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UNCERTAINTY PRINCIPLES AND CALDERÓN'S FORMULAS FOR THE DEFORMED HANKEL L^2_α-MULTIPLIER OPERATORS

Abstract. The main purpose of this paper is to introduce the deformed Hankel L^2_α-multiplier operators and to give some new results related to these operators as Plancherel's, Calderón's reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. Next, using the theory of reproducing kernels, we give best estimates and an integral representation of the extremal functions related to these operators on weighted Sobolev spaces.

Key words: *deformed Hankel transform, Calderón's reproducing formulas, extremal functions, Heisenberg's uncertainty principle, Donoho-Stark's uncertainty principle*

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1. Introduction. In their seminal papers, Hörmander and Mikhlin [5], [12] initiated the study of boundedness of the translation-invariant operators on $\mathbb{R}^d$. The translation-invariant operators on $\mathbb{R}^d$ are characterized using the classical Euclidean Fourier transform $\mathcal{F}(f)$, therefore they also known as Fourier multipliers. Given a measurable function

$$m\colon \mathbb{R}^d \longrightarrow \mathbb{C},$$

its Fourier multiplier is the linear map $\mathcal{T}_m$ given for all $\lambda \in \mathbb{R}^d$ by the relation

$$\mathcal{F}(\mathcal{T}_m(f))(\lambda) = m(\lambda)\mathcal{F}(f)(\lambda). \tag{1}$$

The Hörmander-Mikhlin fundamental condition gives a criterion for L^p-boundedness for all $1 < p < \infty$ of Fourier multipliers $\mathcal{T}_m$ in terms of derivatives of the symbol m; more precisely, if

$$|\partial_\lambda^\gamma m(\lambda)| \lesssim |\lambda|^{-|\gamma|} \quad \text{for} \quad 0 \leqslant |\gamma| \leqslant \left[\frac{d}{2}\right] + 1, \tag{2}$$

then $\mathcal{T}_m$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^d)$ into itself.

The condition (2) imposes m to be a bounded function, smooth over $\mathbb{R}^d\backslash\{0\}$, and satisfying certain local and asymptotic behavior. Locally, m admits a singularity at 0 with a mild control of derivatives around it up to order $\left[\frac{d}{2}\right]+1$. This singularity links to deep concepts in harmonic analysis and justifies the key role of Hörmander-Mikhlin theorem in the Fourier multiplier L_p-theory: this condition defines a large class of Fourier multipliers including Riesz transforms and Littelwood-Paley partitions of unity, which are crucial in Fourier summability or Pseudo-differential operator. The boundedness of Fourier multipliers is useful to solve problems in the area of mathematical analysis and Probability theory: see [11], for stochastic processes see [2], and for the study of nonlinear partial differential equations see [8].

The general theory of reproducing kernels has started with Aronszajn's in [1] in 1950, then the authors in [10], [14], [15] applied this theory to study Tikhonov regularization problem and they obtained approximate solutions for bounded linear operator equations on Hilbert spaces with the viewpoint of numerical solutions by computers. This theory has gained considerable interest in various field of mathematical sciences, especially in Engineering and numerical experiments using computers, see [10], [15]. This paper focuses on the generalized Fourier transform associated with the Dunkl-Laplace operator called the Deformed Hankel transform. More precisely, we consider the following differential operator: Δ_α defined for $\alpha > \frac{-1}{4}$ by

$$\Delta_\alpha := \frac{\partial^2}{\partial x^2} + \frac{2\alpha}{x}\frac{\partial}{\partial x} - \frac{\alpha}{x^2}(1-S),$$

where $S(f)(x) =: f(-x)$.

The operator Δ_α is closely connected with the Dunkl theory, see [4], [1], [6], [17], furthermore, the eigenfunctions of this operator are related to Bessel functions and they satisfies a product formula, which permits to develop a new harmonic analysis associated with this operator; see [17] for more information.

The deformed Hankel transform $\mathcal{F}_\alpha$ generalizes the usual Fourier transform $\mathcal{F}$ and is defined on $L^1_\alpha(\mathbb{R})$ by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} B_\alpha(\lambda x) f(x) d\mu_\alpha(x), \qquad \text{for } \lambda \in \mathbb{R},$$

where μ_α is the measure on $\mathbb{R}$ and $B_\alpha(\lambda.)$ is the Bessel kernel given later. Let σ be a function in $L^2_\alpha(\mathbb{R})$ and $\beta > 0$ be a positive real number. The deformed Hankel L^2_α-multiplier operators [8] is defined for smooth function on $\mathbb{R}$ as

$$\mathcal{T}_{\sigma,\beta}(f)(x) := \mathcal{F}_\alpha^{-1}\left(\sigma_\beta \mathcal{F}_\alpha(f)\right)(x), \tag{3}$$

where the function σ_β is given by

$$\sigma_\beta(\lambda) := \sigma(\lambda\beta). \tag{4}$$

These operators are a generalization of the classical multiplier operators given by the relation (1). The remainder of this paper is arranged as follows: in section 2 we recall the main results concerning the harmonic analysis associated with the deformed Hankel transform, in section 3, we introduce the deformed Hankel L^2_α-multiplier operators $\mathcal{T}_{\sigma,\beta}$ and we give for them a Plancherel's point- wise reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. The last section of this paper is devoted to giving an application of the general theory of reproducing kernels to Fourier multiplier theory and to give the best estimates and an integral representation of the extremal functions related to the deformed Hankel L^2_α-multiplier operators on weighted Sobolev spaces.

2. Harmonic Analysis Associated with the Deformed Hankel Transform. In this section, we set some notation and recall some results in harmonic analysis related to the deformed Hankel transform; for more details, see [9], [13], [17], [18].

- For $\alpha > \frac{-1}{4}$, μ_α is the weighted Lebesgue measure defined on $\mathbb{R}$ by

$$d\mu_\alpha(x) := \frac{x^{2\alpha-1}dx}{2\Gamma(2\alpha)},$$

  where Γ is the Gamma function.

- $L^p_\alpha(\mathbb{R}), 1 \leqslant p \leqslant \infty$ is the space of measurable functions on $\mathbb{R}$, satisfying

$$\|f\|_{p,\mu_\alpha} =: \begin{cases} \left(\int\limits_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty. \end{cases}$$

In particular, $L^2_\alpha(\mathbb{R})$ is a Hilbert space with inner product given by

$$\langle f, g\rangle_\alpha = \int\limits_{\mathbb{R}} f(x)\overline{g(x)}d\mu_\alpha(x).$$

2.1 The Eigenfunctions of the Dunkl-Laplace operator Δ_α. For $\lambda \in \mathbb{R}$, consider the following Cauchy problem:

$$(S):\ \begin{cases} \Delta_\alpha(u)(x) = -\left|\dfrac{\lambda}{x}\right| u(x), \\ u(0) = 1. \end{cases}$$

From [9], [13], [17], the Cauchy problem (S) admits a unique solution $B_\alpha(\lambda.)$ given by

$$B_\alpha(\lambda x) = j_{2\alpha-1}(2\sqrt{|\lambda x|}) - \frac{\lambda x}{2\alpha(2\alpha+1)} j_{2\alpha+1}(2\sqrt{|\lambda x|}), \tag{5}$$

where j_α denotes the normalized Bessel function of order α, see [19] for more information about this function. The function $B_\alpha(\lambda.)$ is infinitely differentiable on $\mathbb{R}$ and we have the following important result:

$$\forall \lambda, x \in \mathbb{R}, \quad |B_\alpha(\lambda x)| \leqslant 1. \tag{6}$$

Furthermore, from [17], the deformed Hankel kernel (5) is multiplicative on $\mathbb{R}$ in the sense

$$\forall \lambda \in \mathbb{R}, x, y \in \mathbb{R}^* \quad B_\alpha(\lambda x) B_\alpha(\lambda y) = \int_{\mathbb{R}} B_\alpha(\lambda z) K_\alpha(x, y, z) d\mu_\alpha(z), \tag{7}$$

where K_α is the Bessel kernel given explicitly in [17], [18]. The function $K_\alpha(x, y, z)$ is unchanged by permutation of the three variables and there exists a constant A_α independent of x,y, such that

$$\int_{\mathbb{R}} |K_\alpha(x, y, z)|\, d\mu_\alpha(z) \leqslant A_\alpha. \tag{8}$$

From [17], the product formula (7) permits to define a translation operator, convolution product, and to develop a new harmonic analysis associated to the Dunkl-Laplace operator Δ_α.

2.2 The Deformed Hankel transform.

Definition 1. *The deformed Hankel transform $\mathcal{F}_\alpha$ is defined on $L^1_\alpha(\mathbb{R})$ by*

$$\mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} B_\alpha(\lambda x) f(x) d\mu_\alpha(x), \qquad \textit{for } \lambda \in \mathbb{R}.$$

Some basic properties of this transform are as follows: (find the proofs in [9], [13], [17], [18]).

Proposition 1.
(1) *For every* $f \in L^1_\alpha(\mathbb{R})$, *we have*

$$\|\mathcal{F}_\alpha(f)\|_{\infty,\mu_\alpha} \leqslant \|f\|_{1,\mu_\alpha}. \tag{9}$$

(2) *(Inversion formula). For* $f \in (L^1_\alpha \cap L^2_\alpha)(\mathbb{R})$, *such that* $\mathcal{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$, *we have*

$$f(x) = \int_{\mathbb{R}} B_\alpha(\lambda x)\mathcal{F}_\alpha(f)(\lambda)d\mu_\alpha(\lambda), \qquad a.e. \quad x \in \mathbb{R}. \tag{10}$$

(3) *(The Parseval formula). For all* $f, g \in L^2_\alpha(\mathbb{R})$ *we have:*

$$\langle f, g\rangle_\alpha = \langle \mathcal{F}_\alpha(f), \mathcal{F}_\alpha(g)\rangle_\alpha, \tag{11}$$

In particular, we have

$$\|f\|_{2,\mu\alpha} = \|\mathcal{F}_\alpha(f)\|_{2,\mu\alpha}. \tag{12}$$

(4) *(The Plancherel theorem). The deformed Hankel transform* $\mathcal{F}_\alpha$ *can be extended to an isometric isomorphism from* $L^2_\alpha(\mathbb{R})$ *into* $L^2_\alpha(\mathbb{R})$.

2.3 The translation operator associated with the deformed Hankel transform. The product formula (7) permits to define the translation operator as follows:

Definition 2. *Let* $x, y \in \mathbb{R}$ *and* f *be a measurable function on* $\mathbb{R}$; *the translation operator is defined by*

$$\tau^x_\alpha f(y) = \int_{\mathbb{R}} f(z)K_\alpha(x, y, z)d\mu_\alpha(z).$$

The following proposition summarizes some properties of the deformed Hankel translation operator see [13], [17].

Proposition 2. *For all* $x, y \in \mathbb{R}$, *we have:*

(i)

$$\tau^x_\alpha f(y) = \tau^y_\alpha f(x). \tag{13}$$

(ii)
$$\int_{\mathbb{R}} \tau_\alpha^x f(y) d\mu_\alpha(y) = \int_{\mathbb{R}} f(y) d\mu_\alpha(y). \tag{14}$$

(iii) *for* $f \in L_\alpha^p(\mathbb{R})$ *with* $p \in [1; +\infty]$ $\tau_\alpha^x f \in L_\alpha^p(\mathbb{R})$ *and we have*
$$\|\tau_\alpha^x f\|_{p,\mu_\alpha} \leqslant A_\alpha \|f\|_{p,\mu_\alpha}, \tag{15}$$
where A_α *is the constant given in* (8).

(iv) *For* $f \in L_\alpha^1(\mathbb{R})$, $\tau_\alpha^x f \in L_\alpha^1(\mathbb{R})$ *and we have*
$$\mathcal{F}_\alpha\left(\tau_\alpha^x f\right)(\lambda) = B_\alpha(\lambda x) \mathcal{F}_\alpha(f)(\lambda), \quad \forall \lambda \in \mathbb{R}. \tag{16}$$

The relation (16) shows that the translation operator τ_α^x is a particular case of the deformed Hankel multiplier operator (3).

By using the translation, we define the generalized convolution product of f, g by
$$\left(f *_\alpha g\right)(x,t) = \int_{\mathbb{K}} \tau_\alpha^x(f)(y) g(y) d\mu_\alpha(y).$$
This convolution is commutative, associative, and satisfies the following properties (see [17]):

Proposition 3.

(i) *(Young's inequality). For all* $p, q, r \in [1; +\infty]$, *such that* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ *and for all* $f \in L_\alpha^p(\mathbb{R}), g \in L_\alpha^q(\mathbb{R})$ *the function* $f *_\alpha g$ *belongs to the space* $L_\alpha^r(\mathbb{R})$ *and we have*
$$\|f *_\alpha g\|_{r,\mu_\alpha} \leqslant A_\alpha \|f\|_{p,\mu_\alpha} \|g\|_{q,\mu_\alpha}. \tag{17}$$

(ii) *For* $f, g \in L_\alpha^2(\mathbb{R})$, *the function* $f *_\alpha g$ *belongs to* $L_\alpha^2(\mathbb{R})$ *if and only if the function* $\mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g)$ *belongs to* $L_\alpha^2(\mathbb{R})$, *and in this case we have:*
$$\mathcal{F}_\alpha\left(f *_\alpha g\right) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g). \tag{18}$$

(iii) *For all* $f, g \in L_\alpha^2(\mathbb{R})$ *we have*
$$\int_{\mathbb{R}} \left|f *_\alpha g(x,t)\right|^2 d\mu_\alpha(x) = \int_{\mathbb{R}} \left|\mathcal{F}_\alpha(f)(\lambda)\right|^2 \left|\mathcal{F}_\alpha(g)(\lambda)\right|^2 d\mu_\alpha(\lambda), \tag{19}$$
where both integrals are simultaneously finite or infinite.

3. Deformed Hankel L^2_α-multiplier operators. The main purpose of this section is to introduce the deformed Hankel L^2_α-multiplier operators on $\mathbb{R}$ and to establish for them some uncertainty principles and Calderon's reproducing formulas.

3.1 Calderon's Reproducing Formulas for the Deformed Hankel L^2_α-multiplier operators.

Definition 3. *Let $\sigma \in L^2_\alpha(\mathbb{R})$ and $\beta > 0$; the deformed Hankel L^2_α-multiplier operators are defined for smooth functions on $\mathbb{R}$ as*

$$\mathcal{T}_{\sigma,\beta}(f)(x) := \mathcal{F}_\alpha^{-1}\left(\sigma_\beta \mathcal{F}_\alpha(f)\right)(x), \tag{20}$$

where the function σ_β is given by

$$\sigma_\beta(\lambda) := \sigma(\beta\lambda),$$

for all $\lambda \in \mathbb{R}$.

By a simple change of variables, we find that for $\beta > 0$, $\sigma_\beta \in L^2_\alpha(\mathbb{R})$ and

$$\|\sigma_\beta\|_{2,\mu_\alpha} = \frac{1}{\beta^{\frac{2\alpha-1}{2}}}\|\sigma\|_{2,\mu_\alpha}. \tag{21}$$

Remark. *According to the relation (18), we find that*

$$\mathcal{T}_{\sigma,\beta}(f)(x) = \left(\mathcal{F}_\alpha^{-1}\left(\sigma_\beta\right) *_\alpha f\right)(x), \tag{22}$$

where

$$\mathcal{F}_\alpha^{-1}\left(\sigma_\beta\right)(x) = \frac{1}{\beta^{2\alpha-1}}\mathcal{F}_\alpha^{-1}(\sigma)\left(\frac{x}{\beta}\right). \tag{23}$$

Let us give some properties of the deformed Hankel L^2_α-multiplier operators.

Proposition 4.

(i) *For every $\sigma \in L^2_\alpha(\mathbb{R})$ and $f \in L^1_\alpha(\mathbb{R})$, the function $\mathcal{T}_{\sigma,\beta}(f)$ belongs to $L^2_\alpha(\mathbb{R})$, and we have*

$$\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leqslant \frac{A_\alpha}{\beta^{\frac{2\alpha-1}{2}}}\|\sigma\|_{2,\mu_\alpha}\|f\|_{1,\mu_\alpha}.$$

(ii) *For every $\sigma \in L^\infty_\alpha(\mathbb{R})$, and for every $f \in L^2_\alpha(\mathbb{R})$, the function $\mathcal{T}_{\sigma,\beta}(f)$ belongs to $L^2_\alpha(\mathbb{R})$, and we have*

$$\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leqslant \|\sigma\|_{\infty,\mu_\alpha}\|f\|_{2,\mu_\alpha}. \tag{24}$$

(iii) *For every* $\sigma \in L^2_\alpha(\mathbb{R})$, *and for every* $f \in L^2_\alpha(\mathbb{R})$, $\mathcal{T}_{\sigma,\beta}(f) \in L^\infty_\alpha(\mathbb{R})$, *and we have*

$$\mathcal{T}_{\sigma,\beta}(f)(x,t) = \int_{\mathbb{R}} \sigma(\beta\lambda, m)\varphi_{\lambda,m}(x,t)\mathcal{F}_\alpha(f)(\lambda,m)d\gamma_\alpha(\lambda,m), \text{ a.e}(x,t) \in \mathbb{K}, \tag{25}$$

and

$$\|\mathcal{T}_{\sigma,\beta}(f)\|_{\infty,\mu_\alpha} \leqslant \frac{A_\alpha}{\beta^{\frac{2\alpha-1}{2}}}\|\sigma\|_{2,\mu_\alpha}\|f\|_{2,\mu_\alpha}.$$

Proof. (i) By the relations (17), (22), we find that

$$\|\mathcal{T}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha} = \left\|\mathcal{F}^{-1}_\alpha(\sigma_\beta) *_\alpha f\right\|^2_{2,\mu_\alpha} \leqslant A^2_\alpha \|f\|^2_{1,\mu_\alpha}\left\|\mathcal{F}^{-1}_\alpha(\sigma_\beta)\right\|^2_{1,\mu_\alpha}.$$

Plancherel's formula (12) and the relation (21) gives the desired result.
(ii) Is a consequence of Plancherel's formula (12).
(iii) Is a consequence of the relations (12), (17), (21), and (22); on the other hand, the relation (25) follows from inversion formula (10). □

In the following result, we give Plancherel's and pointwise reproducing inversion formula for the deformed Hankel L^2_α-multiplier operators.

Theorem 1. *Let* $\sigma \in L^2_\alpha(\mathbb{R})$ *satisfy the admissibility condition:*

$$\int_0^\infty |\sigma_\beta(\lambda)|^2 \frac{d\beta}{\beta} = 1, \quad \lambda \in \mathbb{R}. \tag{26}$$

(i) *(Plancherel's formula). For all* f *in* $L^2_\alpha(\mathbb{R})$, *we have*

$$\int_{\mathbb{R}} |f(x)|^2 d\mu_\alpha(x) = \int_0^\infty \|\mathcal{T}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha}\frac{d\beta}{\beta}. \tag{27}$$

(ii) *(First Calderón's formula). Let* $f \in L^1_\alpha(\mathbb{R})$, *such that* $\mathcal{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$; *then we have*

$$f(x) = \int_0^\infty \left(\mathcal{T}_{\sigma,\beta}(f) *_\alpha \mathcal{F}^{-1}_\alpha(\overline{\sigma_\beta})\right)(x)\frac{d\beta}{\beta}, \quad \text{a.e. } x \in \mathbb{R}.$$

Proof. (i) By using the Fubini theorem and relations (19) and (22), we get

$$\int_0^\infty \|\mathcal{T}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta} = \int_0^\infty \left[\int_{\mathbb{R}} |\mathcal{T}_{\sigma,\beta}(f)(x)|^2 \, d\mu_\alpha(x) \right] \frac{d\beta}{\beta} =$$
$$= \int_0^\infty \left[\int_{\mathbb{R}} \left|\mathcal{F}_\alpha^{-1}(\sigma_\beta) *_\alpha f(x)\right|^2 d\mu_\alpha(x) \right] \frac{d\beta}{\beta} =$$
$$= \int_0^\infty \left[\int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\lambda)|^2 \, d\mu_\alpha(\lambda) \right] |\sigma_\beta(\lambda)|^2 \frac{d\beta}{\beta}.$$

The admissibility condition (26) and Plancherel's formula (12) give the desired result.
(ii) Let $f \in L^1_\alpha(\mathbb{R})$, such that $\mathcal{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$; by using the Fubini theorem and the relations (11), (16), we find that

$$\int_0^\infty \left(\mathcal{T}_{\sigma,\beta}(f) *_\alpha \mathcal{F}_\alpha^{-1}(\overline{\sigma_\beta})\right)(x) \frac{d\beta}{\beta} =$$
$$= \int_0^\infty \Big[\int_{\mathbb{R}} \mathcal{T}_{\sigma,\beta}(f)(y) \overline{\tau^x_\alpha\left(\mathcal{F}_\alpha^{-1}(\sigma_\beta)\right)(y)} d\mu_\alpha(y) \Big] \frac{d\beta}{\beta} =$$
$$= \int_0^\infty \Big[\int_{\mathbb{R}} |\sigma_\beta(\lambda)|^2 \mathcal{F}_\alpha(f)(\lambda) B_\alpha(\lambda x) d\mu_\alpha(\lambda) \Big] \frac{d\beta}{\beta}.$$

Admissibility condition (26) and inversion formula (10) give the desired result. □

To establish the second Calderon reproducing formula for the deformed Hankel L^2_α-multiplier operators, we need the following technical result.

Proposition 5. *Let $\sigma \in L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ satisfy the admissibility condition (26); then the function defined by*

$$\Phi_{\gamma,\delta}(\lambda) = \int_\gamma^\delta |\sigma_\beta(\lambda)|^2 \frac{d\beta}{\beta}$$

belongs to $L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ *for all* $0 < \gamma < \delta < \infty$.

Proof. Using Hölder's inequality for the measure $\frac{d\beta}{\beta}$ and the Fubini theorem, we get

$$\|\Phi_{\gamma,\delta}\|^2_{2,\gamma_\alpha} \leqslant \log(\delta/\gamma)\|\sigma\|^2_{\infty,\mu_\alpha} \int\limits_\gamma^\delta \|\sigma\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}.$$

Using the relation (26), we find that

$$\|\Phi_{\gamma,\delta}\|^2_{2,\mu_\alpha} \leqslant \log(\delta/\gamma)\|\sigma\|^2_{\infty,\mu_\alpha}\|\sigma\|^2_{2,\gamma_\alpha} \int\limits_\gamma^\delta \frac{d\beta}{\beta^{2\alpha}} < \infty.$$

So, $\Phi_{\gamma,\delta} \in L^2_\alpha(\mathbb{R})$. Furthermore, using relation (26), we get $\|\Phi_{\gamma,\delta}\|_{\infty,\gamma_\alpha} < \infty$. Therefore, $\Phi_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$. $\square$

Theorem 2. *(Second Calderón's formula). Let* $f \in L^2_\alpha(\mathbb{R})$ *and* $\sigma \in L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ *satisfy the admissibility condition* (26) *and* $0 < \gamma < \delta < \infty$. *Then the function*

$$f_{\gamma,\delta}(x) = \int\limits_\gamma^\delta \left(\mathcal{T}_{\sigma,\beta}(f) *_\alpha \mathcal{F}^{-1}_\alpha\left(\overline{\sigma_\beta}\right)\right)(x)\frac{d\beta}{\beta}, \quad x \in \mathbb{R},$$

belongs to $L^2_\alpha(\mathbb{R})$ *and satisfies*

$$\lim_{(\gamma,\delta)\to(0,\infty)} \|f_{\gamma,\delta} - f\|_{2,\mu_\alpha} = 0. \tag{28}$$

Proof. By a simple computation, we find that

$$f_{\gamma,\delta}(x) = \int\limits_{\mathbb{R}} \Phi_{\gamma,\delta}(\lambda)B_\alpha(\lambda x)\mathcal{F}_\alpha(f)(\lambda)d\mu_\alpha(\lambda) = \mathcal{F}^{-1}_\alpha\left(\Phi_{\gamma,\delta}\mathcal{F}_\alpha(f)\right)(x).$$

Using Proposition 5, we find that $\Phi_{\gamma,\delta} \in L^\infty_\alpha(\mathbb{R})$. Then we have $f_{\gamma,\delta} \in L^2_\alpha(\mathbb{R})$ and

$$\mathcal{F}_\alpha\left(f_{\gamma,\delta}\right)(\lambda) = \Phi_{\gamma,\delta}(\lambda, m)\mathcal{F}_\alpha(f)(\lambda).$$

On the other hand, using Plancherel's formula (12), we find that

$$\lim_{(\gamma,\delta)\to(0,\infty)} \|f_{\gamma,\delta} - f\|^2_{2,\mu_\alpha} = \lim_{(\gamma,\delta)\to(0,\infty)} \int\limits_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\lambda)|^2 \left(1 - \Phi_{\gamma,\delta}(\lambda)\right)^2 d\mu_\alpha(\lambda).$$

Using the admissibility condition (26), relation (28) follows from the dominated convergence theorem. □

3.2 Uncertainty principles for the Deformed Hankel L^2_α-multiplier operators. The main purpose of this subsection is to establish Heisenberg's and Donoho-Stark's uncertainty principles for the deformed Hankel L^2_α-multiplier operators $\mathcal{T}_{\sigma,\beta}$.

3.2.1 Heisenberg's uncertainty principle for $\mathcal{T}_{\sigma,\beta}$.
In [6], the authors proved the following Heisenberg's inequality for $\mathcal{F}_\alpha$: there exist a positive constant c, such that for all $f \in L^2_\alpha(\mathbb{R})$ we have:

$$\|f\|^2_{2,\mu_\alpha} \leqslant c \left\||x|^2 f\right\|_{2,\mu_\alpha} \left\||\lambda|^2 \mathcal{F}_\alpha(f)\right\|_{2,\mu_\alpha}. \tag{29}$$

We generalize this inequality for $\mathcal{T}_{\sigma,\beta}$:

Theorem 3. *There exists a constant $c > 0$, such that for all $f \in L^2_\alpha(\mathbb{R})$ we have*

$$\|f\|^2_{2,\mu_\alpha} \leqslant c \left\||\lambda|^2 \mathcal{F}_\alpha(f)\right\|_{2,\mu_\alpha} \Big[\int_0^\infty \left\||x|^2 \mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}\Big]^{\frac{1}{2}}.$$

Proof. Suppose that $\||\lambda|^2 \mathcal{F}_\alpha(f)\|_{2,\mu_\alpha} + \left[\int_0^\infty \||x|^2 \mathcal{T}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}\right] < \infty$. Using relation (29), we have:

$$\int_{\mathbb{R}} |\mathcal{T}_{\sigma,\beta}(f)(x)|^2 d\mu_\alpha(x) \leqslant c \left\||x|^2 \mathcal{T}_{\sigma,\beta}(f)\right\|_{2,\mu_\alpha} \left\||\lambda|^2 \sigma_\beta \mathcal{F}_\alpha(f)\right\|_{2,\mu_\alpha}.$$

Integrating over $]0,+\infty[$ with respect to measure $\frac{d\beta}{\beta}$ and using Plancherel's formula (27) and Schwartz's inequality, we get

$$\|f\|^2_{2,\mu_\alpha} \leqslant$$
$$\leqslant c\Big[\int_0^\infty \left\||x|^2 \mathcal{T}_{\sigma,\beta}(f)\right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}\Big]^{\frac{1}{2}} \Big[\int_0^\infty \Big[\int_{\mathbb{R}} \left||\lambda|^4 \sigma_\beta(\lambda)\right|^2 |\mathcal{F}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\Big] \frac{d\beta}{\beta}\Big].$$

The Fubini theorem and the admissibility condition (26) gives the desired result. □

3.2.2 Donoho-Stark's uncertainty principle for $\mathcal{T}_{\sigma,\beta}$.
Developing the ideas of Donoho and Stark [3], the main purpose of this

subsection is to give an uncertainty inequality of concentration type in $L^2_\theta(\mathbb{R})$, where $L^2_\theta(\mathbb{R})$ is the space of measurable functions on $]0,+\infty[\times\mathbb{R}$, such that

$$\|f\|_{2,\theta_\alpha} = \Big[\int_0^\infty \|f(\beta,.)\|^2_{2,\mu_\alpha}\frac{d\beta}{\beta}\Big]^{\frac{1}{2}}.$$

Denote by θ_α the measure defined on $]0,+\infty[\times\mathbb{R}$ by

$$d\theta_\alpha(\beta,x) = d\mu_\alpha(x)\otimes\frac{d\beta}{\beta}.$$

Definition 4. [3]
(i) *Let E be a measurable subset of $\mathbb{R}$. We say that the function $f\in L^2_\alpha(\mathbb{R})$ is ε-concentrated on E if*

$$\|f-\chi_E f\|_{2,\mu_\alpha}\leqslant\varepsilon\|f\|_{2,\mu_\alpha}, \tag{30}$$

where χ_E is the indicator function of the set E.
(ii) *Let F be a measurable subset of $]0,+\infty[\times\mathbb{R}$. We say that the function $\mathcal{T}_{\sigma,\beta}(f)$ is ρ-concentrated on F if*

$$\|\mathcal{T}_{\sigma,\beta}(f)-\chi_F\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}\leqslant\rho\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}. \tag{31}$$

We have the following result:

Theorem 4. *Let $f\in L^2_\alpha(\mathbb{R})$ and $\sigma\in L^2_\alpha(\mathbb{R})\cap L^1_\alpha(\mathbb{R})$ satisfy the admissibility condition (26). If f is ε-concentrated on E and $\mathcal{T}_{\sigma,\beta}(f)$ is ρ-concentrated on F, then we have*

$$\|\sigma\|_{1,\mu_\alpha}(\mu_\alpha(E))^{\frac{1}{2}}\Big[\int_F\frac{d\theta_\alpha(\beta,x)}{\beta^{4\alpha-2}}\Big]^{\frac{1}{2}}\geqslant 1-(\varepsilon+\rho).$$

Proof. Let $f\in L^2_\alpha(\mathbb{R})$ and $\sigma\in L^2_\alpha(\mathbb{R})\cap L^\infty_\alpha(\mathbb{R})$ satisfy (26). Assume that $\mu_\alpha(E)<\infty$ and $\Big[\int_F\frac{d\theta_\alpha(\beta,x)}{\beta^{4\alpha-2}}\Big]^{\frac{1}{2}}<\infty$. According to the relations (30) and (31), we have

$$\begin{aligned}\|\mathcal{T}_{\sigma,\beta}(f)-\chi_F\mathcal{T}_{\sigma,\beta}(\chi_E f)\|_{2,\theta_\alpha}&\leqslant\|\mathcal{T}_{\sigma,\beta}(f)-\chi_F\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}+\\&+\|\chi_F\mathcal{T}_{\sigma,\beta}(f-\chi_E f)\|_{2,\theta_\alpha}\leqslant\rho\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}+\|\mathcal{T}_{\sigma,\beta}(f-\chi_E f)\|_{2,\theta_\alpha}.\end{aligned}$$

Using Plancherel's relation (27), we get

$$\|\mathcal{T}_{\sigma,\beta}(f) - \chi_F \mathcal{T}_{\sigma,\beta}(\chi_E f)\|_{2,\theta_\alpha} \leqslant (\varepsilon + \rho)\|f\|_{2,\mu_\alpha}.$$

So, we get

$$\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} \leqslant \|\mathcal{T}_{\sigma,\beta}(f) - \chi_F \mathcal{T}_{\sigma,\beta}(\chi_E f)\|_{2,\theta_\alpha} + \|\chi_F \mathcal{T}_{\sigma,\beta}(\chi_E f)\|_{2,\theta_\alpha} \leqslant$$

$$\leqslant (\varepsilon + \rho)\|f\|_{2,\mu_\alpha} + \|\chi_F \mathcal{T}_{\sigma,\beta}(\chi_E f)\|_{2,\theta_\alpha}. \tag{32}$$

On the other hand, by relations (9), (25), and Hölder's inequality, we find that

$$|\mathcal{T}_{\sigma,\beta}(\chi_E f)(x)|^2 \leqslant \frac{1}{\beta^{4\alpha-2}}\|f\|^2_{2,\mu_\alpha}\|\sigma\|^2_{1,\gamma_\alpha}\mu_\alpha(E),$$

so we arrive at

$$\|\chi_F \mathcal{T}_{\sigma,\beta}(\chi_E f)\|_{2,\theta_\alpha} \leqslant \|f\|_{2,\mu_\alpha}\|\sigma\|_{1,\gamma_\alpha}(\mu(E))^{\frac{1}{2}}\Big[\int\limits_F \frac{d\theta_\alpha(\beta, x)}{\beta^{4\alpha-2}}\Big]^{\frac{1}{2}}. \tag{33}$$

By the relations (32) and (33), we deduce that

$$\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} \leqslant \|f\|_{2,\mu_\alpha}\left[(\varepsilon + \rho) + \|\sigma\|_{1,\gamma_\alpha}(\mu_\alpha(E))^{\frac{1}{2}}\Big[\int\limits_F \frac{d\theta_\alpha(\beta, x}{\beta^{4\alpha-2}}\Big]^{\frac{1}{2}}\right].$$

Plancherel's formula (27) for $\mathcal{T}_{\sigma,\beta}$ gives the desired result. □

4. Extremal functions associated with the deformed Hankel L^2_α-multiplier operators.

In the following, we study the extremal functions associated with the deformed Hankel L^2_α-multiplier operators.

Definition 5. *Let ψ be a positive function on $\mathbb{R}$ satisfying the following conditions:*

$$\frac{1}{\psi} \in L^1_\alpha(\mathbb{R}) \tag{34}$$

and

$$\psi(\lambda) \geqslant 1, \quad (\lambda) \in \mathbb{R}. \tag{35}$$

We define the Sobolev-type space $\mathcal{H}_\psi(\mathbb{R})$ by

$$\mathcal{H}_\psi(\mathbb{R}) = \Big\{f \in L^2_\alpha(\mathbb{R})\colon \sqrt{\psi}\mathcal{F}_\alpha(f) \in L^2_\alpha(\mathbb{R})\Big\}$$

equipped with inner product

$$\langle f, g\rangle_\psi = \int_{\mathbb{R}} \psi(\lambda, m)\mathcal{F}_\alpha(f)(\lambda)\overline{\mathcal{F}_\alpha(g)(\lambda)}d\mu_\alpha(\lambda),$$

and the norm

$$\|f\|_\psi = \sqrt{\langle f, f\rangle_\psi}\,.$$

Proposition 6. *Let σ be a function in $L^\infty_\alpha(\mathbb{R})$. The deformed Hankel L^2_α multiplier operators $\mathcal{T}_{\sigma,\beta}$ are bounded and linear from $\mathcal{H}_\psi(\mathbb{R})$ into $L^2_\alpha(\mathbb{R})$, and we have for all $f \in \mathcal{H}_\psi(\mathbb{R})$:*

$$\|\mathcal{T}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leqslant \|\sigma\|_{\infty,\gamma_\alpha}\|f\|_\psi. \tag{36}$$

Proof. Using relations (12), (24), (35), we get the result. □

Definition 6. *Let $\eta > 0$ and let σ be a function in $L^\infty_\alpha(\mathbb{R})$. Denote by $\langle f, g\rangle_{\psi,\eta}$ the inner product defined on the space $\mathcal{H}_\psi(\mathbb{R})$ by*

$$\langle f, g\rangle_{\psi,\eta} = \int_{\mathbb{R}} \left(\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2\right) \mathcal{F}_\alpha(f)(\lambda)\overline{\mathcal{F}_\alpha(g)(\lambda)}d\mu_\alpha(\lambda),$$

and the norm

$$\|f\|_{\psi,\eta} = \sqrt{\langle f, f\rangle_{\psi,\eta}}\,.$$

In the following results, we show that the norm $\|\cdot\|_{\psi,\eta}$ can be expressed as a function of norm of the Hilbert space $\mathcal{H}_\psi(\mathbb{R})$ and the norm of deformed Hankel L^2_α-multiplier operators. Moreover, we show the equivalence between the norms $\|\cdot\|_{\psi,\eta}$ and $\|\cdot\|_\psi$.

Proposition 7. *Let σ be a function in $L^\infty_\alpha(\mathbb{R})$ and $f \in \mathcal{H}_\psi(\mathbb{R})a$. Then*
(i) *the norm $\|\cdot\|_{\psi,\eta}$ satisfies*

$$\|f\|^2_{\psi,\eta} = \|f\|^2_\psi + \|\mathcal{T}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha}\,.$$

(ii) *The norms $\|\cdot\|_{\psi,\eta}$ and $\|\cdot\|_\zeta$ are equivalent and*

$$\sqrt{\eta}\|f\|_\psi \leqslant \|f\|_{\psi,\eta} \leqslant \sqrt{\eta + \|\sigma\|^2_{\infty,\gamma_\alpha}}\|\varphi\|_{\psi,\eta}.$$

Proof. The results follow from Plancherel's formula (12) and relation (36). □

Theorem 5. *Let $\sigma \in L^\infty_\alpha(\mathbb{R})$, the Sobolev-type space $(\mathcal{H}_\psi(\mathbb{R}))\,, \langle \cdot, \cdot \rangle_{\psi,\eta})$ be a reproducing kernel Hilbert space with kernel*

$$\mathcal{K}_{\psi,\eta}(x,y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} d\mu_\alpha(\lambda),$$

that is
(i) *For all $y \in \mathbb{R}$, the function $x \mapsto \mathcal{K}_{\psi,\eta}\,(x,y)$ belongs to $\mathcal{H}_\psi(\mathbb{R})$.*
(ii) *For all $f \in \mathcal{H}_\psi(\mathbb{R})$ and $y \in \mathbb{R}$, we have the reproducing property:*

$$f(y) = \langle f, \mathcal{K}_{\psi,\eta}(\cdot, (y))\rangle_{\psi,\eta}\,.$$

Proof. (i) Let $y \in \mathbb{R}$. From relations (6), (34), we see that the function

$$g_y \colon \lambda \longrightarrow \frac{B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}$$

belongs to $L^1_\alpha(\mathbb{R}) \cap L^2_\alpha((\mathbb{R})$. Hence, the function $\mathcal{K}_{\psi,\eta}$ is well-defined, and by the inversion formula (10), we get

$$\mathcal{K}_{\psi,\eta}(xy) = \mathcal{F}_\alpha^{-1}(g_y)(x).$$

Using Plancherel's theorem for $\mathcal{F}_\alpha$, we find that $\mathcal{K}_{\psi,\eta}(\cdot, y)$ belongs to $L^2_\alpha(\mathbb{R})$, and we have

$$\mathcal{F}_\alpha(\mathcal{K}_{\psi,\eta}(\cdot, y))(\lambda) = \frac{B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}. \tag{37}$$

Using relations (6), (34), and (37), we find that

$$\|\sqrt{\psi}\mathcal{F}_\alpha(\mathcal{K}_{\psi,\eta}(\cdot, y))\|_{2,\mu_\alpha} \leqslant \frac{1}{\eta^2}\left\|\frac{1}{\psi}\right\|_{1,\mu_\alpha} < \infty.$$

This proves that for every $y \in \mathbb{R}$ the function $x \mapsto \mathcal{K}_{\psi,\eta}\,(x,y)$ belongs to $\mathcal{H}_\psi(\mathbb{R})$.
(ii) Using the relation (37), we find that for all $f \in \mathcal{H}_\psi(\mathbb{R})$,

$$\langle f, \mathcal{K}_{\psi,\eta}\,(\cdot, y)\rangle_{\psi,\eta} =$$
$$= \int_{\mathbb{R}} \left(\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2\right) \mathcal{F}_\alpha(f)(\lambda)\overline{\mathcal{F}_\alpha(\mathcal{K}_{\psi,\eta}\,(\cdot,y)\,(\lambda)}d\mu_\alpha(\lambda) =$$

$$= \int_{\mathbb{R}} B_\alpha(\lambda y)\mathcal{F}_\alpha(f)(\lambda)d\mu_\alpha(\lambda),$$

and inversion formula (10) gives the desired result. □

Taking σ a null function and $\eta = 1$, we find the following result:

Corollary 1. *The Sobolev-type space $(\mathcal{H}_\psi(\mathbb{R})), \langle\cdot,\cdot\rangle_\psi)$ is a reproducing kernel Hilbert space with kernel*

$$\mathcal{K}_\psi(x,y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda)}d\mu_\alpha(\lambda).$$

The main result of this section can be stated as follows:

Theorem 6. *Let $\sigma \in L^\infty_\alpha(\mathbb{R})$ and $\beta > 0$, for any $h \in L^2_\alpha(\mathbb{R})$ and for any $\eta > 0$, there exists a unique function $f^*_{\eta,\beta,h}$, where the infimum*

$$\inf_{f\in\mathcal{H}_\psi(\mathbb{R})} \left\{\eta\|f\|^2_\psi + \|h - \mathcal{T}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha}\right\} \tag{38}$$

is attained. Moreover, the extremal function $f^*_{\eta,\beta,h}$ is given by

$$f^*_{\eta,\beta,h}(y) = \int_{\mathbb{R}} h(x)\overline{\Theta_{\eta,\beta}(x,y)}d\mu_\alpha(x),$$

where $\Theta_{\eta,\beta}$ is given by

$$\Theta_{\eta,\beta}(x,y) = \int_{\mathbb{R}} \frac{\sigma_\beta(\lambda)B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}d\mu_\alpha(\lambda).$$

Proof. The existence and the unicity of the extremal function $f^*_{\eta,\beta,h}$ satisfying (38) is given in [7], [10], [14]. Furthermore, $f^*_{\eta,\beta,h}$ is given by

$$f^*_{\eta,\beta,h}(y) = \langle h, \mathcal{T}_{\sigma,\beta}(\mathcal{K}_{\psi,\eta}(\cdot,y))\rangle_{\mu_\alpha}.$$

Using inversion formula (10) and the relation (37), we get

$$\mathcal{T}_{\sigma,\beta}(\mathcal{K}_{\psi,\eta}(\cdot,y))(x) = \int_{\mathbb{R}} \frac{\sigma_\beta(\lambda)B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}d\mu_\alpha(\lambda) = \Theta_{\eta,\beta}(x,y)$$

and the proof is complete. □

Theorem 7. *Let $\sigma \in L^\infty_\alpha(\mathbb{R})$ and $h \in L^2_\alpha(\mathbb{R})$. The function $f^*_{\eta,\beta,h}$ satisfies the following properties:*

$$\mathcal{F}_\alpha(f^*_{\eta,\beta,h})(\lambda) = \frac{\overline{\sigma_\beta(\lambda)}}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}\mathcal{F}_\alpha(h)(\lambda) \tag{39}$$

and

$$\|f^*_{\eta,\beta,h}\|_\psi \leqslant \frac{1}{\sqrt{2\eta}}\|h\|_{2,\mu_\alpha}.$$

Proof. Let $y \in \mathbb{R}$. The function

$$k_y \colon (\lambda) \longrightarrow \frac{\sigma_\beta(\lambda)B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}$$

belongs to $L^2_\alpha(\mathbb{R}) \cap L^1_\alpha(\mathbb{R})$. Using inversion formula (10), we get

$$\Theta_{\eta,\beta}(x, y) = \mathcal{F}_\alpha^{-1}(k_y)(x).$$

Using Plancherel's theorem and Parseval's relation (11), we find that $\Theta_{\eta,\beta}(\cdot, y) \in L^2_\alpha(\mathbb{R})$ and

$$f^*_{\eta,\beta,h}(y) = \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(\lambda)\overline{k_y(\lambda)}d\mu_\alpha(\lambda) = \int_{\mathbb{R}} \frac{\overline{\sigma_\beta(\lambda)}}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}\mathcal{F}_\alpha(h)(\lambda)d\mu_\alpha(\lambda).$$

On the other hand, the function

$$F \colon \lambda \longrightarrow \frac{\overline{\sigma_\beta(\lambda)}\mathcal{F}_\alpha(h)(\lambda)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}$$

belongs to $L^1_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$. Using inversion formula (10) and Plancherel's theorem, we find that $f^*_{\eta,\beta,h}$ belongs to $L^2_\alpha(\mathbb{R})$ and

$$\mathcal{F}_\alpha(f^*_{\eta,\beta,h})(\lambda) = F(\lambda).$$

On the other hand, we have

$$|\mathcal{F}_\alpha(f^*_{\eta,\beta,h})(\lambda)|^2 = \frac{|\sigma_\beta(\lambda)|^2}{\left(\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2\right)^2}|\mathcal{F}_\alpha(h)(\lambda)|^2 \leqslant \frac{1}{2\eta\psi(\lambda)}|\mathcal{F}_\alpha(h)(\lambda)|^2.$$

By Plancherel's formula (12), we find that

$$\|f^*_{\eta,\beta,h}\|_\psi \leqslant \frac{1}{\sqrt{2\eta}}\|h\|_{2,\mu_\alpha}.$$

The proof is completed. □

Theorem 8. *(Third Calderón's formula). Let $\sigma \in L^{\infty}_{\alpha}(\mathbb{R})$ and $f \in \mathcal{H}_{\psi}(\mathbb{R})$. The extremal function given by*

$$f^{*}_{\eta,\beta,h}(y) = \int_{\mathbb{R}} \mathcal{T}_{\sigma,\beta}(f)(x)\overline{\Theta_{\eta,\beta}(x,y)}d\mu_{\alpha}(x),$$

satisfies

$$\lim_{\eta \to 0^{+}} \left\| f^{*}_{\eta,\beta} - f \right\|_{2,\mu_{\alpha}} = 0. \tag{40}$$

Moreover, we have $f^{*}_{\eta,\beta} \longrightarrow f$ uniformly when $\eta \longrightarrow 0^{+}$.

Proof. Let $f \in \mathcal{H}_{\psi}(\mathbb{R})$. Put $h = \mathcal{T}_{\sigma,\beta}(f)$ and $f^{*}_{\eta,\beta,h} = f^{*}_{\eta,\beta}$ in relation (39) to find that

$$\mathcal{F}_{\alpha}(f^{*}_{\eta,\beta,h} - f)(\lambda) = \frac{-\eta\psi(\lambda)\mathcal{F}_{\alpha}(f)(\lambda)}{\eta\psi(\lambda) + |\sigma_{\beta}(\lambda)|^{2}}. \tag{41}$$

Therefore,

$$\left\| f^{*}_{\eta,\beta} - f \right\|^{2}_{\psi} = \int_{\mathbb{R}} \frac{\eta^{2}\left(\psi(\lambda)\right)^{3}}{\eta\psi(\lambda) + |\sigma_{\beta}(\lambda)|^{2}} \left|\mathcal{F}_{\alpha}(f)(\lambda)\right|^{2} d\mu_{\alpha}(\lambda).$$

On the other hand, we have

$$\frac{\eta^{2}\left(\psi(\lambda)\right)^{3}}{\eta\psi(\lambda) + |\sigma_{\beta}(\lambda)|^{2}} \left|\mathcal{F}_{\alpha}(f)(\lambda)\right|^{2} \leqslant \psi(\lambda)\left|\mathcal{F}_{\alpha}(f)(\lambda)\right|^{2}. \tag{42}$$

The result (40) follows from (42) and the dominated convergence theorem. Now, for all $f \in \mathcal{H}_{\psi}(\mathbb{R})$ we have $\mathcal{F}_{\alpha}(f) \in L^{2}_{\alpha}(\mathbb{R}) \cap L^{1}_{\alpha}(\mathbb{R})$. Using relations (10) and (41), we find that

$$f^{*}_{\eta,\beta(y,s)} - f(y) = \int_{\mathbb{R}} \frac{-\eta\psi(\lambda)\mathcal{F}_{\alpha}(f)(\lambda)}{\eta\psi(\lambda) + |\sigma_{\beta}(\lambda)|^{2}} B_{\alpha}(\lambda y)d\mu_{\alpha}(\lambda)$$

and

$$\left| \frac{-\eta\psi(\lambda)\mathcal{F}_{\alpha}(f)(\lambda)}{\eta\psi(\lambda) + |\sigma_{\beta}(\lambda)|^{2}} B_{\alpha}(\lambda y) \right| \leqslant \left|\mathcal{F}_{\alpha}(f)(\lambda, m)\right|. \tag{43}$$

Using relation (43) and the dominated convergence theorem, we deduce that

$$\lim_{\eta \to 0^{+}} \left| f^{*}_{\eta,\beta}(y) - f(y) \right| = 0.$$

This completes the proof of the theorem. □

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