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A *d*-ORTHOGONAL POLYNOMIAL SET OF MEIXNER TYPE

Abstract. In this contribution, a new set of d-orthogonal polynomials of Meixner type is introduced. Some properties of these polynomials, including an explicit formula, hypergeometric representation, as well as higher-order recurrence relation, and difference equation, are analyzed.

Key words: *d-orthogonality, Meixner polynomials, generating function, recurrence relation, difference equation*

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1. Introduction and basic background. The vector space of polynomials with coefficients in $\mathbb{C}$ is denoted by $\mathcal{P}$, and its algebraic dual is denoted by $\mathcal{P}'$, We state that a polynomial sequence $\{P_n\}_{n\geqslant 0}$ in $\mathcal{P}$ is a polynomial set (PS, for short) if $\deg P_n = n$, $n \geqslant 0$. It is also assumed that P_n is a polynomial of a single variable and a monic when its leading coefficient is equal to one.

The associated dual sequence $\{u_n\}_{n\geqslant 0}$ in $\mathcal{P}'$ of a PS $\{P_n\}_{n\geqslant 0}$ is defined by

$$\langle u_n, P_m \rangle = \delta_{n,m} = \begin{cases} 1, \ n = m, \\ 0, \ n \neq m, \end{cases} \quad n, m \geqslant 0.$$

Here $\delta_{n,m}$ is the Kronecker delta, where $\langle u, f \rangle$ is the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$.

Multiple orthogonal polynomials, whether of type **I** or **II**, extend the concept of standard orthogonal polynomials. Initially developed within the framework of the Hermite-Pade approximation to achieve a simultaneous rational approximation to a vector of functions [19], these polynomials became a focal point of research. While numerous type **II** multiple orthogonal polynomials have been explicitly discovered, particularly those related to discrete variables, a wealth of literature exists on the subject, found in several contributions,such as articles [1–3], [19] and books [14, Chapter 23], [17], among others.

Here we give a brief introduction to type **II** multiple orthogonal polynomials. For further information, please refer to the above references.

Let $r \geqslant 1$ and let $\vec{\mu} = (\mu_0, \dots, \mu_{r-1})$ be a vector of measures supported on the real line with finite moments. Let also $\vec{n} = (n_0, \dots, n_{r-1}) \in \mathbb{N}_0^r$ be a multi-index of length $|\vec{n}| = n_0 + \cdots + n_{r-1}$. The monic polynomial $P_{\vec{n}}$ is called the type **II** multiple orthogonal polynomial if its degree is $|\vec{n}|$ and it satisfies the simultaneous orthogonality conditions:

$$\int x^k P_{\vec{n}}(x) d\mu_j(x) = 0, \quad 0 \leqslant k \leqslant n_j - 1, \quad 1 \leqslant j \leqslant r.$$

However, the existence of $P_{\vec{n}}$ is not guaranteed but it holds under some additional conditions imposed on the moments of the measures. Clearly, these conditions extend the notion of standard orthogonality when $r = 1$.

In the same vein of generalizing orthogonal polynomials, Maroni and Van Iseghem [15], [21] introduced a subclass of type **II** multiple orthogonal polynomials on the step-line, known in the literature as d-orthogonal polynomials ($d \geqslant 1$), as follows.

Let $\{P_n\}_{n\geqslant 0}$ be a PS in $\mathcal{P}$, $\{P_n\}_{n\geqslant 0}$ is a d-orthogonal polynomial set (d-OPS) with respect to the d-dimensional functional vector $\Gamma = (u_0, u_1, \dots, u_{d-1})$ if the following conditions hold:

$$\begin{cases} \langle u_k, P_n(x)P_m(x)\rangle = 0, & n \geqslant md + k + 1,\ m \geqslant 0, \\ \langle u_k, P_m(x)P_{md+k}(x)\rangle \neq 0, & m \geqslant 0,\ 0 \leqslant k \leqslant d - 1. \end{cases} \tag{1}$$

This is equivalent to the existence of coefficients $\{\beta_{n+1}\}_{n\geqslant 0}$, $\{\gamma_k^n\}_{n\geqslant 0}$, $k = 1, \dots, d$, satisfying the following $(d+1)$-order recurrence relation [15], [21]:

$$\alpha_{n+1} P_{n+1}(x) = (x - \beta_{n+1}) P_n(x) + \sum_{k=1}^{d} \gamma_k^n P_{n-k}(x),\ n \geqslant 0, \tag{2}$$

with $\alpha_{n+1} \neq 0$, $n \geqslant 0$, $\gamma_d^n \neq 0$, $n \geqslant d$, and $P_{-n}(x) = 0$, $n \geqslant 1$.

When $d = 1$, (1) reduces to standard orthogonality, while the formula (2) represents the three-term recurrence relation characterizing orthogonal polynomial sequences (Favard's theorem), see [12].

In this paper, we introduce a new sequence of d-orthogonal polynomials of Meixner type, where a parameter β remains fixed throughout. In the case $d = 1$, they are expected to yield the discrete Meixner polynomials, which are well-known for being orthogonal with respect to the Pascal probability distribution [12], [16]. This approach contrasts with [2], [3], [14], [23], where type **II** multiple orthogonal polynomials are explored with varying parameters of β.

The operational rules governing lowering and raising operators have been instrumental in our studies because many properties of polynomial sets can be inferred by applying these rules. These rules, associated with the operators σ and ρ, are known as the quasi-monomiality principle. A polynomial set $\{P_n\}_{n\geqslant 0}$ is called quasi-monomial under the action of σ and ρ if

$$\sigma P_n(x) = nP_{n-1}(x), \quad \rho P_n(x) = P_{n+1}(x),\, n \geqslant 0. \tag{3}$$

1.1. Sheffer type polynomials.

Definition 1. [14], [18] *A PS $\{P_n\}_{n\geqslant 0}$ is called of Sheffer type if it is generated by a function of the form:*

$$\sum_{n=0}^{+\infty} P_n(x)\frac{t^n}{n!} = A(t)\exp\left(xH(t)\right), \tag{4}$$

where

$$A(t) = \sum_{k=0}^{+\infty} a_k t^k, \;\; A(0) \neq 0, \quad H(t) = \sum_{k=1}^{+\infty} c_k t^k, \;\; H'(0) \neq 0. \tag{5}$$

Let us now introduce a fundamental result about Sheffer type polynomials that we need for the remainder of this paper.

Proposition 1. [6] *Let $A(t)$ and $H(t)$ be as in (5) and the PS of Sheffer type generated by*

$$\sum_{n=0}^{+\infty} P_n(x)\frac{t^n}{n!} = A(t)\left(1 + H(t)\right)^x.$$

Then the lowering operator σ and the raising operator ρ of the PS $\{P_n\}_{n\geqslant 0}$ are given in the sense of formal power series in Δ, respectively, by

$$\sigma = H^*(\Delta), \quad \rho = \frac{A'(\sigma)}{A(\sigma)} + x\frac{H'(\sigma)}{1+\Delta}. \tag{6}$$

Here H^* denotes the compositional inverse of H, i. e.,

$$H\left(H^*(t)\right) = H^*\left(H(t)\right) = t$$

and Δ is the forward difference operator

$$\Delta f(x) = f\left(x+1\right) - f(x),\ f \in \mathcal{P}.$$

In the notation of [12], the standard Meixner polynomial set is given by:

$$M_n(x;\beta,c) = (\beta)_n\ {}_2F_1\left(\begin{array}{c} -n, -x \\ \beta \end{array}; 1-\frac{1}{c}\right),\quad \beta > 1,\ 0 < c < 1,$$

where ${}_2F_1$ is a particular case of the well-known generalized hypergeometric function (see [18])

$${}_pF_q = \left(\begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \mu_1, \mu_2, \ldots, \mu_q \end{array}; x\right) = \sum_{n=0}^{+\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_p)_n}{(\mu_1)_n (\mu_2)_n \ldots (\mu_q)_n} \frac{x^n}{n!},$$

where $\mu_j \neq -1, -2, -3, \ldots$ for $j = 1, \ldots, q$.

Here $(a)_n$ denotes the Pochhammer symbol defined by:

$$(a)_n := \begin{cases} 1, & n = 0, \\ a\left(a+1\right)\left(a+2\right)\ldots\left(a+n-1\right), & n = 1, 2, 3, \ldots. \end{cases}$$

Using the Pochhammer symbol, we obtain:

$$(-n)_k = \begin{cases} \dfrac{(-1)^n\, n!}{(n-k)!}, & 0 \leqslant k \leqslant n, \\ 0, & k \geqslant n+1, \end{cases} \quad \text{and} \quad (-x)_k = (-1)^k \left(x-k+1\right)_k.$$

Whenever one of the numerator parameters is a negative integer or zero, the generalized hypergeometric series terminate. The series ${}_pF_q$ converges for all finite x if $p \leqslant q$, converges for $|x| < 1$ if $p = q+1$ and diverges for all x, $x \neq 0$, if $p > q+1$.

The Meixner polynomials, a significant subset of Sheffer polynomials, can be defined through their generating function as follows (see [14]):

$$\sum_{n=0}^{+\infty} \frac{(\beta)_n}{n!} M_n\left(x;\beta,c\right) t^n = \left(1-\frac{t}{c}\right)^x \left(1-t\right)^{-x-\beta},\quad \beta > 1,\ \ 0 < c < 1.$$

The multiple orthogonal polynomial set generalizing Meixner polynomials has been given [3]. Moreover, several generalizations to d-orthogonality have been treated [11], [22].

The paper is structured as follows: Section 2 introduces a new d-OPS of Meixner type denoted by $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$. Additionally, we explore various aspects of $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$, including its explicit formula, hypergeometric representation, connection formulas, addition formula, and other related properties. We prove that $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$ is a d-orthogonal polynomial set, provide the $(d+1)$-order recurrence relation, deduce the classical property. Moreover, we derive the corresponding d-dimensional functional vector, and the $(d+1)$-order difference equation, respectively.

2. d-OPS of Meixner type. In this section, we define a new d-orthogonal polynomial set of Meixner type by using a generating function of Sheffer type (which satisfies conditions (4) and (5)). Many of their algebraic properties are analyzed therein. Meixner polynomials can be generalized as follows:

$$G(x,t)=\sum_{n=0}^{+\infty} M_n^{(\beta;\,d)}(x)\frac{t^n}{n!}=$$
$$=(1-t)^{-\beta}\Big[1+\frac{c-1}{c}\left((1-t)^{-d}-1\right)\Big]^x,\ c\notin\{0,1\}\,. \quad (7)$$

In the following theorem, we present the explicit formula, hypergeometric representation, some connection formulas, and addition formula of $M_n^{(\beta;\,d)}$.

Theorem 1.

(1) *Explicit formula and hypergeometric representation:*

$$M_n^{(\beta;\,d)}(x)=\sum_{k=0}^{n}\sum_{i=0}^{k}\frac{(-x)_k}{k!}\frac{(-k)_i}{i!}\left(\frac{c-1}{c}\right)^k(id+\beta)_n\,; \quad (8)$$

$$M_n^{(\beta;\,d)}(x)=(\beta)_n\left(\frac{1}{c}\right)^x {}_{d+1}F_d\left(\begin{matrix}-x,\frac{\beta+n}{d},\ldots,\frac{\beta+n+d-1}{d}\\ \frac{\beta}{d},\ldots,\frac{\beta+d-1}{d}\end{matrix};1-c\right), \quad (9)$$

where ${}_{d+1}F_d$ denotes the generalized hypergeometric function.

(2) *Connection formulas:*

$$M_n^{(\beta;\,d)}(x)=\sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha)_{(n-k)}\ M_k^{(\alpha;\,d)}(x), \quad (10)$$

$$\Delta^m M_n^{(\beta;\,d)}(x) = \\ = \left(\frac{c-1}{c}\right)^m \sum_{k=0}^{n-m} \Big[\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \binom{n}{k} (di)_{(n-k)} \Big] M_k^{(\beta;\,d)}(x), m \geqslant 1. \tag{11}$$

(3) *Addition formula:*

$$M_n^{(\beta+\alpha;\,d)}(x+z) = \sum_{k=0}^{n} \binom{n}{k} M_k^{(\beta;\,d)}(x)\, M_{n-k}^{(\alpha;\,d)}(z)\,. \tag{12}$$

Proof. (1) According to the generating function (7) and by using binomial expansion and shifting indices, we get

$$\begin{aligned} \sum_{n=0}^{+\infty} M_n^{(\beta;\,d)}(x)\frac{t^n}{n!} &= \sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\sum_{i=0}^{k} \frac{(-x)_k}{k!}\frac{(-k)_i}{i!}\left(\frac{c-1}{c}\right)^k (id+\beta)_n \frac{t^n}{n!} = \\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{n}\sum_{i=0}^{k} \frac{(-x)_k}{k!}\frac{(-k)_i}{i!}\left(\frac{c-1}{c}\right)^k (id+\beta)_n \frac{t^n}{n!}. \end{aligned}$$

Identifying the coefficients of $\dfrac{t^n}{n!}$ in both sides, we obtain (8).

The following characteristics of Pochhammer's symbol [20]:

$$\begin{aligned} (\lambda)_{m+n} &= (\lambda+m)_n\,(\lambda)_m\,, \\ (\lambda)_{mn} &= m^{nm} \prod_{j=1}^{m} \left(\tfrac{\lambda+j-1}{m}\right)_n, \end{aligned}$$

give us

$$\begin{aligned} M_n^{(\beta;\,d)}(x) &= \sum_{k=0}^{+\infty} \frac{(-x)_k}{k!}\left(\frac{c-1}{c}\right)^k \sum_{i=0}^{k} \frac{(-k)_i}{i!} \frac{(\beta)_n\,(\beta+n)_{id}}{(\beta)_{id}} = \\ &= (\beta)_n \sum_{k=0}^{+\infty} \frac{(-x)_k}{k!}\left(\frac{c-1}{c}\right)^k \sum_{i=0}^{k} \frac{(-k)_i}{i!} \prod_{j=1}^{d} \left[\frac{\left(\dfrac{\beta+n+j-1}{d}\right)_i}{\left(\dfrac{\beta+j-1}{d}\right)_i} \right]. \end{aligned}$$

Then, changing the summation order and shifting k, we get

$$M_n^{(\beta;\,d)}(x) =$$

$$= (\beta)_n \sum_{i=0}^{+\infty} \frac{(-x)_i}{i!} \left(\frac{1-c}{c}\right)^i \prod_{j=1}^{d} \left[\frac{\left(\frac{\beta+n+j-1}{d}\right)_i}{\left(\frac{\beta+j-1}{d}\right)_i} \right] \sum_{k=0}^{+\infty} (-x+i)_k \frac{\left(\frac{c-1}{c}\right)^k}{k!} =$$

$$= (\beta)_n \sum_{i=0}^{+\infty} \frac{(-x)_i}{i!} \left(\frac{1-c}{c}\right)^i \prod_{j=1}^{d} \left[\frac{\left(\frac{\beta+n+j-1}{d}\right)_i}{\left(\frac{\beta+j-1}{d}\right)_i} \right] \left(1 - \frac{c-1}{c}\right)^{x-i}.$$

Thus, we derive (9).

(2) The generating function (7) can be written in the form:

$$\sum_{n=0}^{+\infty} M_n^{(\beta;\,d)}(x) \frac{t^n}{n!} = (1-t)^{\alpha-\beta} (1-t)^{-\alpha} \left[1 + \frac{c-1}{c} \left((1-t)^{-d} - 1\right)\right]^x =$$

$$= \sum_{m=0}^{+\infty} \frac{(\beta-\alpha)_m}{m!} t^m \sum_{k=0}^{+\infty} M_k^{(\alpha;\,d)}(x) \frac{t^k}{k!} =$$

$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \binom{n}{k} (\beta-\alpha)_{(n-k)} M_k^{(\alpha;\,d)}(x) \frac{t^n}{n!}.$$

Then, by comparing the coefficients of $\frac{t^n}{n!}$, we get (10).

Further, we have

$$\sum_{n=0}^{+\infty} \Delta^m M_n^{(\beta;\,d)}(x) \frac{t^n}{n!} =$$

$$= \left(\frac{c-1}{c}\right)^m \left((1-t)^{-d} - 1\right)^m (1-t)^{-\beta} \left[1 + \frac{c-1}{c} \left((1-t)^{-d} - 1\right)\right]^x =$$

$$= \left(\frac{c-1}{c}\right)^m \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (1-t)^{-di} \sum_{k=0}^{+\infty} M_k^{(\beta;\,d)}(x) \frac{t^k}{k!}.$$

Using the binomial expansion, we obtain

$$\sum_{n=0}^{+\infty} \Delta^m M_n^{(\beta;\,d)}(x) \frac{t^n}{n!} =$$

$$= \left(\frac{c-1}{c}\right)^m \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left[\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \binom{n}{k} (di)_{(n-k)} \right] M_k^{(\beta;\,d)}(x) \frac{t^n}{n!}. \quad (13)$$

Since $\deg \Delta^m M_n^{(\beta;\, d)} = n - m$, we get

$$\sum_{n=0}^{+\infty} \Delta^m M_n^{(\beta;\, d)}(x) \frac{t^n}{n!} =$$
$$= \left(\frac{c-1}{c}\right)^m \sum_{n=0}^{+\infty} \sum_{k=0}^{n-m} \left[\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \binom{n}{k} (di)_{(n-k)}\right] M_k^{(\beta;\, d)}(x) \frac{t^n}{n!}.$$

This yields (11).
(3) From (7), we have

$$\sum_{n=0}^{+\infty} M_n^{(\beta+\alpha;\, d)}(x+z) \frac{t^n}{n!} = (1-t)^{-\beta-\alpha} \left[1 + \frac{c-1}{c}\left((1-t)^{-d} - 1\right)\right]^{x+z} =$$
$$= \sum_{k=0}^{+\infty} M_k^{(\beta;\, d)}(x) \frac{t^k}{k!} \sum_{m=0}^{+\infty} M_m^{(\alpha;\, d)}(z) \frac{t^m}{m!} =$$
$$= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \binom{n}{k} M_k^{(\beta;\, d)}(x)\, M_{n-k}^{(\alpha;d)}(z) \frac{t^n}{n!}.$$

Hence, by identification, we obtain (12). □

Remark 1.

(1) *When $d = 1$ in (9), we get the standard Meixner orthogonal polynomials, according to [20, p. 33].*

(2) *If $\alpha = \beta + 1$, then (10) becomes*

$$M_n^{(\beta;\, d)}(x) = M_n^{(\beta+1;\, d)}(x) - n M_{n-1}^{(\beta+1;\, d)}(x). \tag{14}$$

(3) *If $\alpha = \beta$ in (12), then we obtain*

$$M_n^{(2\beta;\, d)}(x+z) = \sum_{k=0}^{n} \binom{n}{k} M_k^{(\beta;\, d)}(x)\, M_{n-k}^{(\beta;\, d)}(z)\,.$$

Moreover, the case $z = -x$ of the last formula gives

$$\sum_{k=0}^{n} \binom{n}{k} M_k^{(\beta;\, d)}(x)\, M_{n-k}^{(\beta;\, d)}(-x) = (2\beta)_n\,.$$

Similarly, in [4], [10], the connection formula allows us to derive a linear operator to transform any Meixner type polynomial set into another Meixner type polynomial set. Thus, we have

Proposition 2. *For fixed β, we define the operator $\mathcal{C}_{\gamma}^{\beta}$ by:*

$$\mathcal{C}_{\lambda}^{\beta} M_n^{(\beta;\, d)} = \tau_{-\lambda} \circ M_n^{(\beta+\lambda;\, d)},$$

where

$$\tau_{-\lambda} f(x) = f\left(x+\lambda\right), \ f \in \mathcal{P}.$$

Thus, $\mathcal{C}_{\lambda}^{\beta}$ is independent of β for any λ; so, we write $\mathcal{C}_{\lambda} := \mathcal{C}_{\lambda}^{\beta}$.

Proof. The connection formula permits us to write

$$\begin{aligned}\mathcal{C}_{\lambda}^{\alpha} M_n^{(\beta;\, d)} &= \sum_{k=0}^{n} \binom{n}{k} (\beta-\alpha)_{(n-k)}\ \mathcal{C}_{\lambda}^{\alpha} M_k^{(\alpha;\, d)} = \\ &= \tau_{-\lambda} \circ \sum_{k=0}^{n} \binom{n}{k} (\beta-\alpha)_{(n-k)}\ M_k^{(\alpha+\lambda;\, d)} = \\ &\qquad = \tau_{-\lambda} \circ M_n^{(\beta+\lambda;\, d)} = \mathcal{C}_{\lambda}^{\beta} M_n^{(\beta;\, d)}.\end{aligned}$$

Hence, since $\left\{M_n^{(\beta;\, d)}\right\}_{n\geqslant 0}$ forms a basis of $\mathcal{P}$, the desired result becomes clear. □

Furthermore, from (6) we deduce, respectively, the expressions of the lowering operator σ and the raising operator ρ associated with $\left\{M_n^{(\beta;\, d)}\right\}_{n\geqslant 0}$.

Proposition 3. *The PS $\left\{M_n^{(\beta;\, d)}\right\}_{n\geqslant 0}$ is quasi-monomial under the action of the operators*

$$\sigma = 1 - \left(1 - \frac{c\Delta}{1-c}\right)^{-\frac{1}{d}}, \tag{15}$$

$$\rho = \beta\left(1 - \frac{c\Delta}{1-c}\right)^{\frac{1}{d}} + \frac{(c-1)\,d}{c}\,\frac{x}{1+\Delta}\left(1 - \frac{c\Delta}{1-c}\right)^{\frac{1+d}{d}}. \tag{16}$$

In the next proposition, we deduce several relations that the polynomials $M_n^{(\beta;\, d)}$, $n \geqslant 0$, satisfy.

Proposition 4. *For $n \geqslant 0$, we have:*

$$dx M_n^{(\beta+d+1;\, d)}\ (x-1) = \frac{c}{c-1}\left[M_{n+1}^{(\beta;\, d)}\ (x) - \beta M_n^{(\beta+1;\, d)}\ (x)\right], \tag{17}$$

$$M_n^{(\beta;\,d)}\;(x+1) = \frac{1}{c}\left[M_n^{(\beta;\,d)}\;(x) + (c-1)\,M_n^{(\beta+d;\,d)}\;(x)\right], \tag{18}$$

$$\begin{aligned} xM_n^{(\beta+1;\,d)}(x) = \frac{1}{d\,(c-1)}\sum_{k=1}^{d}\binom{n}{k}(-d)_k\left(M_{n-k+1}^{(\beta;\,d)}(x) - \beta M_{n-k}^{(\beta+1;\,d)}(x)\right) + \\ + \frac{c}{d\,(c-1)}\left(M_{n+1}^{(\beta;\,d)}\;(x) - \beta M_n^{(\beta+1;\,d)}(x)\right). \end{aligned} \tag{19}$$

Proof. First, we apply the operator $\dfrac{\partial}{\partial t}$ to each side of (7) and obtain

$$\sum_{n=0}^{+\infty} M_{n+1}^{(\beta;\,d)}(x)\frac{t^n}{n!} = \frac{\beta}{(1-t)}\sum_{n=0}^{+\infty} M_n^{(\beta;\,d)}(x)\frac{t^n}{n!} + xH_1'(t)\sum_{n=0}^{+\infty} M_n^{(\beta;\,d)}(x)\frac{t^n}{n!}, \tag{20}$$

where

$$H_1\,(t) = \ln\left(1 + \frac{c-1}{c}\left((1-t)^{-d} - 1\right)\right).$$

Then

$$\sum_{n=0}^{+\infty} M_{n+1}^{(\beta;\,d)}(x)\frac{t^n}{n!} = \beta\sum_{n=0}^{+\infty} M_n^{(\beta+1;\,d)}(x)\frac{t^n}{n!} + \frac{d(c-1)x}{c}\sum_{n=0}^{+\infty} M_n^{(\beta+d+1;\,d)}(x-1)\frac{t^n}{n!}.$$

Thus, we get the first relation.

From the generating function (7) we deduce (18) in a straightforward way.

Multiplying both sides of (20) by

$$\frac{1}{(1-t)\,H_1'\,(t)} = \frac{(1-t)^d}{d\,(c-1)} + \frac{1}{d},$$

we get

$$\begin{aligned} x\sum_{n=0}^{+\infty} M_n^{(\beta+1;\,d)}(x)\frac{t^n}{n!} &= \\ &= \left(\frac{1}{d(c-1)}(1-t)^d + \frac{1}{d}\right)\sum_{n=0}^{+\infty}\left(M_{n+1}^{(\beta;\,d)}(x) - \beta M_n^{(\beta+1;\,d)}(x)\right)\frac{t^n}{n!} = \\ &= \sum_{n=0}^{+\infty}\left[\frac{1}{d\,(c-1)}\sum_{k=1}^{d}\binom{n}{k}(-d)_k\left(M_{n-k+1}^{(\beta;\,d)}(x) - \beta M_{n-k}^{(\beta+1;\,d)}(x)\right) + \right. \end{aligned}$$

$$+\frac{c}{d\,(c-1)}\left(M_{n+1}^{(\beta;\,d)}(x)-\beta M_{n}^{(\beta+1;\,d)}(x)\right)\Bigg]\frac{t^n}{n!}.$$

Thus, identifying the coefficients of $\frac{t^n}{n!}$ in both sides, we get (19). □

Next, we present some relations involving $M_n^{(\beta;\,d)}$ and $\Delta^k M_n^{(\beta;\,d)}$, $0\leqslant k\leqslant n$ in the following proposition:

Proposition 5. *For $n\geqslant 0$, the following relations hold:*

$$\left(\frac{c}{c-1}\right)^m \Delta^m M_n^{(\beta;\,d)}(x)=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\,M_n^{(\beta+di;\,d)}(x), \tag{21}$$

$$M_n^{(\beta+nd;\,d)}(x)=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{c}{c-1}\right)^k\Delta^k M_n^{(\beta;\,d)}(x), \tag{22}$$

$$M_n^{(\beta-1;\,d)}(x)=\sum_{k=0}^{n}\frac{\left(\frac{1}{d}\right)_k}{k!}\left(\frac{c}{1-c}\right)^k\Delta^k M_n^{(\beta;\,d)}(x), \tag{23}$$

$$\begin{aligned}M_n^{(\beta;\,d)}(x)&=\frac{c}{d(c-1)}\Delta M_{n+1}^{(\beta-1;\,d)}(x)-\\&\quad-\frac{(\beta-1+d)c}{d(c-1)}\Delta M_n^{(\beta;\,d)}(x)-x\Delta M_n^{(\beta+d;\,d)}(x-1).\end{aligned} \tag{24}$$

Proof. Replacing β by $\beta+di$ and α by β in (10), we obtain

$$M_n^{(\beta+id;\,d)}(x)=\sum_{k=0}^{n}\binom{n}{k}(di)_{(n-k)}\,M_k^{(\beta;\,d)}(x).$$

By insertion in (13), we conclude (21).

Choosing $m=1$ in (21), we get

$$\left(\frac{c}{c-1}\right)\Delta M_n^{(\beta;\,d)}(x)=M_n^{(\beta+d;\,d)}(x)-M_n^{(\beta;\,d)}(x). \tag{25}$$

Or, equivalently,

$$M_n^{(\beta+d;\,d)}(x)=\left[\left(\frac{c}{c-1}\right)\Delta+I\right]M_n^{(\beta;\,d)}(x). \tag{26}$$

Iterating (26), we obtain

$$M_n^{(\beta+nd;\,d)}(x) = \left[\left(\frac{c}{c-1}\right)\Delta + I\right]^n M_n^{(\beta;\,d)}(x) =$$
$$= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{c}{c-1}\right)^k \Delta^k M_n^{(\beta;\,d)}(x).$$

From (3) and (15), we have

$$nM_{n-1}^{(\beta;\,d)}(x) = M_n^{(\beta;\,d)}(x) - \sum_{k=0}^{n} \frac{\left(\frac{1}{d}\right)_k}{k!} \left(\frac{c}{1-c}\right)^k \Delta^k M_n^{(\beta;\,d)}(x),\ n \geqslant 0.$$

Thereafter, using (14) ($\beta \to \beta - 1$), we find (23).

By changing β by $\beta - 1$ in (17) and inserting the operator Δ, we get

$$\frac{c}{c-1} \Delta M_{n+1}^{(\beta-1;\,d)}(x) =$$
$$= \frac{(\beta-1)\,c}{c-1} \Delta M_n^{(\beta;\,d)}(x) + dx\Delta M_n^{(\beta+d;\,d)}\ (x-1) + dM_n^{(\beta+d;\,d)}(x).$$

Thus, when $dM_n^{(\beta+d;\,d)}(x)$ is removed, by using (25), we see that the statement holds. $\square$

Remark 2.

(1) *Notice that, for $d = 1$ (14) and (25) yield a classical result for standard Meixner polynomials* [12], [14]

$$\Delta M_n\,(x; \beta, c) = n\left(\frac{c-1}{c}\right) M_{n-1}\,(x; \beta+1, c)\,.$$

(2) (22) *and* (23) *are connection formulas.*

3. Connection between $M_n^{(\beta;\,d)}$ and some known polynomial sets. First, we aim to obtain a relation between $M_n^{(\beta;\,d)}$ and $\ell_n^{(\alpha_1,\,\dots,\,\alpha_d)}$, where $\left\{\ell_n^{(\alpha_1,\,\dots,\,\alpha_d)}\right\}_{n\geqslant 0}$ is the d-OPS of Laguerre type defined by [8]:

$$\ell_n^{(\alpha_1,\,\dots,\,\alpha_d)}(x) = {}_1F_d\left(\begin{matrix} -n \\ \alpha_1+1, \dots, \alpha_d+1 \end{matrix}\,;x\right).$$

Taking into account (see [13])

$$
{}_{p+r}F_{q+s}\left(\begin{matrix}\boldsymbol{\alpha_p},\boldsymbol{c_r}\\ \boldsymbol{b_q},\boldsymbol{d_s}\end{matrix};zw\right)=
$$

$$
=\sum_{k=0}^{+\infty}\frac{(-1)^k\left(\boldsymbol{\alpha_p}\right)_k z^k}{k!\left(\boldsymbol{b_q}\right)_k}\;{}_pF_q\left(\begin{matrix}\boldsymbol{\alpha_p}+\boldsymbol{k}\\ \boldsymbol{b_q}+\boldsymbol{k}\end{matrix};z\right)\times\;{}_{r+1}F_s\left(\begin{matrix}-k,\boldsymbol{c_r}\\ \boldsymbol{d_s}\end{matrix};w\right),
$$

where $\boldsymbol{\alpha_p}$ is an abbreviation for $\alpha_1,\ldots,\alpha_p$, and setting $p=d+1$, $s=d$, $q=r=0$, $z=1$, $w=1-c$, $\alpha_1=-x$, $\alpha_{j+1}=\dfrac{\beta+n+j-1}{d}$, $d_j=\dfrac{\beta+j-1}{d}$, $j=1,\ldots,d$, we obtain

$$
M_n^{(\beta;\,d)}(x)=\left(\frac{1}{c}\right)^x\sum_{k=0}^{+\infty}\frac{(-1)^k\left(-x\right)_k\left(\beta\right)_n}{k!}\prod_{j=1}^{d}\left(\frac{\beta+n+j-1}{d}\right)_k\times
$$

$$
\times_{d+1}F_0\left(\begin{matrix}-x+k,\frac{\beta+n+kd}{d},\ldots,\frac{\beta+n+kd+d-1}{d}\\ -\end{matrix};1\right)\ell_k^{\left(\frac{\beta}{d}-1,\ldots,\frac{\beta+d-1}{d}-1\right)}(1-c).
$$

Next, if $r=s=d$, $p=1$, $z=\dfrac{1}{x}$, $w=(1-c)x$, $\lambda=\gamma$, $c_j=\dfrac{\beta+n+j-1}{d}$, $d_j=\dfrac{\beta+j-1}{d}$, $j=1,\ldots,d$, in (see [13])

$$
{}_{p+r}F_{q+s}\left(\begin{matrix}\boldsymbol{\alpha_p},\boldsymbol{c_r}\\ \boldsymbol{b_q},\boldsymbol{d_s}\end{matrix};zw\right)=\sum_{k=0}^{+\infty}\frac{(-1)^k\left(\boldsymbol{\alpha_p}\right)_k(\lambda)_k z^k}{k!\left(\boldsymbol{b_q}\right)_k(\gamma+k)_k}\times
$$

$$
\times\;{}_{p+1}F_{q+1}\left(\begin{matrix}\lambda+k,\boldsymbol{\alpha_p}+\boldsymbol{k}\\ \gamma+2k+1,\boldsymbol{b_q}+\boldsymbol{k}\end{matrix};z\right)\times\;{}_{r+2}F_{s+1}\left(\begin{matrix}-k,\gamma+k,\boldsymbol{c_r}\\ \alpha,\boldsymbol{d_s}\end{matrix};w\right),
$$

with $p=q+1$, $z\neq1$, $|1-w|<\left|1-\frac{1}{z}\right|$, $\mathrm{Re}\,(c)\leqslant1$, we have

$$
M_n^{(\beta;\,d)}(x)=\left(\frac{1}{c}\right)^x\sum_{k=0}^{+\infty}\frac{\left(\dfrac{-1}{x}\right)^k(-x)_k(\beta)_n}{(\gamma+k)_k\,k!}\times
$$

$$
\times\;{}_2F_1\left(\begin{matrix}\gamma+k,k-x\\ \gamma+2k+1\end{matrix};\frac{1}{x}\right)\times\mathbf{S}_k^{(\gamma)}\left((1-c)\,x\right).
$$

Here $\mathbf{S}_k^{(\gamma)}(x)$ are the Srivastava–Pathan polynomials defined by [7]:

$$
\mathbf{S}_k^{(\gamma)}(x)=\;(\gamma)_k\;{}_{d+2}F_{d+1}\left(\begin{matrix}-k,\gamma+k,\frac{\beta+n}{d},\ldots,\frac{\beta+n+d-1}{d}\\ \gamma,\frac{\beta}{d},\ldots,\frac{\beta+d-1}{d}\end{matrix};x\right).
$$

4. Recurrence relation.

Theorem 2. *The PS* $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$ *satisfies the following* $(d+1)$*-order recurrence relation:*

$$M_0^{(\beta;\,d)}(x) = 1,$$

$$\frac{c}{(c-1)\,d} M_{n+1}^{(\beta;\,d)}(x) = \left[x + \frac{1}{d}\left(\frac{\beta c}{c-1} + n\left(\frac{d+1}{c-1}+1\right)\right)\right] M_n^{(\beta;\,d)}(x) + \\ + \frac{1}{(c-1)\,d}\sum_{k=1}^{d}(-d)_k\left[\beta\binom{n}{k} + (d+1)\binom{n}{k+1}\right] M_{n-k}^{(\beta;\,d)}(x),$$

with $M_{-n}^{(\beta;d)} = 0$, $n \geqslant 1$.

Proof. If we multiply both sides of (20) by $\dfrac{1}{H_1'(t)}$, then we get

$$\frac{1}{H_1'(t)}\sum_{n=0}^{+\infty} M_{n+1}^{(\beta;\,d)}(x)\frac{t^n}{n!} = \frac{\beta}{(c-1)\,d}(1-t)^d \sum_{n=0}^{+\infty} M_n^{(\beta;\,d)}(x)\frac{t^n}{n!} + \\ + \left(\frac{\beta}{d} + x\right)\sum_{n=0}^{+\infty} M_n^{(\beta;\,d)}(x)\frac{t^n}{n!}.$$

Using the binomial expansion and shifting indices, we have

$$\frac{1}{H_1'(t)}\sum_{n=0}^{+\infty} M_{n+1}^{(\beta;\,d)}(x)\frac{t^n}{n!} = \\ = \frac{\beta}{(c-1)\,d}\sum_{n=0}^{+\infty}\left[\sum_{k=1}^{d}(-d)_k\binom{n}{k} M_{n-k}^{(\beta;\,d)}(x) + \left(\frac{\beta c}{(c-1)d} + x\right) M_n^{(\beta;\,d)}(x)\right]\frac{t^n}{n!}.$$

On the other hand,

$$\frac{1}{H_1'(t)}\sum_{n=0}^{+\infty} M_{n+1}^{(\beta;\,d)}(x)\frac{t^n}{n!} = \\ = \frac{1}{(c-1)\,d}(1-t)^{d+1}\sum_{n=0}^{+\infty} M_{n+1}^{(\beta;\,d)}(x)\frac{t^n}{n!} + \frac{1}{d}(1-t)\sum_{n=0}^{+\infty} M_{n+1}^{(\beta;\,d)}(x)\frac{t^n}{n!}.$$

By employing the binomial expansion and shifting indices, we obtain

$$\frac{1}{H_1'(t)}\sum_{n=0}^{+\infty} M_{n+1}^{(\beta;\,d)}(x)\frac{t^n}{n!} = \sum_{n=0}^{+\infty}\left[\frac{1}{(c-1)d}\sum_{k=2}^{d+1}\binom{n}{k}(-d-1)_k M_{n-k+1}^{(\beta;\,d)}(x) - \right.$$
$$\left. -\frac{1}{d}\left(1+\frac{d+1}{c-1}\right) nM_n^{(\beta;\,d)}(x) + \frac{c}{(c-1)d}M_{n+1}^{(\beta;\,d)}(x)\right]\frac{t^n}{n!}.$$

Thus, by identification of the coefficients of the monomials $\frac{t^n}{n!}$, the $(d+1)$-order recurrence relation is deduced. □

Remark 3.

(1) *Under the assumption*

$$\beta \neq -n,\ n \geqslant 0,$$

the order of the above recurrence relation is $(d+1)$.

(2) *When* $d = 1$, *we retrieve the well-known three-term recurrence relation of standard Meixner polynomials* [12]*:*

$$cM_{n+1}(x;\beta,c) = ((c-1)x + \beta c + n(c+1))\, M_n(x;\beta,c) -$$
$$- [\beta n + n(n-1)]\, M_{n-1}\ (x;\beta,c)\,,\ n \geqslant 0.$$

By using the above theorem, the following result holds:

Proposition 6. *The polynomial sets* $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$ *and* $\left\{\Delta M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$ *satisfy the following limit relations:*

$$\lim_{c\to 1} M_n^{(\beta;\,d)}\left(\frac{x}{1-c}\right) = \mathcal{L}_n^{(\beta-1,\,d)}(x),$$
$$\lim_{c\to 1}\left(\frac{c}{1-c}\right)\Delta M_n^{(\beta;\,d)}\left(\frac{x}{1-c}\right) = D\mathcal{L}_n^{(\beta-1,\,d)}(x).$$

Here $\left\{\mathcal{L}_n^{(\alpha,\,d)}\right\}_{n\geqslant 0}$ *denotes the d-OPS of Laguerre type, see* [10],

$$\sum_{n=0}^{+\infty}\mathcal{L}_n^{(\alpha,\,d)}(x)\,\frac{t^n}{n!} = (1-t)^{-\alpha-1}\exp\left[-x\left((1-t)^{-d}-1\right)\right].$$

In addition, we recall that

$$D\mathcal{L}_n^{(\alpha,\,d)}(x) = \mathcal{L}_n^{(\alpha,\,d)}(x) - \mathcal{L}_n^{(\alpha+d,\,d)}(x).$$

When $d = 1$, we get a very well-known limit result for standard Meixner polynomials [12, 14]:

$$\lim_{c\to 1} M_n\left(\frac{x}{1-c};\beta,c\right) = n!L_n^{(\beta-1)}(x),$$

where $\left\{L_n^{(\alpha)}\right\}_{n\geqslant 0}$ denotes the standard Laguerre PS.

In the following, we deal with establishing an operator $\mathcal{T}$ in $\mathcal{P}$, such that the PS $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$ satisfy the classical property with respect to it.

5. The classical property.

Theorem 3. *The PS* $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$ *is classical with respect to the operator*

$$\mathcal{T} = \frac{c}{d\,(c-1)}\Big[\Delta\rho\sigma\,(I-\sigma) - (\beta-1+d)\,\Delta\sigma - x\Delta\sigma\Big(d\Delta + \frac{(c-1)\,d}{c}I\Big)\tau_1\Big],$$

where τ_1 *denotes the operator given by*

$$\tau_1 f(x) = f\,(x-1)\,,\ \ f\in\mathcal{P}.$$

Proof. Multiplying (24) by $(n+1)$, we get

$$\begin{aligned}(n+1)\,M_n^{(\beta;\,d)}(x) &= \frac{c}{d\,(c-1)}\Delta\rho\sigma M_{n+1}^{(\beta-1;\,d)}(x) - \\ &- \frac{(\beta-1+d)\,c}{d\,(c-1)}\Delta\sigma\ M_{n+1}^{(\beta;\,d)}(x) - x\Delta\sigma M_{n+1}^{(\beta+d;\,d)}\,(x-1)\,.\end{aligned}$$

Furthermore, (14) gives us

$$M_n^{(\beta;\,d)}(x) = (I-\sigma)\,M_n^{(\beta+1;\,d)}(x).$$

From this identity and relation (26), we obtain

$$\begin{aligned}(n+1)\,M_n^{(\beta;\,d)}(x) &= \frac{c}{d\,(c-1)}\Delta\rho\sigma\,(I-\sigma)\,M_{n+1}^{(\beta;\,d)}(x) - \\ -\frac{(\beta-1+d)c}{d(c-1)}\Delta\sigma M_{n+1}^{(\beta;\,d)}(x) &- x\Delta\sigma\Big(\frac{c}{(c-1)}\Delta + I\Big)\tau_1 M_{n+1}^{(\beta;\,d)}(x) = \mathcal{T}M_{n+1}^{(\beta;\,d)}(x).\end{aligned}$$

Then we derive the result. □

6. The canonical d-dimensional functional vector.

Theorem 4. *The components of the d-dimensional vector of functionals associated with the d-OPS $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$ are given as the following representation:*

$$u_r = \frac{1}{r!}\sum_{j=0}^{+\infty}\sum_{i=0}^{r}\binom{r}{i}(-1)^i\left(1+\frac{c}{1-c}\right)^{-\left(\frac{\beta+i}{d}+j\right)}\left(\frac{\beta+i}{d}\right)_j\frac{\left(\frac{c}{1-c}\right)^j}{j!}\delta_j,$$
$$r = 0,1,\ldots,d-1.$$

Here δ_j is the Dirac mass defined by $\langle\delta_j, f\rangle = f(j)$, $f \in \mathcal{P}$.

Proof. The components of the d-dimensional vector of functionals associated with the d-OPS $\left\{M_n^{(\beta;\,d)}\right\}_{n\geqslant 0}$ are given by [5], [9]:

$$\langle u_r, f\rangle = \frac{1}{r!}\left[\frac{\sigma^r}{(1-\sigma)^{-\beta}}f(x)\right]_{x=0}, \; r = 0,1,\ldots,d-1, \; f \in \mathcal{P}.$$

Therefore, we have

$$\langle u_r, f\rangle = \frac{1}{r!}\left[\sum_{i=0}^{r}\binom{r}{i}(-1)^i\left(1-\frac{c\Delta}{1-c}\right)^{-\frac{i}{d}}\left(1-\frac{c\Delta}{1-c}\right)^{-\frac{\beta}{d}}f(x)\right]_{x=0} =$$
$$= \frac{1}{r!}\sum_{i=0}^{r}\binom{r}{i}(-1)^i\sum_{n=0}^{+\infty}\frac{\left(\frac{i+\beta}{d}\right)_n}{n!}\left(\frac{c}{1-c}\right)^n\triangle^n f(0).$$

Taking into account

$$\triangle^n f(0) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(j),$$

we obtain

$$\langle u_r, f\rangle = \frac{1}{r!}\sum_{j=0}^{+\infty}\sum_{i=0}^{r}\binom{r}{i}(-1)^i\left(1+\frac{c}{1-c}\right)^{-\left(\frac{\beta+i}{d}+j\right)}\left(\frac{\beta+i}{d}\right)_j\frac{\left(\frac{c}{1-c}\right)^j}{j!}f(j).$$

This gives the desired result. $\square$

7. Higher-order difference equation. Here, we rely on the lowering and rasing operators to determine the difference equation of order $(d+1)$:

Theorem 5. *The d-OPS* $\left\{M_n^{(\beta;\, d)}\right\}_{n\geqslant 0}$ *satisfies the following* $(d+1)$*-order difference equation:*

$$\left[\left((\beta+dx)\Delta+\beta+\frac{(c-1)d}{c}x\right)\left(\sum_{k=1}^{d}\frac{\left(-\frac{1}{d}\right)_k}{k!}\left(\frac{c}{1-c}\right)^k\Delta^k\right)-n\Delta\right]M_n^{(\beta;\, d)}= \\ = nM_n^{(\beta;\, d)},\ n\geqslant 0.$$

Proof. From (3), we get

$$nM_n^{(\beta;\, d)}(x)=\rho\sigma M_n^{(\beta;\, d)}(x)\,.$$

Then, replacing (15) and (16) and applying $(1+\Delta)$ in both hand sides, we get

$$\begin{aligned} & n\,(1+\Delta)\,M_n^{(\beta;\, d)}(x)= \\ &=\left(\beta(1+\Delta)+x\frac{(c-1)\,d}{c}\left(1-\frac{c\Delta}{1-c}\right)\right)\left(\left(1-\frac{c\Delta}{1-c}\right)^{\frac{1}{d}}-1\right)M_n^{(\beta;\, d)}(x)= \\ &=\left((\beta+dx)\,\Delta+\beta+\frac{(c-1)\,d}{c}x\right)\left(\sum_{k=1}^{d}\frac{\left(-\frac{1}{d}\right)_k}{k!}\left(\frac{c}{1-c}\right)^k\Delta^k\right)M_n^{(\beta;\, d)}(x). \end{aligned}$$

Thus the statement follows. □

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