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A NEW GENERALIZATION OF GEORGE & VEERAMANI-TYPE FUZZY METRIC SPACE

Abstract. In this article, a concept of fuzzy $\mathscr{F}$-metric space, which is a generalization of George & Veeramani-type fuzzy metric space, is introduced. Concepts of convergent sequence, Cauchy sequence, completeness etc. are given, and we study some properties in such spaces. Finally, some topological results are established.

Key words: *t-norm, fuzzy metric, fuzzy b-metric, fuzzy $\mathscr{F}$-metric*

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1. Introduction. In the recent years, many interesting extensions of the concept of metric spaces have been developed and studied. b-metric by Czerwik [1], G-metric by Mustafa & Sims [12], S-metric by Sedghi et al. [15], cone metric by Huang & Zhang [6], F-metric by Jleli & Samet [7], ϕ-metric and generalized parametric metric by Das et al. [2], [3] are some outcomes of this research process; see also [8], [14], [5] etc. Among these generalized metric spaces, F-metric space [7] is quite an attractive generalization. Here authors have used a family $\mathcal{F}$ of real-valued function $g\colon (0,\infty) \to \mathbb{R}$ that satisfies the following conditions:

($\mathcal{F}1$) g is nondecreasing on $(0,\infty)$;

($\mathcal{F}2$) for every sequence $\{x_n\} \subset (0,\infty)$, $\lim\limits_{n\to\infty} x_n = 0 \iff \lim\limits_{n\to\infty} g(x_n) = -\infty$.

In an F-metric space (X, D), by using the members of $\mathcal{F}$, Jleli & Samet [7] relaxed the "triangle inequality" of the metric axiom as

($D3$) for every $x, y \in X$, for every $N \in \mathbb{N}$, $N \geqslant 2$ and $\{a_n\}_{n=1}^{N} \subset X$,

$$D(x,y) > 0 \implies g(D(x,y)) \leqslant g \cup \left(\sum_{i=1}^{N-1} D(a_i,a_{i+1})\right) + \beta, \text{ where}$$

$g \in \mathcal{F}$ and $\beta \in [0,\infty)$.

On the other side, different authors introduced several generalized fuzzy metric spaces followed by the definition of the fuzzy metric space given by Kramosil & Michalek [11]. Later George & Veeramani [4] modified the definition by Kramosil & Michalek [11] to define a Hausdorff topology on a fuzzy metric space and proved some other basic results. An appropriate notion of generalized fuzzy metric space is fuzzy b-metric space developed by Nadaban [13]. Fuzzy b-metric space extends the concept of b-metric space following in the setting of Kramosil & Michalek type fuzzy metric space.

In this paper, we introduce a new concept, known as the fuzzy $\mathscr{F}$-metric space, as a generalization of George & Veeramani-type fuzzy metric space [4], by involving some special kind of functions $f\colon [0,1] \to [0,1]$. We provide examples to show that fuzzy $\mathscr{F}$-metric space is a proper generalization of George & Veeramani-type fuzzy metric space. But the concept of fuzzy $\mathscr{F}$-metric space and fuzzy b-metric space are totally different. We confirm this by showing that there exists a b-metric which is not a fuzzy $\mathscr{F}$-metric, and vice versa. Some basic topological properties of fuzzy $\mathscr{F}$-metric space have been studied. We prove that if a fuzzy $\mathscr{F}$-metric M is continuous with respect to t, then George & Veeramani-type fuzzy metric m is induced from M, which plays a crucial role in developing more results in fuzzy $\mathscr{F}$-metric space. The notion of fuzzy $\mathscr{F}$-boundedness is introduced, which will help in the future to study several characterizations of fuzzy $\mathscr{F}$-metric spaces. $\mathscr{F}$-convergent and $\mathscr{F}$-Cauchy sequence are defined, and some related results are studied. This paper is organized as follows. In Section 2, we provide some basic notation and results from the existing literature, necessary to develop the main results. In Section 3, definition and examples of fuzzy $\mathscr{F}$-metric spaces are given and relations with other generalized fuzzy metric spaces are studied. In Section 4, we show that the George & Veeramani-type fuzzy metric is induced from the fuzzy $\mathscr{F}$-metric. Definition of the fuzzy $\mathscr{F}$-bounded set, $\mathscr{F}$-convergent sequence, and $\mathscr{F}$-Cauchy sequence are given. This Section also proposes the concepts of open ball, Hausdorff topology, and some related results.

2. Preliminaries. We start this Section with the definition of the t-norm.

Definition 1. [10] *A binary operation $\star\colon [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:*

(i) $*$ *is associative and commutative;*

(ii) $\alpha * 1 = \alpha, \ \forall \alpha \in [0,1]$*;*

(iii) $\alpha * \gamma \leqslant \beta * \delta$ *whenever* $\alpha \leqslant \beta$ *and* $\gamma \leqslant \delta$, $\forall \alpha, \beta, \gamma, \delta \in [0,1]$.

If $*$ *is continuous, then it is called a continuous* t*-norm.*

Here are examples of some t-norms given in [10]:

1) Standard intersection: $\alpha * \beta = \min\{\alpha, \beta\}$.
2) Algebraic product: $\alpha * \beta = \alpha\beta$.
3) Bounded difference: $\alpha * \beta = \max\{0, \alpha + \beta - 1\}$.

Definition 2. [4] *The 3-tuple* $(X, M, \star)$ *is said to be a (George & Veeramani-type) fuzzy metric space if the fuzzy set* M *on* $X^2 \times (0, \infty)$ *satisfies the following conditions:*
(M1) $M(x, y, t) > 0$,
(M2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(M3) $M(x, y, t) = M(y, x, t)$,
(M4) $M(x, y, t) \star M(y, z, s) \leqslant M(x, z, s + t)$,
(M5) $M(x, y, .)\colon (0, \infty) \to [0, 1]$ *is continuous,*
for all $x, y, z \in X$ *and* $t, s > 0$.

Lemma 1. [4] *George & Veeramani fuzzy metric* $M(x, y, \cdot)$ *over a nonempty set* X *is non-decreasing with respect to* $t > 0$ *for all* $x, y \in X$.

Remark 1. [4] *In fuzzy metric space* $(X, M, \star)$, *George & Veeramani considered the following:*

1) *If* $M(x, y, t) > 1 - r$ *for all* $x, y \in X$, $t > 0$, $0 < r < 1$, *we can find a* $t_0, 0 < t_0 < t$, *such that* $M(x, y, t_0) > 1 - r$.
2) *For any* $r_1 > r_2$ *in* $(0, 1)$, *we can find* $r_3 \in (0, 1)$, *such that* $r_1 \star r_3 \geqslant r_2$ *and for any* $r_4 \in (0,1)$ *we can find* $r_5 \in (0,1)$, *such that* $r_5 \star r_5 \geqslant r_4$.

Before passing to definition of fuzzy b-metric space by Nadaban [13], we recall the definition of b-metric space by Czerwik [1] and consider an example.

Definition 3. [1] *Let* X *be a nonempty set and* $k \geqslant 1$ *be a given real number. A function* $d\colon X \times X \to [0, \infty)$ *is said to be a* b*-metric on* X *if for all* $x, y, z, \in X$ *the following conditions hold:*
$(b1)$ $d(x, y) = 0$ *if and only if* $x = y$;
$(b2)$ $d(x, y) = d(y, x)$;
$(b3)$ $d(x, z) \leqslant k[d(x, y) + d(y, z)]$.

The triple (X, d, k) *is called a* b*-metric space with the constant coefficient* k.

Example 1. [9] Let $X = [0, 1]$ and let $d : X \times X \to [0, \infty)$ be a mapping defined by
$$d(x, y) = (x - y)^2, \quad \forall x, y \in X.$$
Then d is a b-metric on X with the constant coefficient $k = 2$.

Definition 4. [13] *Let X be a nonempty set, $k \geqslant 1$ be a given real number, and $\star$ be a continuous t-norm. A fuzzy set M in $X \times X \times [0, \infty)$ is called a fuzzy b-metric if M satisfies the following conditions:*
$(bM1)$ $M(x, y, 0) = 0$;
$(bM2)$ $M(x, y, t) = 1, \forall t > 0$, *if and only if* $x = y$;
$(bM3)$ $M(x, y, t) = M(y, x, t), \forall\ t \geqslant 0$;
$(bM4)$ $M(x, z, k(t + s)) \geqslant M(x, y, t) \star M(y, z, s), \forall\ t, s \geqslant 0$;
$(bM5)$ $M(x, y, .) \colon [0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{t\to\infty} M(x, y, t) = 1$.
for all $x, y, z \in X$ and $s, t > 0$. The quadruple $(X, M, \star, k)$ is said to be a fuzzy b- metric space.

Example 2. [13] Let (X, d, k) be a b-metric space. A function $M_d \colon X \times X \times [0, \infty) \to [0, 1]$ defined by
$$M_d(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)} & if\ t > 0, \\ 0 & if\ t = 0 \end{cases}$$
is a fuzzy b-metric on X with the constant coefficient k under the t-norm ‘ min’.

3. Introduction to fuzzy $\mathscr{F}$-metric space. In this Section, we introduce the idea of fuzzy $\mathscr{F}$-metric space and provide some examples. A comparison between fuzzy $\mathscr{F}$-metric space and fuzzy metric as well as fuzzy b-metric space is studied.

Throughout the article, $\mathscr{F}$ denotes the set of all functions $f \colon [0, 1] \to [0, 1]$ that satisfy the following conditions:

($\mathscr{F}1$) f is strictly increasing in $[0, 1)$;
($\mathscr{F}2$) for every sequence $\{t_n\}$ in $[0, 1]$, $\lim_{n\to\infty} t_n = 1 \Longleftrightarrow \lim_{n\to\infty} f(t_n) = 1$.

Example 3. Here are some examples of elements of $\mathscr{F}$:
(i) $f(x) = x^n, \ \forall x \in [0, 1], n \in \mathbb{N}$;
(ii) $f(x) = \sqrt{x}, \ \forall x \in [0, 1]$.

Now, using the functions $f \in \mathscr{F}$, we define the fuzzy $\mathscr{F}$-metric space by relaxing the axiom $(M4)$ of George & Veeramani-type fuzzy metric space.

Definition 5. *Let X be a nonempty set and $M\colon X\times X\times(0,\infty)\to[0,1]$ be a mapping, and $\star$ be a continuous t-norm. If there exists $(f,\alpha)\in\mathscr{F}\times(0,1]$, such that M satisfies the following conditions:*

($\mathscr{F}M1$) $M(x,y,t)>0,\ \forall x,y\in X$ *and* $t>0$;

($\mathscr{F}M2$) $M(x,y,t)=1,\forall t,>0$ *iff* $x=y$;

($\mathscr{F}M3$) $M(x,y,t)=M(y,x,t),\ \forall x,y\in X$ *and* $t>0$;

($\mathscr{F}M4$) for every $(x,y)\in X\times X$, *for every* $N\in\mathbb{N}$, $N\geqslant 2$, *and for every* $\{u_i\}_i^N\subseteq X$ *with* $u_1=x$ *and* $u_N=y$, *we have* $M(x,y,t)<1$ *implies*

$$(f(M(x,y,t)))^\alpha\geqslant f(M(u_1,u_2,t_1)\star M(u_2,u_3,t_2)\star\cdots\star M(u_{N-1},u_N,t_{N-1})),$$

where $t=t_1+t_2+\ldots+t_{N-1}$; $t_i>0$ *for* $i=1,2,\cdots,(N-1)$, *then M is said to be a fuzzy $\mathscr{F}$-metric on X and the 5-tuple $(X,M,f,\alpha,\star)$ is said to be a fuzzy $\mathscr{F}$-metric space.*

Remark 2. *It has been observed that every George & Veeramani-type fuzzy metric space is a fuzzy $\mathscr{F}$-metric space, since if $(X,M,\star)$ is a George & Veeramani-type fuzzy metric space, then, clearly, M satisfies $(\mathscr{F}M1)-(\mathscr{F}M3)$. We only verify the condition $(\mathscr{F}M4)$. For $x,y,z\in X$ with $x\neq y$ and $s,t>0$, we have, from $(M4)$,*

$$\begin{aligned}
&M(x,y,s+t)\geqslant M(x,z,s)\star M(z,y,t)\\
\Longrightarrow\ &f(M(x,y,s+t))\geqslant f(M(x,z,s)\star M(z,y,t)) \qquad (\textit{using } (\mathscr{F}1))\\
\Longrightarrow\ &(f(M(x,y,s+t)))^\alpha\geqslant f(M(x,z,s)\star M(z,y,t)) \quad (\textit{since } \alpha\in(0,1]).
\end{aligned}$$

If we write $s+t=T$ *and* $f(t)=t$ *for all* $t\in[0,1]$ *and* $\alpha=1$, *we get*

$$(f(M(x,y,T)))^\alpha\geqslant f(M(x,z,s)\ \star\ M(z,y,t)).$$

Thus, M is a fuzzy $\mathscr{F}$-metric on X with $f(t)=t,\ \forall t\in[0,1]$ and $\alpha=1$.

Remark 3. *The family of fuzzy $\mathscr{F}$-metric spaces is larger than the family of fuzzy metric spaces over a non-empty set. In this context, we present the following example of a fuzzy $\mathscr{F}$-metric, which is not a George-Veeramani-type fuzzy metric.*

Example 4. Let $X=\mathbb{R}$ and define a function $M\colon X\times X\times(0,\infty)\to[0,1]$ by

$$M(x,y,t)=\Big(\frac{t}{t+1}\Big)^{|x-y|^2}\quad\text{for all } (x,y)\in X\times X \text{ and } t\in(0,\infty).$$

We now prove that M is a fuzzy $\mathscr{F}$-metric on X with respect to the t-norm '$\star$' as 'product'.
Clearly, M satifies $(\mathscr{F}M1)-(\mathscr{F}M3)$. We only verify the condition $(\mathscr{F}M4)$. Let us choose $x, y, z \in X$, such that $x \neq y$. Then the definition of fuzzy $\mathscr{F}$-metric implies that $M(x,y,t) < 1$ for all $t > 0$. Now we take $s, t > 0$ arbitrarily and consider a mapping $f(x) = x^2,\ x \in [0,1]$. Then, for $\alpha = \frac{1}{2}$,

$$\begin{aligned}
&(f(M(x,y,s+t)))^{\alpha} - f(M(x,z,s) \star M(z,y,t)) = \\
&\quad = \left((M(x,y,s+t))^2\right)^{\frac{1}{2}} - f\left(M(x,z,s) \cdot M(z,y,t)\right) = \\
&\quad = M(x,y,s+t) - (M(x,z,s))^2.(M(z,y,t))^2 = \\
&\qquad\qquad = \left(\frac{(s+t)}{(s+t)+1}\right)^{|x-y|^2} - \left(\left(\frac{s}{s+1}\right)^{|x-z|^2}\right)^2 . \left(\left(\frac{t}{t+1}\right)^{|z-y|^2}\right)^2 .
\end{aligned}$$

Again,

$$|x-y|^2 \leqslant 2(|x-z|^2 + |z-y|^2) \tag{1}$$

holds for any $x, y, z \in \mathbb{R}$. Now, for $s, t > 0$, we have the following cases:
Case I : Suppose $s = t$. Then we have $\dfrac{(s+t)}{(s+t)+1} = \dfrac{2t}{2t+1}$.
Again, $\dfrac{2t}{2t+1} > \dfrac{t}{t+1}$ holds for all $t > 0$. This implies

$$\begin{aligned}
&\left(\frac{2t}{2t+1}\right)^{|x-y|^2} > \left(\frac{t}{t+1}\right)^{|x-y|^2} > \\
&\qquad\qquad > \left(\frac{t}{t+1}\right)^{2(|x-z|^2+|z-y|^2)} \quad \left(\text{Since } 0 < \left(\frac{t}{t+1}\right) < 1\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\Big(f(M(x,y,s+t))\Big)^{\alpha} - f(M(x,z,s) \star M(z,y,t)) > \\
&\qquad > \left(\frac{2t}{2t+1}\right)^{|x-y|^2} - \left(\frac{t}{t+1}\right)^{2|x-z|^2} \left(\frac{t}{t+1}\right)^{2|z-y|^2} > \\
&\qquad\qquad > \left(\frac{t}{t+1}\right)^{|x-y|^2} - \left(\frac{t}{t+1}\right)^{2(|x-z|^2+|z-y|^2)} > 0.
\end{aligned}$$

Hence, $(f(M(x,y,s+t)))^{\alpha} > f(M(x,z,s) \star M(z,y,t))$ holds.
Case II : Suppose $s < t$. Then we have $\dfrac{(s+t)}{(s+t)+1} > \dfrac{t}{t+1}$, which implies

$$\left(\frac{(s+t)}{(s+t)+1}\right)^{|x-y|^2} > \left(\frac{t}{t+1}\right)^{|x-y|^2}.$$

Again,

$$\left(\frac{s}{s+1}\right) < \left(\frac{t}{t+1}\right) \implies \left(\frac{s}{s+1}\right)^{2|x-z|^2} < \left(\frac{t}{t+1}\right)^{2|x-z|^2}.$$

Thus, we have

$$\begin{aligned}
&\Big(f(M(x,y,s+t))\Big)^{\alpha} - f(M(x,z,s) \star M(z,y,t)) = \\
&= \left(\frac{(s+t)}{(s+t)+1}\right)^{|x-y|^2} - \left(\left(\frac{s}{s+1}\right)^{2|x-z|^2}\right) \cdot \left(\left(\frac{t}{t+1}\right)^{2|z-y|^2}\right) > \\
&> \left(\frac{t}{t+1}\right)^{|x-y|^2} - \left(\left(\frac{t}{t+1}\right)^{2|x-z|^2}\right)\left(\left(\frac{t}{t+1}\right)^{2|z-y|^2}\right) = \\
&= \left(\frac{t}{t+1}\right)^{|x-y|^2} - \left(\frac{t}{t+1}\right)^{2(|x-z|^2+|z-y|^2)} > 0.
\end{aligned}$$

Hence, $(f(M(x,y,s+t)))^{\alpha} > f(M(x,z,s) \star M(z,y,t))$ holds.

Case III : Suppose $t < s$. This case is similar to Case II.

Therefore, M is a fuzzy $\mathscr{F}$-metric on X with '$\star$ = product'. However M does not satisfy the inequality $(M4)$, since for $x = 0,\ y = 2,\ z = 1$ and $s = t = 1$, we have

$$M(0,2,1+1) = (\frac{2}{3})^4 < \frac{1}{4} = (\frac{1}{2}).(\frac{1}{2}) = M(0,1,1).M(1,2,1).$$

Therefore, M is not George & Veeramani-type fuzzy metric.

Remark 4. *As we know, every George & Veeramani-type fuzzy metric is a fuzzy $\mathscr{F}$-metric and every George & Veeramani-type fuzzy metric is an increasing function with respect to $t > 0$. So a fuzzy $\mathscr{F}$-metric is often an increasing function with respect to $t > 0$. But our next example shows that there exist fuzzy $\mathscr{F}$-metric that is neither increasing nor decreasing with respect to $t > 0$.*

Example 5. Consider two metrics d_1 and d_2 on $\mathbb{R}^2$, defined by:

$$d_1(a,b) = \max\{|a_i - b_i| : i = 1,2\} \text{ and } d_2(a,b) = \sum_{i=1}^{2} |a_i - b_i|$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{R}^2$. Then, clearly, $d_1(a, b) \leqslant d_2(a, b)$ for all $a, b \in \mathbb{R}^2$. Now define a function $M : \mathbb{R}^2 \times \mathbb{R}^2 \times (0, \infty) \to [0, 1]$ by

$$M(a, b, t) = \begin{cases} 1/2 & \text{if } 0 < t \leqslant d_1(a, b), \\ 1/4 & \text{if } d_1(a, b) < t \leqslant d_2(a, b), \\ \dfrac{t}{t + d_2(a, b)} & \text{if } d_2(a, b) < t < \infty \end{cases}$$

for all $a, b \in \mathbb{R}^2$ and $t > 0$.

We claim that M is a fuzzy $\mathscr{F}$-metric on $\mathbb{R}^2$ with respect to the t-norm $\star = \min$. Now observe that M satisfies $(\mathscr{F}M1)$–$(\mathscr{F}M3)$. We only verify the condition $(\mathscr{F}M4)$.

Let $a, b, c \in \mathbb{R}^2$ and $T > 0$, such that $T = s + t$ where $s, t > 0$. Then for $s, t > 0$ we have the following cases:

Case I : $s > d_2(a, c)$, $t > d_2(b, c)$.
Therefore, $s+t > d_2(a, c)+d_2(b, c) \geqslant d_2(a, b)$, which implies $M(a, b, s+t) = \frac{s+t}{(s+t)+d_2(a,b)}$.

Again, we have $M(a, c, s) = \frac{s}{s+d_2(a,c)}$ and $M(b, c, t) = \frac{t}{t+d_2(b,c)}$. Since $M(a, b, t) = \frac{t}{t+d(a,b)}$ for all $a, b \in \mathbb{R}^2$ and $t > 0$ is a fuzzy metric on $\mathbb{R}^2$ whenever d is a metric on $\mathbb{R}^2$(please see [4]), we obtain

$$\min\{M(a, c, s), M(b, c, t)\} \leqslant M(a, b, s + t). \tag{2}$$

Again, since $M(a, b, r) \in [0, 1]$ for all $a, b \in \mathbb{R}^2$ and $r > 0$, so

$$M(a, b, s + t) \leqslant M(a, b, s + t)^{\frac{1}{2}}. \tag{3}$$

Hence, by taking $f(x) = x$, $\forall x \in [0, 1]$, $\alpha = \frac{1}{2}$ and using (2) and (3), we have

$$(f(M(a, c, s) \star M(b, c, t))) \leqslant f((M(a, b, T)))^{\frac{1}{2}}.$$

Case II : $s > d_2(a, c)$, $d_1(b, c) < t \leqslant d_2(b, c)$.
Therefore, $s + t > d_2(a, c) + d_1(b, c) \geqslant d_1(a, c) + d_1(b, c) \geqslant d_1(a, b)$ and, hence,

$$M(a, b, s + t) = \begin{cases} 1/4 & \text{if } d_1(a, b) < s + t \leqslant d_2(a, b), \\ \dfrac{s + t}{(s + t) + d_2(a, b)} & \text{if } d_2(a, b) < s + t. \end{cases}$$

Again, $M(a,c,s) = \frac{s}{s+d_2(a,c)}$ and $M(b,c,t) = \frac{1}{4}$ implies

$$\min\{M(a,c,s), M(b,c,t)\} < M(a,b,s+t)$$

or

$$f(M(a,c,s) \star M(b,c,t)) < (f(M(a,b,s+t)))^{\frac{1}{2}}.$$

Case III : $s > d_2(a,c),\ 0 < t < d_1(b,c)$. Then $s+t > d_2(a,c) > 0$. Therefore,

$$M(a,b,s+t) = \begin{cases} 1/2 & \text{if } \ 0 < s+t \leqslant d_1(a,b), \\ 1/4 & \text{if } \ d_1(a,b) < s+t \leqslant d_2(a,b), \\ \dfrac{s+t}{(s+t)d_2(a,b)} & \text{if } \ d_2(a,b) < s+t < \infty. \end{cases}$$

Again, $M(a,c,s) = s/(s+d_2(a,c))$ and $M(b,c,t) = \frac{1}{2}$ implies $\min\{M(a,c,s), M(b,c,t)\} = \frac{1}{2}$. Thus, if $0 < s+t \leqslant d_1(a,b)$ or $d_2(a,b) < s+t < \infty$, then $M(a,b,s+t) \geqslant M(a,c,s) \star M(b,c,t)$. But if $d_1(a,b) < s+t \leqslant d_2(a,b)$, then

$$(M(a,b,s+t))^{\frac{1}{2}} = (\frac{1}{4})^{\frac{1}{2}} = M(a,c,s) \star M(b,c,t),$$
$$i.e. \ \ (M(a,b,s+t))^{\frac{1}{2}} = M(a,c,s) \star M(b,c,t).$$

The other cases can be verified similarly.

Therefore, for all $a, b \in \mathbb{R}^2$ and $T = s+t (> 0)$, M satisfies

$$(f(M(a,b,T)))^{\alpha} \geqslant f(M(a,c,s) \star M(b,c,t)),$$

where $f(x) = x$, $\forall x \in [0,1]$ and $\alpha = \frac{1}{2}$. Hence, M is a fuzzy $\mathscr{F}$-metric on $\mathbb{R}^2$ with respect to $f(x) = x$, for all $x \in [0,1]$, $\alpha = \frac{1}{2}$ and $\star = \min$.

Remark 5. *Though the family of fuzzy $\mathscr{F}$-metric spaces contains the family of fuzzy metric spaces, there is a fuzzy b-metric which may not be a fuzzy $\mathscr{F}$-metric, and vice versa. The fuzzy $\mathscr{F}$-metric space in the Example 5 above is not a fuzzy b-metric space. In the next example we prove the converse part.*

Example 6. Consider the b-metric space of the Example 1. Then, by Example 2,

$$M_d(x,y,t) = \begin{cases} \dfrac{t}{t+d(x,y)}, & \text{if } t > 0, \\ 0, & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$ and $t > 0$, is a fuzzy b-metric space with the constant coefficient $k = 2$ and t-norm $\star = \min$. If possible to suppose that M_d is a fuzzy $\mathscr{F}$-metric on X. Then there exists $(f, \alpha) \in \mathscr{F} \times (0, 1]$, such that M satisfies $(\mathscr{F}M4)$.

Let $n \in \mathbb{N}$ and $u_i = i/n$ for $i = 0, 1, \ldots, n$. Then, taking $u_0 = 0,\ u_n = 1$ and $t = t_0 + t_1 + \cdots + t_{n-1}$, from $(\mathscr{F}M4)$ we have:

$$(f(M_d(0,1,t)))^{\alpha} \geqslant \\ \geqslant f(\min\{M_d(0,u_1,t_0), M_d(u_1,u_2,t_1), \ldots, M_d(u_{n-1},1,t_{n-1})\}). \quad (4)$$

Suppose for some $i = r$,
$\min\{M_d(0,u_1,t_0), M_d(u_1,u_2,t_1), \ldots, M_d(u_{n-1},1,t_{n-1})\} = M_d(u_{r-1},u_r,t_{r-1})$.
Then the relation (4) gives

$$\left(f\left(\frac{t}{t+1}\right)\right)^{\alpha} \geqslant f\left(\frac{t_{r-1}}{t_{r-1}+\frac{1}{n}}\right) \to 1 \quad \text{as } n \to \infty$$
$$\implies \left(f\left(\frac{t}{t+1}\right)\right)^{\alpha} = 1 \implies f\left(\frac{t}{t+1}\right)$$

which gives $t/(t+1) = 1$. This is absurd.

4. Characterization of fuzzy $\mathscr{F}$-metric space. In this Section, first we induce George & Veeramani-type fuzzy metric from fuzzy $\mathscr{F}$-metric. Then we define the notion of convergent and Cauchy sequences and prove some related results. Later we study some topological properties and define fuzzy $\mathscr{F}$-boundedness for a set in the setting of fuzzy $\mathscr{F}$-metric space.

The next proposition shows that a fuzzy $\mathscr{F}$- metric induces a George & Veeramani-type fuzzy metric under certain conditions.

Proposition 1. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space. Define a function $m\colon X \times X \times (0,\infty) \to [0,1]$ by*

$$m(x,y,t) = \sup\{M(u_1,u_2,t_1) \star \ldots \star M(u_{N-1},u_N,t_{N-1})\colon N \in \mathbb{N}, N \geqslant 2; \\ \text{with } (u_1,u_N) = (x,y)\} \quad (5)$$

for all $x, y \in X$ and $t > 0$, where $t = t_1 + t_2 + \ldots + t_{N-1}$.

If $M(x,y,\cdot)$ is a continuous and non-decreasing function of t for all $x, y \in X$, then $(X, m, \star)$ is a George & Veeramani-type fuzzy metric space.

Proof. Since M is a fuzzy $\mathscr{F}$-metric space, there exists a pair $(f,\alpha) \in \mathscr{F} \times (0,1]$ with respect to which M satisfies the condition $(\mathscr{F}M4)$. Now, m satisfies $(M1), (M3)$ and $(M5)$ trivially. We only verify $(M2)$ and $(M4)$. Now,
(i) If $x = y$, then $M(x,y,t) = 1$ for all $t > 0$ and, hence, $m(x,y,t) = 1$ for all $t > 0$.

Conversly, if possible, suppose that there exists $x, y (x \neq y) \in X$, such that $m(x,y,t) = 1$ for all $t > 0$. Then there exists $t_0 > 0$i, such that

$$M(x,y,t_0) < 1. \tag{6}$$

Let $0 < \varepsilon < 1$. Then, by definition of m, there exists $N \in \mathbb{N}$, $N \geqslant 2$ and $\{u_i\}_i^N \subset X$ with $(u_1, u_N) = (x,y)$, such that

$$\begin{aligned}
&1 - \epsilon < M(u_1,u_2,t_1) \star \ldots \star M(u_{N-1},u_N,t_{N-1}), t_0 = t_1 + t_2 + \ldots + t_{N-1},\\
&\Longrightarrow f(1-\epsilon) < f(M(u_1,u_2,t_1) \star \ldots \star M(u_{N-1},u_N,t_{N-1})) \leqslant\\
&\qquad \leqslant (f(M(x,y,t_0)))^{\alpha} \quad (\mathit{by}\ (\mathscr{F}1)\ \$\ (\mathscr{F}M4)).
\end{aligned}$$

Since $0 < \epsilon < 1$ is choosen arbitrarily, we have

$$(f(M(x,y,t_0)))^{\alpha} \geqslant f(1) = 1 \Longrightarrow (f(M(x,y,t_0))) = 1 \Longrightarrow M(x,y,t_0) = 1.$$

This contradicts the relation (6). Hence, $m(x,y,t) = 1$ for all $t > 0$ implies $x = y$.
(ii) Let $x, y, z \in X$ and $0 < \epsilon < 1$. Then, by the definition of m, there exist two chains of points $x = u_1, u_2, \ldots, u_n = y$, $y = u_n, u_{n+1}, \ldots, u_N = z$, such that

$$\begin{aligned}
&m(x,y,s) - \epsilon < M(u_1,u_2,t_1) \star \ldots \star M(u_{n-1},u_n,t_{n-1})\\
\text{and}\quad &m(y,z,t) - \epsilon < M(u_n,u_{n+1},t_n) \star \ldots \star M(u_{N-1},u_N,t_{N-1}),
\end{aligned}$$

where $s = t_1 + \ldots + t_{n-1}$, $t = t_n + t_{n+1} + \ldots + t_{N-1}$; $t_i > 0$, $i = 1, 2, \ldots, N-1$. Therefore,

$$\begin{aligned}
&m(x,z,T) \geqslant M(u_1,u_2,t_1) \star \ldots \star M(u_{n-1},u_n,t_{n-1}) \star M(u_n,u_{n+1},t_n) \star\\
&\star \ldots \star M(u_{N-1},u_N,t_{N-1}) >\\
&> (m(x,y,s) - \varepsilon) \star (m(y,z,t) - \varepsilon) \quad \text{where } T = s + t.
\end{aligned}$$

Since $0 < \varepsilon < 1$ is arbitrary, we obtain as $\varepsilon \to 0^+$:

$$m(x,z,T) \geqslant m(x,y,s) \star m(y,z,t).$$

Thus m satisfies the inequality $(M4)$. Moreover, since M is a non-decreasing function of t, so, from the definition of m, it follows that $m(x,y,t)$ is also non-decreasing w.r.t t, for all $x,y \in X$. Therefore, m is a George-Veeramani-type fuzzy metric on X with t-norm '$\star$'. $\square$

Remark 6. *We call the obtained George-Veeramani-type fuzzy metric m the induced by the fuzzy $\mathscr{F}$-metric M.*

The following example is for detailed demonstration of the Proposition 1.

Example 7. Consider the fuzzy $\mathscr{F}$-metric space of Example 4. Then, using Proposition 1, we can define the function 'm' as

$$\begin{aligned} m(x,y,t) &= \\ &= \sup\Big\{M(u_1,u_2,t_1)\cdot\ldots\cdot M(u_{N-1},u_N,t_{N-1})\colon N\in\mathbb{N}, N\geqslant 2;\ \{u_i\}_i^N\subset\mathbb{R} \\ &\qquad \text{with } (u_1,u_N)=(x,y)\Big\} \\ &= \sup\Big\{\Big(\frac{t_1}{t_1+1}\Big)^{|u_1-u_2|^2}\cdot\ldots\cdot\Big(\frac{t_{N-1}}{t_{N-1}+1}\Big)^{|u_{N-1}-u_N|^2} : N\in\mathbb{N}, N\geqslant 2; \\ &\qquad \{u_i\}_i^N\subset\mathbb{R} \text{ with } (u_1,u_N)=(x,y)\Big\} \end{aligned}$$

for all $x,y\in\mathbb{R}$ & $t>0$, where $t=t_1+t_2+\ldots+t_{N-1}$. Now,

(i) Clearly, $m(x,y,t)>0$ $\forall x,y\in\mathbb{R}$ and $t>0$, since $M(u,v,r)>0$ $\forall u,v\in\mathbb{R}$ and $r>0$. Thus $(M1)$ holds.

(ii) Let $x,y\in\mathbb{R}$, such that $x=y$. Then, from the definition of m, it directly follows that $m(x,y,t)=1$ for all $x,y\in\mathbb{R}$ and $t>0$. Again, suppose $x,y\in\mathbb{R}$, such that

$$\begin{aligned} & m(x,y,t)=1 \quad \forall\, t>0 \\ & \implies \sup\Big\{\Big(\frac{t_1}{t_1+1}\Big)^{|u_1-u_2|^2}\cdot\ldots\cdot\Big(\frac{t_{N-1}}{t_{N-1}+1}\Big)^{|u_{N-1}-u_N|^2} : \\ & N\in\mathbb{N}, N\geqslant 2;\ \{u_i\}_i^N\subset\mathbb{R} \text{ with } (u_1,u_N)=(x,y)\}=1 \quad \forall\, t>0 \\ & \implies \Big(\frac{t_i}{t_i+1}\Big)^{|u_i-u_{i+1}|^2}=1 \quad \text{for each } i=1,\cdots N-1 \\ & \implies u_1=u_2,\ \cdots\ u_{N-1}=u_N \implies x=y. \end{aligned}$$

Therefore, m satisfies $(M2)$.

(iii) Observe that $(M3)$ holds trivially. Using the 'product' t-norm instead of t-norm '$\star$' in the third point of the proof of Proposition 1, we can check that m satisfies the inequality $(M4)$. Moreover, since $\left(\frac{t}{t+1}\right)^{|u-v|^2}$ is a continuous function of t for all $u, v \in \mathbb{R}$ and $t > 0$, from the definition of m it follows that $m(x,y,t)$ is also continuous w.r.t t, for all $x, y \in \mathbb{R}$. Hence, m satisfies $(M5)$.

Therefore, m is a George-Veeramani-type fuzzy metric on $\mathbb{R}$ with respect to the t-norm 'product'.

Theorem 1. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space, such that f is continuous from the left, and suppose that $M(x,y,\cdot)$ is a continuous and non-decreasing function of t, for all $x, y \in X$. If m is the respective induced George & Veeramani-type fuzzy metric on X with respect to '$\star$', then the following holds:*

$$x, y \in X \text{ with } M(x,y,t) < 1 \ \forall t > 0 \implies \\ f(M(x,y,t)) \leqslant f(m(x,y,t)) \leqslant (f(M(x,y,t)))^{\alpha}. \quad (7)$$

Proof. Let $(x,y) \in X \times X$ be such that $M(x,y,t) < 1$ for all $t > 0$. From the definition of m, it is clear that $m(x,y,t) \geqslant M(x,y,t)$; this implies

$$f(m(x,y,t)) \geqslant f(M(x,y,t)) \quad (\text{by } (\mathscr{F}1)). \quad (8)$$

Let $0 < \varepsilon < 1$ be arbitrary. Then, by the definition of m, there exists $N \in \mathbb{N}$, $N \geqslant 2$, $\{u_i\}_{i=1}^{N} \subseteq X$ with $u_1 = x$, $u_N = y$, such that for $t = t_1 = t_2 + \ldots + t_{N-1}$:

$$\begin{aligned} & m(x,y,t) - \varepsilon < M(u_1,u_2,t_1) \star \ldots \star M(u_{N-1},u_N,t_{N-1}) \\ \implies & f(m(x,y,t) - \varepsilon) < f(M(u_1,u_2,t_1) \star \ldots \star M(u_{N-1},u_N,t_{N-1})) \\ \implies & (f(M(x,y,t)))^{\alpha} > f(m(x,y,t) - \varepsilon) \quad (\textit{by } (\mathscr{F}M4)). \end{aligned}$$

Since $0 < \epsilon < 1$ is arbitrary, by letting $\varepsilon \to 0^+$ we obtain

$$(f(M(x,y,t)))^{\alpha} \geqslant f(m(x,y,t)). \quad (9)$$

The relations (8), (9) together give

$$f(M(x,y,t)) \leqslant f(m(x,y,t)) \leqslant (f(M(x,y,t)))^{\alpha}.$$

□

Now let us study the topology in fuzzy $\mathscr{F}$-metric space induced by the $\mathscr{F}$-open ball.

Definition 6. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space. For some $x \in X$ and $r > 0$, $t > 0$, define $\mathscr{F}$-open ball as*

$$B_{\mathscr{F}}(x, r, t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

Proposition 2. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space. Then*

$$\tau_{\mathscr{F}} = \{A \subseteq X : \text{ for each } x \in A, \exists r > 0, t > 0 \text{ such that } B_{\mathscr{F}}(x, r, t) \subseteq A\}$$

is a topology on X.

Proof. The proof is similar to George & Veeramani-type fuzzy metric space [4, Result 3.3]. □

Now let us prove that $\tau_{\mathscr{F}}$ is a Hausdorff topology.

Theorem 2. *Every fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ is a Hausdorff space.*

Proof. Let $x, y \in X$ with $x \neq y$. Then there exists $t_0 > 0$, such that $M(x, y, t_0) < 1$.

Let $a_n = 1 - M(x, y, t_0)/n$, $\forall n$. Then $a_n \to 1$ as $n \to \infty$.

Since for each $n \in \mathbb{N}$, $0 < a_n < 1$, then, by Remark 1, there exists $0 < b_n < 1$, such that $b_n \star b_n \geqslant a_n$. Now, consider for some $k \in \mathbb{N}$ two open sets $U = \mathrm{int}(\mathrm{B}(\mathrm{x}, 1-\mathrm{b_k}, \frac{t_0}{2}))$ and $V = \mathrm{int}(\mathrm{B}(\mathrm{y}, 1-\mathrm{b_k}, \frac{t_0}{2}))$ containing x and y, respectively. If $U \cap V = \phi$, then the proof is done. If possible, suppose that there exists $z \in U \cap V$. Then $M(x, z, \frac{t_0}{2}) > b_k$ and $M(y, z, \frac{t_0}{2}) > b_k$ and hence we get

$$(f(M(x, y, t_0)))^{\alpha} \geqslant f\Big(M(x, z, \frac{t_0}{2}) \star M(y, z, \frac{t_0}{2})\Big) > f(b_k \star b_k) \geqslant f(a_k).$$

Letting $k \to \infty$, we obtain

$$\begin{aligned} f(M(x, y, t_0)))^{\alpha} \geqslant 1 &\implies f(M(x, y, t_0)))^{\alpha} = 1 \\ &\implies f(M(x, y, t_0)) = 1 \implies M(x, y, t_0) = 1. \end{aligned}$$

This contradicts our assumption. Hence the proof is complete. □

Definition 7. *A subset G in a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ is said to be $\mathscr{F}$-open if $G \in \tau_{\mathscr{F}}$ and $\mathscr{F}$-closed if $X \backslash G \in \tau_{\mathscr{F}}$.*

Next we define $\mathscr{F}$-convergent and $\mathscr{F}$-Cauchy sequence in a fuzzy $\mathscr{F}$-metric space.

Definition 8. *Let $\{x_n\}$ be a sequence in a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$. Then $\{x_n\}$ is said to be*

1) *$\mathscr{F}$-convergent if there exists $x \in X$, such that for any $0 < r < 1$, there exists a natural number $N \in \mathbb{N}$ such that for all $t > 0$, $M(x_n, x, t) > 1 - r \;\; \forall n \geqslant N$.*
2) *$\mathscr{F}$-Cauchy sequence if for each $t > 0$ and $0 < r < 1$, there exists a natural number $N \in \mathbb{N}$, such that $M(x_n, x_m, t) > 1 - r \;\; \forall m, n \geqslant N$.*
3) *X is said to be $\mathscr{F}$-complete if every $\mathscr{F}$-Cauchy sequence in X converges to some point in X.*

The folowing results can be proved easily.

Proposition 3. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space, $\{x_n\} \subseteq X$ be a sequence, and $x \in X$. Then*

1) *$\{x_n\}$ is $\mathscr{F}$-convergent to x iff $\lim\limits_{n\to\infty} M(x_n, x, t) = 1 \;\; \forall\, t > 0$.*
2) *$\{x_n\}$ is $\mathscr{F}$-Cauchy iff $\lim\limits_{m,n\to\infty} M(x_n, x_m, t) = 1 \;\; \forall\, t > 0$.*

Proposition 4. *Limit of an $\mathscr{F}$-convergent sequence in $(X, M, f, \alpha, \star)$ is unique.*

Proof. Let $x, y \in X$ be such that a sequence $\{x_n\}$ converges to both x and y. Then

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(x_n, y, t) \;\; \forall t > 0.$$

Since M is a fuzzy $\mathscr{F}$-metric, there exists $(f, \alpha) \in \mathscr{F} \times (0, 1]$ satisfying $(\mathscr{F}M4)$.

Now suppose $x \neq y$. Then there exists $t_0 > 0$, such that $M(x, y, t_0) < 1$ and, hence, by $(\mathscr{F}M4)$:

$$\begin{aligned}(f(M(x, y, t_0)))^\alpha \geqslant f(M(x, x_n, t_1) \star M(y, x_n, t_2)) \;\; \forall n \in \mathbb{N}, \quad t_0 = t_1 + t_2,\\ \implies (f(M(x, y, t_0)))^\alpha \geqslant \lim_{n\to\infty} f(M(x, x_n, t_1) \star M(y, x_n, t_2)).\end{aligned}$$

On the other hand, by $(\mathscr{F}2)$ we have $\lim\limits_{n\to\infty} f(M(x, x_n, t_1) \star M(y, x_n, t_2)) = 1$, which implies

$$\begin{aligned}(f(M(x, y, t_0)))^\alpha \geqslant 1 \implies (f(M(x, y, t_0)))^\alpha = 1\\ \implies f(M(x, y, t_0)) = 1 \implies M(x, y, t_0) = 1 \quad (by\ (\mathscr{F}2)).\end{aligned}$$

This is a contradiction to our assumption. □

Proposition 5. *In a fuzzy $\mathscr{F}$-metric space, every $\mathscr{F}$-convergent sequence is an $\mathscr{F}$-Cauchy sequence.*

Proof. Let $\{x_n\}$ be an $\mathscr{F}$-convergent sequence in $(X, M, f, \alpha, \star)$ converging to $x \in X$. Then

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ for all } t > 0. \tag{10}$$

Let $(f, \alpha) \in \mathscr{F} \times (0, 1]$ be such that $(\mathscr{F}M4)$ holds.
Let $0 < \epsilon < 1$ be fixed. By $(\mathscr{F}2)$, there exists $0 < \delta < 1$, such that

$$1 - \delta < t < 1 \implies 1 - \epsilon < f(t) < 1. \tag{11}$$

By Remark 1, for $\delta \in (0, 1)$, we can choose $\beta \in (0, 1)$, such that $(1 - \beta) \star (1 - \beta) \geqslant (1 - \delta)$.

Again, (10) implies that for $t_1 > 0$ and $t_2 > 0$ there exist $N_1(t_1)$ and $N_2(t_2) \in \mathbb{N}$, such that

$$M(x_n, x, t_1) > 1 - \beta \text{ for all } n \geqslant N_1(t_1),$$

$$M(x_m, x, t_2) > 1 - \beta \text{ for all } m \geqslant N_2(t_2).$$

Let $t = t_1 + t_2$ and $N(t) = \max\{N_1(t_1), N_2(t_2)\}$. Then using (11) and $(\mathscr{F}M4)$, we have

$$\begin{aligned}
&M(x_n, x, t_1) \star M(x_m, x, t_2) > (1 - \beta) \star (1 - \beta) \geqslant (1 - \delta)\ \forall m, n \geqslant N(t)\\
&\implies f(M(x_n, x_m, t))^\alpha > 1 - \epsilon \quad \text{for all } \ m, n \geqslant N(t) \quad \text{for all } t > 0\\
&\implies \lim_{m,n\to\infty} f(M(x_n, x_m, t)) = 1 \quad \text{for all } t > 0\\
&\implies \lim_{m,n\to\infty} M(x_n, x_m, t) = 1 \quad \text{for all } t > 0.
\end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in $(X, M, f, \alpha, \star)$. □

The next result proves that convergence nature with converging point and cauchyness of a sequence remain invariant in fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ and George & Veeramani-type metric space $(X, m, \star)$, where m is induced by M as in relation (5).

Theorem 3. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space, such that $M(x, y, \cdot)$ is continuous and non-decreasing with respect to t for all $x, y \in X$ and $t > 0$ and m be the George-Veeramani-type fuzzy metric space induced in the relation (5). Let (f, α) in $\mathscr{F} \times (0, 1]$ with respect to which $(\mathscr{F}M4)$ holds. Then*
(i) $\{x_n\}$ is $\mathscr{F}$-convergent to $x \in X$ in $(X, M, f, \alpha, \star)$ $\iff$ $\{x_n\}$ converges

to x in $(X, m, \star)$.
(ii) $\{x_n\}$ is a $\mathscr{F}$-Cauchy sequence in $(X, M, f, \alpha, \star)$ $\iff$ $\{x_n\}$ is a Cauchy sequence in $(X, m, \star)$.
(iii) X is $\mathscr{F}$-complete $\iff$ X is complete with respect to the fuzzy metric m.

Proof. (i) First suppose that $\{x_n\}$ converges to $x \in X$ in $(X, M, f, \alpha, \star)$. Then for any $1 > \varepsilon > 0$, for each $t > 0$, there exists $N(t) \in \mathbb{N}$, such that

$$M(x_n, x, t) > 1 - \varepsilon \;\text{ for all } n \geqslant N(t).$$

From the definition of m, we get

$$m(x_n, x, t) \geqslant M(x_n, x, t) \text{ for all } n \quad\text{or}\quad m(x_n, x, t) > 1-\varepsilon \;\text{ for all } n \geqslant N(t).$$

Therefore, $\{x_n\}$ converges to $x \in X$ with respect to the fuzzy metric m.

Conversly, suppose that $\{x_n\}$ converges to x in the fuzzy metric space $(X, m, \star)$ and let $0 < \varepsilon < 1$.
Then, by $(\mathscr{F}2)$, for $0 < f(1-\varepsilon) < 1$ there exists $\delta > 0$, such that

$$1 - \frac{\delta}{2} < t < 1 \implies f(1-\varepsilon) < f(t) < 1.$$

Again, for each $t > 0$ there exists $N(t) \in \mathbb{N}$, such that

$$m(x_n, x, t) > 1 - \frac{\delta}{4} \quad \text{for all } n \geqslant N(t).$$

By the definition of m, we can write

$$m(x_n, x, t) - \frac{\delta}{4} < M(x_n, x, t) < 1 \implies 1 - \frac{\delta}{2} < M(x_n, x, t) < 1 \; \forall\, n \geqslant N(t),$$

which implies

$$f(1-\varepsilon) < f(M(x_n, x, t)) < 1 \;\text{ for all } n \geqslant N(t) \quad\text{or}$$

$$M(x_n, x, t) > 1 - \varepsilon \;\text{ for all } n \geqslant N(t).$$

This proves that $\{x_n\}$ converges to x in $(X, M, f, \star)$ also.

Proof of (ii) can be done similarly and (iii) is straightforward. □

Next define fuzzy $\mathscr{F}$-boundedness as follows.

Definition 9. *Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space. A subset A of X is said to be fuzzy $\mathscr{F}$-bounded if and only if there exist $t > 0$ and $0 < r < 1$, such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.*

Now we have the following result.

Theorem 4. *In fuzzy $\mathscr{F}$-metric space, every $\mathscr{F}$-convergent sequence is bounded.*

Proof. Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space and $\{x_n\}$ be a sequence in X, such that $x_n \to x$ as $n \to \infty$.
Let $0 < \epsilon < 1$. Then, by $(\mathscr{F}2)$, there exists $\delta \in (0,1)$, such that

$$1 - \delta < t \leqslant 1 \implies 1 - \epsilon < f(t) \leqslant 1. \tag{12}$$

Since $x_n \to x$ as $n \to \infty$, so, for $\delta \in (0, 1)$, there exists $N \in \mathbb{N}$, such that for all $t > 0$, $M(x_n, x, t_0) > 1 - \delta, \quad \forall n \geqslant N$.
In particular, for a given $t_0 > 0$, we have

$$M(x_N, x, t_0) > 1 - \delta. \tag{13}$$

Now, by Remark 1, there exists $r \in (0, 1)$, such that

$$M(x_N, x, t_0) \star (1 - r) \geqslant 1 - \delta. \tag{14}$$

Next, consider a sequence $\{\alpha_n\}$ in (0,1), such that $\alpha_n \to 1$ as $n \to \infty$
If possible, suppose that $\{x_n\}$ is unbounded. So, for a given t_0, for each α_k there exists x_{n_k} in $\{x_n\}$, such that

$$M(x_{n_k}, x_N, 2t_0) \leqslant 1 - \alpha_k. \tag{15}$$

Since $x_n \to x$ as $n \to \infty$, so $x_{n_k} \to x$ as $k \to \infty$. So, for t_0, there exists $m(t_0) \in \mathbb{N}$, such that

$$M(x_{n_k}, x, t_0) > 1 - r \quad \forall\, k \geqslant m(t_0).$$

Hence, we can write

$$\begin{aligned} M(x_N, x, t_0) \star M(x_{n_k}, x, t_0) &> M(x_N, x, t_0) \star (1 - r)\ \forall\, k \geqslant m(t_0) \\ &\geqslant (1 - \delta) \qquad\qquad \forall\, k \geqslant m(t_0). \end{aligned}$$

This implies

$$f(M(x_N, x, t_0) \star M(x_{n_k}, x, t_0)) > 1 - \epsilon \quad \forall\, k \geqslant m(t_0) \quad (\text{using } (12))$$

$$
\begin{aligned}
&\text{or } \left(f(M(x_N, x_{n_k}, 2t_0))\right)^\alpha > 1-\epsilon \quad \forall\, k \geqslant m(t_0) \quad (\text{using } (\mathscr{F}M4)) \\
&\text{or } \left(f(1-\alpha_k)\right)^\alpha > 1-\epsilon \quad \forall\, k \geqslant m(t_0) \quad (\text{using } (15)) \\
&\text{or } \lim_{k\to\infty} \left(f(1-\alpha_k)\right)^\alpha \geqslant 1-\epsilon.
\end{aligned}
$$

Since $0 < \epsilon < 1$ is an arbitrary, letting $\epsilon \to 0$ we get

$$
\begin{aligned}
&\lim_{k\to\infty} \left(f(1-\alpha_k)\right)^\alpha = 1 \implies \lim_{k\to\infty} f(1-\alpha_k) = 1 \\
\implies &\lim_{k\to\infty} (1-\alpha_k) = 1 \quad (\text{using } (\mathscr{F}2)) \implies \lim_{k\to\infty} \alpha_k = 0, \text{ a contradiction}
\end{aligned}
$$

Hence, every $\mathscr{F}$-convergent sequence is bounded. □

Conclusion. A new idea of generalized fuzzy metric space known as fuzzy $\mathscr{F}$-metric space is introduced by considering a family of functions. The inequality $(\mathscr{F}M4)$ in fuzzy $\mathscr{F}$-metric axiom is totally different form other fuzzy metric spaces. Justification of this generalization is justified by counterexamples. Since the definition of fuzzy $\mathscr{F}$-metric space is given in a new approach, there is a huge scope of research to develop the results of fuzzy functional analysis by using such generalized fuzzy metric space.

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