\mathcal{K}} - \lim \xi_k = l$.

Proof. (i) Since $\mathcal{I}^{\mathcal{K}_1} - \lim \xi_k = l$ and $\mathcal{I}^{\mathcal{K}_2} - \lim \xi_k = l$, so there exists $M, N \in \mathcal{F}(\mathcal{I})$, such that for all $\varepsilon, \delta > 0$, $\{k \in M \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}_1$ and $\{k \in N \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \delta\} \in \mathcal{K}_2$.

By the hereditary property of $\mathcal{K}_1$ and $\mathcal{K}_2$, $\forall \varepsilon > 0$, $\delta > 0$, we have,

$$\{k \in M \cap N \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}_1, \quad \{k \in M \cap N \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \delta\} \in \mathcal{K}_2. \tag{5}$$

Let $\eta > 0$ be arbitrary. Then, from (5), choosing $\varepsilon = \delta = \eta$, we get $\{k \in M \cap N \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \eta\} \in \mathcal{K}_1 \vee \mathcal{K}_2$.

As $M \cap N \in \mathcal{F}(\mathcal{I})$, so the sequence (ω_k) defined by $\omega_k = \begin{cases} \xi_k, & k \in M \cap N, \\ l, & k \notin M \cap N, \end{cases}$
is $\mathcal{K}_1 \vee \mathcal{K}_2$-convergent to l. Hence, $\mathcal{I}^{\mathcal{K}_1 \vee \mathcal{K}_2} - \lim \xi_k = l$.

(ii) Since $\mathcal{I}_1^{\mathcal{K}} - \lim \xi_k = l$ and $\mathcal{I}_2^{\mathcal{K}} - \lim \xi_k = l$, so there exists $M \in \mathcal{F}(\mathcal{I}_1)$ and $N \in \mathcal{F}(\mathcal{I}_2)$, such that for all $\varepsilon, \delta > 0$, $\{k \in M \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}$ and $\{k \in N \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \delta\} \in \mathcal{K}$.

By the Hereditary property of $\mathcal{K}$, we have for all $\varepsilon > 0$, $\delta > 0$, $\{k \in M \cap N \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \varepsilon\} \in \mathcal{K}$ and $\{k \in M \cap N \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \delta\} \in \mathcal{K}$.

Let $\eta > 0$ be arbitrary. Choosing $\varepsilon = \delta = \eta$, we get,

$$\forall \eta > 0, \{k \in M \cap N \colon \|\xi_k - l\|_{\mathbb{C}_2} \geqslant \eta\} \in \mathcal{K}.$$

As $M \cap N \in \mathcal{F}(\mathcal{I}_1 \vee \mathcal{I}_2)$, so we can conclude that $(\mathcal{I}_1 \vee \mathcal{I}_2)^{\mathcal{K}} - \lim \xi_k = l$. □

4. Conclusion. The main contribution of this paper is to provide the notion of $\mathcal{I}^{\mathcal{K}}$-convergence of sequences of bi-complex numbers and study some of its properties and identify the relationships between newly introduced notion and its relationship with other convergence methods of sequences of bi-complex numbers. These ideas and results are expected to be a source for researchers in the area of convergence of sequences of bi-complex number. Also, these concepts can be generalized and applied for further studies.

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