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ON THE CLASS OF $m(\phi)$ BOUNDED VARIATION SEQUENCES OF FUZZY REAL NUMBERS

Abstract. In this article, we introduce the sequence space $bv(\phi, M, p)$, for $1 \leqslant p < \infty$ of fuzzy real numbers. We verify and establish some algebraic and topological properties like solidness, monotonicity, convergence-free etc. and prove some inclusion relations of this class of sequences.

Key words: *Orlicz function, bounded variation, fuzzy real number, solid, monotonicity, convergence-free*

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1. Introduction. In 1965, the concept of fuzzy set and its application was introduced for the first time by Zadeh [21]. Later, theory related to boundedness and convergent properties of fuzzy sequences was studied. Subsequently, Altinok, Altin and Et [1], Tripathy and Dutta [16], Savas [11], and several other authors have introduced the different classes of fuzzy real number sequences from various aspects and applied in different applications, such as fuzzy topology, classical metrics, fuzzy mathematical programming, fuzzy ordering, fuzzy metrics etc.

Kizmaz [4] introduced the concept of the class of difference sequences in 1981 for crisp sets, where he defined $\Delta(y_t) = y_t - y_{t+1}, \forall t \in \mathbb{N}$. In the recent past, various authors ([2], [3], [16], [15], [14]) applied the idea in different aspects to construct some new difference sequence spaces and different properties were studied. The sequence space $m(\phi)$ was introduced by Sargent [10] in 1960. Later, Rath and Tripathy [9], Tripathy and Dutta [14], Tripathy and Borgohain [13] have extended this concept to study this space from different classes of fuzzy sequences.

A fuzzy real number X is a mapping $X\colon \mathbb{R} \to I(= [0,1])$ associating each real number t with its grade of membership $X(t)$. X is said to be *normal* if $X(t_0) = 1$ holds for some $t_0 \in \mathbb{R}$. It is also called *upper semi-continuous*, if for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, is open for all $a \in I$ in the usual topology of $\mathbb{R}$. If $X(t) \geqslant X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$, then X is called convex.

The class of all *upper semi-continuous, normal, convex* fuzzy real numbers is denoted by $\mathbb{R}(I)$.

The set $\mathbb{R}$ of all real numbers can be embedded into $\mathbb{R}(I)$ as given below: for each $r \in \mathbb{R}$,

$$\bar{r}(t) = \begin{cases} 1 & \text{for } t = r, \\ 0 & \text{for } t \neq r, \end{cases}$$

where $\bar{0}$ and $\bar{1}$ denote the additive identity and multiplicative identity of $\mathbb{R}(I)$, respectively.

The *α-level* set of a fuzzy real number Y is denoted $[Y]^\alpha$, $0 < \alpha \leqslant 1$, where $[Y]^\alpha = \{t \in \mathbb{R} \colon Y(t) \geqslant \alpha\}$. The 0-*level* set is the closure of $\{t \in \mathbb{R} \colon Y(t) > 0\}$.

Let $X, Y \in \mathbb{R}(I)$; then $X \leqslant Y$ if and only if for any $\alpha \in (0,1]$, $x_1^\alpha \leqslant y_1^\alpha$ and $x_2^\alpha \leqslant y_2^\alpha$, where $[X]^\alpha = [x_1^\alpha, x_2^\alpha]$ and $[Y]^\alpha = [y_1^\alpha, y_2^\alpha]$.

The arithmetic operations for *α-level* sets can be defined as follows:

$$[X+Y]^\alpha = [x_1^\alpha + y_1^\alpha, x_2^\alpha + y_2^\alpha],$$

$$[X-Y]^\alpha = [x_1^\alpha - y_2^\alpha, x_2^\alpha - y_1^\alpha],$$

$$[X \times Y]^\alpha = \left[\min_{m,\, n \in \{1,2\}} x_m^\alpha y_n^\alpha, \max_{m,\, n \in \{1,2\}} x_m^\alpha y_n^\alpha\right],$$

$$[X^{-1}]^\alpha = \left[\frac{1}{x_2^\alpha}, \frac{1}{x_1^\alpha}\right], \quad 0 \notin [x_1^\alpha, x_2^\alpha].$$

For each $\alpha \in (0,1]$, X^α is a non-empty compact subset of $\mathbb{R}$. Also, The closure of $\{t \in \mathbb{R} \colon X(t) > 0\}$ is compact.

The absolute value of $Y \in \mathbb{R}(I)$ is defined by

$$|Y|(t) = \begin{cases} \max\{Y(t), Y(-t)\}, & \text{when } t \geqslant 0, \\ 0, & \text{when } t < 0. \end{cases}$$

Let $X = [X^L, X^R]$ and $Y = [Y^L, Y^R]$ be two elements in the set of all closed and bounded intervals of $\mathbb{R}$, denoted by D, and we define a metric $d(X, Y)$ on D as follows: $d(X,Y) = \max\left\{|X^L - Y^L|, |X^R - Y^R|\right\}$, then it is clear that (D, d) is a complete metric space.

Again, we define the mapping $\bar{d}\colon \mathbb{R}(I) \times \mathbb{R}(I) \to \mathbb{R}$ by $\bar{d}(X,Y) = \sup_{0\leqslant\alpha\leqslant 1} d(X^\alpha, Y^\alpha)$ for $X, Y \in \mathbb{R}(I)$. Then it is clear that $\big(\mathbb{R}(I), \bar{d}\big)$ is a complete metric space.

A fuzzy real-valued sequence (X_t) is said to be convergent to a fuzzy real number R if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\bar{d}(X_t, R) < \varepsilon$, $\forall t \geqslant n$.

Let $m_t, n_t \in \mathbb{C}$ and $p = (p_t)$ be a bounded sequence of positive real numbers. We use the following inequality in this paper:

$$|m_t + n_t|^{p_t} \leqslant \max(1, 2^{L-1})(|m_t|^{p_t} + |n_t|^{p_t}),$$

where $0 < p_t \leqslant \sup p_t = L$.

Throughout this paper, w and w^F denote the class of all sequences and all fuzzy real-valued sequences respectively.

2. Definitions and Preliminaries. In this section we give some definitions.

Definition 1. *An Orlicz function is a non-decreasing convex continuous function $M\colon [0,\infty) \to [0,\infty)$ that satisfies the following properties: $M(0) = 0$, $t > 0 \implies M(t) > 0$ and $M(t) \to \infty$ when $t \to \infty$.*

The Orlicz sequence space was defined by Lindenstrauss and Tzafriri [5] as given below:

$$\ell_M = \Big\{ y = (y_t) \in w\colon \sum_{t=1}^{\infty} M\left(\frac{|y_t|}{m}\right) < \infty, \quad \text{for some } m > 0 \Big\},$$

(where w denotes the space of all sequences). Under the following norm, the defined space is also a Banach space:

$$\|y\| = \inf \Big\{ m > 0\colon \sum_{t=1}^{\infty} M\left(\frac{|y_t|}{m}\right) \leqslant 1 \Big\}.$$

The Orlicz function M is said to satisfy Δ_2-condition, if for any $\delta > 1$, there exists a positive constant $C(\delta) > 0$, such that $M(\delta y) \leqslant C(\delta) M(y)$, $\forall y \geqslant 0$.

Remark 1. *It is well-known that for any Orlicz function M, the property $M(\beta x) \leqslant \beta M(x), \forall x \geqslant 0$ and $0 < \beta < 1$ holds.*

Definition 2. *Consider $P_s = \Big\{ \sigma \in P\colon \sum_{t=1}^{\infty} p_t(\sigma) \leqslant s \Big\}$, where P denotes the class containing only finite sets of positive integers which are distinct.*

Also, for any $\sigma \in P$, each term of the sequence $\{p_t(\sigma)\}$ gives the value 1 if $t \in \sigma$ and 0 if $t \notin \sigma$. Further, $\phi = (\phi_t)$ is an non-decreasing sequence of positive real numbers that satisfies the condition $t\phi_{(t+1)} \leqslant (t+1)\phi_t, \forall t \in \mathbb{N}$. In 1960, the $m(\phi)$ space was introduced by Sargent [10] as given below:

$$m(\phi) = \Big\{y = (y_t) \in w\colon \sup_{s \geqslant 1,\, \sigma \in P_s} \frac{1}{\phi_s} \sum_{t \in \sigma} |y_t| < \infty\Big\}$$

(where w is the space of all sequences).

After that, several authors have studied and extended this space. An extension of $m(\phi)$ space to $m(\phi, p)$ was done by Tripathy and Sen [20] as given below:

$$m(\phi, p) = \Big\{y = (y_t) \in w\colon \sup_{s \geqslant 1,\, \sigma \in P_s} \frac{1}{\phi_s} \sum_{t \in \sigma} |y_t|^p < \infty, 0 < p < \infty\Big\}.$$

Definition 3. *Let $K = \{t_n\colon n \in \mathbb{N};\ t_1 < t_2 < t_3 < \dots\} \subseteq \mathbb{N}$ and E be a fuzzy sequence space. A K-step space of E is a sequence space $\lambda_K^E = \big\{(X_{t_n}) \in w^F : (X_t) \in E\big\}$.*

A canonical pre-image of a sequence $(X_t) \in E$ is a sequence $(Y_r) \in w^F$ defined as follows:

$$Y_r = \begin{cases} X_r, & \text{if } r \in K, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Definition 4. *A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.*

Definition 5. *A class of fuzzy real-valued sequences E is said to be monotone if the canonical pre-images of all its step sets are contained in E.*

Definition 6. *A class of fuzzy real-valued sequences E is said to be solid if $(X_t^1) \in E$, whenever $|X_t^1| \leqslant |X_t^2|$ and $(X_t^2) \in E$, $\forall t \in \mathbb{N}$.*

Definition 7. *A class of fuzzy real-valued sequences E is said to be convergence-free if $(X_t^1) \in E$, whenever $(X_t^2) \in E$ and $X_t^2 = \bar{0}$ implies $X_t^1 = \bar{0}$.*

Definition 8. *A class of fuzzy real-valued sequences E is said to be symmetric if $(X_t) \in E \implies (X_{\pi(t)}) \subset E, \forall (X_t) \in E$, where π denotes a permutation of $\mathbb{N}$.*

Remark 2. *If a fuzzy real-valued sequence space E is solid then the space is also monotone.*

Definition 9. *A sequence (X_t) is said to be of bounded variation if it satisfies the property $\sum_{t=1}^{\infty} |\Delta x_t| < \infty$, where $\Delta x_t = x_t - x_{t+1}, \forall t \in \mathbb{N}$. The bounded variation sequence space is denoted by bv, which is given below:*

$$bv = \Big\{(x_t) \in w \colon \sum_{t=1}^{\infty} |\Delta x_t| < \infty\Big\}.$$

Tripathy and Dutta [16] introduced and defined the class of lacunary bounded variation fuzzy real-valued sequences as follows:

$$bv_{\theta}^{F} = \Big\{(X_t) \in w^F \colon \sum_{r=1}^{\infty} \Big(\frac{1}{h_r} \sum_{t \in I_r} \bar{d}(\Delta X_t, \bar{0})\Big) < \infty\Big\}.$$

Also, Tripathy and Das [15] formed the p-bounded variation sequences of fuzzy real numbers bv_p^F, as given below:

$$bv_p^F = \Big\{(X_t) \in w^F \colon \sum_{t=1}^{\infty} \big\{\bar{d}(\Delta X_t, \bar{0})\big\}^p < \infty, \text{ for } 1 \leqslant p < \infty\Big\}.$$

In this paper, we introduce a new sequence space $bv(\phi, M, p)$ by using Orlicz function M, $m(\phi)$ space, and the bounded variation fuzzy real-valued sequences; this space is given below:

$$bv(\phi, M, p) = \Big\{(X_t) \in w^F \colon \sum_{s=1}^{\infty} \frac{1}{\phi_s} \Big(\sum_{\sigma \in P_s, t \in \sigma} \Big\{M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m}\right)\Big\}^p\Big) < \infty\Big\}$$

for some $m > 0$ and $1 \leqslant p < \infty$.

Also, we have constructed the Cesaro-type summable bounded variation sequence spaces of fuzzy real numbers as follows:

$$bv(C, M, p) = \Big\{(X_t) \in w^F \colon \sum_{n=1}^{\infty} \frac{1}{n+1} \Big(\sum_{t=1}^{n} \Big\{M\Big(\frac{\bar{d}(\Delta X_t, \bar{0})}{m}\Big)\Big\}^p\Big) < \infty\Big\}$$

for some $m > 0$ and $1 \leqslant p < \infty$.

3. Main Results.

Theorem 1. *The class of sequences $bv(\phi, M, p)$ is linear over the field $\mathbb{C}$.*

Proof. Let α, β be two arbitrary elements of $\mathbb{C}$ and $X = (X_t)$, $Y = (Y_t) \in bv(\phi, M, p)$. Then, for some $m_1 > 0$ and $m_2 > 0$, we have

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s}\Big(\sum_{\sigma \in P_s, t \in \sigma} \Big\{ M\Big(\frac{\bar{d}(\Delta X_t, \bar{0})}{m_1}\Big)\Big\}^p\Big) < \infty$$

and

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s}\Big(\sum_{\sigma \in P_s, t \in \sigma} \Big\{ M\Big(\frac{\bar{d}(\Delta Y_t, \bar{0})}{m_2}\Big)\Big\}^p\Big) < \infty.$$

Let $m_3 = \max\{2|\alpha|m_1, 2|\beta|m_2\}$. Again, M is a non-decreasing convex continuous function. Therefore,

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s}\Big(\sum_{\sigma \in P_s, t \in \sigma} \Big\{ M\Big(\frac{\bar{d}(\Delta \alpha X_t + \Delta \beta Y_t, \bar{0})}{m_3}\Big)\Big\}^p\Big) \leqslant$$

$$\leqslant \max(1, 2^{p-1})\Big[\sum_{s=1}^{\infty} \frac{1}{\phi_s}\Big(\sum_{\sigma \in P_s, t \in \sigma} \Big\{ M\Big(\frac{\bar{d}(\Delta X_t, \bar{0})}{m_1}\Big)\Big\}^p\Big) +$$

$$+ \sum_{s=1}^{\infty} \frac{1}{\phi_s}\Big(\sum_{\sigma \in P_s, t \in \sigma} \Big\{ M\Big(\frac{\bar{d}(\Delta Y_t, \bar{0})}{m_2}\Big)\Big] < \infty.$$

This implies $(\alpha X + \beta Y) \in bv(\phi, M, p)$. Therefore, the sequence space $bv(\phi, M, p)$ is linear over $\mathbb{C}$. □

Theorem 2. *The class of sequences $bv(\phi, M, p)$ is a complete metric space under the metric*

$$\beta(X, Y) =$$

$$= \bar{d}(X_1, Y_1) + \inf\Big\{ m > 0 \colon \sum_{s=1}^{\infty} \frac{1}{\phi_s}\Big[\sum_{\sigma \in P_s, t \in \sigma} \Big\{ M\Big(\frac{\bar{d}(\Delta X_t, \Delta Y_t)}{m}\Big)\Big\}^p\Big]^{\frac{1}{p}} \leqslant 1\Big\}.$$

Proof. It is very easy to show that $bv(\phi, M, A, p)$ is a metric space with respect to the given metric. Next, we prove that $bv(\phi, M, A, p)$ is a complete metric space.

Consider a Cauchy sequence $(X_t^{(u)})_{t=1}^{\infty} \in bv(\phi, M, A, p)$. We choose $z > 0$, such that for a fixed positive number $y > 0$ we get $M\Big(\frac{yz}{2}\Big) \geqslant 1$. Let $\varepsilon > 0$ be given. Then there exists a non-negative integer $s = s(\varepsilon) > 0$, such that $\forall u, v \geqslant s$, $\beta(X^{(u)}, X^{(v)}) < \frac{\varepsilon}{yz}$. This implies

$$\bar{d}(X_1^{(u)}, X_1^{(v)})+$$
$$+\inf\left\{m>0:\sum_{s=1}^{\infty}\frac{1}{\phi_s}\Big[\sum_{\sigma\in P_s,t\in\sigma}\Big\{M\Big(\frac{\bar{d}(\Delta X_t^{(u)},\Delta X_t^{(v)})}{m}\Big)\Big\}^p\Big]^{\frac{1}{p}}\leqslant 1\right\}\leqslant\varepsilon,$$

for $\forall u,v\geqslant s$. Therefore, $\bar{d}(X_1^{(u)}, X_1^{(v)})<\varepsilon$ and

$$\inf\left\{m>0:\sum_{s=1}^{\infty}\frac{1}{\phi_s}\Big[\sum_{\sigma\in P_s,t\in\sigma}\Big\{M\Big(\frac{\bar{d}(\Delta X_t^{(u)},\Delta X_t^{(v)})}{m}\Big)\Big\}^p\Big]^{\frac{1}{p}}\leqslant 1\right\}<\varepsilon,$$

for $\forall u,v\geqslant s$. Here we see that $(X_1^{(u)})$ is a Cauchy sequence in $\mathbb{R}(I)$ and, since $\mathbb{R}(I)$ is a complete metric space, $(X_1^{(u)})$ is convergent in $\mathbb{R}(I)$.

Let

$$\lim_{u\to\infty}X_1^{(u)}=X_1. \tag{1}$$

Also,

$$\sum_{s=1}^{\infty}\frac{1}{\phi_s}\Big[\sum_{\sigma\in P_s,t\in\sigma}\Big\{M\Big(\frac{\bar{d}(\Delta X_t^{(u)},\Delta X_t^{(v)})}{m}\Big)\Big\}^p\Big]^{\frac{1}{p}}\leqslant 1.$$

Now, taking $s=1$, $\forall u,v\geqslant s$

$$\Big[\sum_{\sigma\in P_s,t\in\sigma}\Big\{M\Big(\frac{\bar{d}(\Delta X_t^{(u)},\Delta X_t^{(v)})}{\beta(X^{(u)},X^{(v)})}\Big)\Big\}^p\Big]^{\frac{1}{p}}\leqslant\phi_1\implies$$

$$\implies\sum_{\sigma\in P_s,t\in\sigma}\Big\{M\Big(\frac{\bar{d}(\Delta X_t^{(u)},\Delta X_t^{(v)})}{\beta(X^{(u)},X^{(v)})}\Big)\Big\}^p\leqslant{\phi_1}^p\implies$$

$$\implies M\Big(\frac{\bar{d}(\Delta X_t^{(u)},\Delta X_t^{(v)})}{\beta(X^{(u)},X^{(v)})}\Big)\leqslant\phi_1\leqslant M\left(\frac{yz}{2}\right)\implies$$

$$\implies\bar{d}(\Delta X_t^{(u)},\Delta X_t^{(v)})<\frac{yz}{2}\cdot\frac{\varepsilon}{yz}$$

(since M is a continuous non-decreasing function). Then we have

$$\bar{d}(\Delta X_t^{(u)},\Delta X_t^{(v)})<\frac{\varepsilon}{2}.$$

Here, $\Delta(X_t^{(u)})$ is a Cauchy sequence in $\mathbb{R}(I)$ and, since $\mathbb{R}(I)$ is a complete metric space, $\Delta(X_t^{(u)})$ is convergent in $\mathbb{R}(I)$.

Let

$$\lim_{u\to\infty} \Delta X_t^{(u)} = Y_t \in \mathbb{R}(I), \ \forall t \in \mathbb{N}. \tag{2}$$

From equation (1) and equation (2), we get $\lim_{u\to\infty} X_{t+1}^{(u)} = X_{t+1}, \forall t \geqslant 1$. This implies $\lim_{u\to\infty} \Delta X_t^{(u)} = \Delta X_t, \forall t \in \mathbb{N}$.

Now, keeping u fixed, taking $v \to \infty$, and using the continuity of M we get

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \left[\sum_{\sigma\in P_s, t\in\sigma} \left\{ M\left(\frac{\bar{d}(\Delta X_t^{(u)}, \Delta X_t)}{m} \right) \right\}^p \right]^{\frac{1}{p}} \leqslant 1,$$

for some $m > 0$ and $\forall u \geqslant s$.

After taking the infimum of such m's together, we have

$$\inf \left\{ m > 0 \colon \sum_{s=1}^{\infty} \frac{1}{\phi_s} \left[\sum_{\sigma\in P_s, t\in\sigma} \left\{ M\left(\frac{\bar{d}(\Delta X_t^{(u)}, \Delta X_t)}{m} \right) \right\}^p \right]^{\frac{1}{p}} \leqslant 1 \right\} < \varepsilon,$$

for $\forall u \geqslant s$.

Therefore, $\beta(X^{(u)}, X) < 2\varepsilon, \forall u \geqslant s$. Hence, $\lim_{u\to\infty} X^{(u)} = X$.

Now we need to show $X \in bv(\phi, M, A, p)$. Here $\beta(X, \bar{0}) \leqslant \beta(X, X^{(u)}) + {} + \beta(X^{(u)}, \bar{0}) < \infty, \forall u \geqslant s$. This implies $X \in bv(\phi, M, A, p)$. Hence, $bv(\phi, M, A, p)$ is a complete metric space. □

Theorem 3. *The class of sequences $bv(\phi, M, p)$ is not monotone in general.*

Proof. The theorem can be proved by the following example.

Consider $M(x) = x^4$, $\forall x \in [0, \infty)$ and $\phi_s = s$, $\forall s \in \mathbb{N}$. Let $p = 1$ and $X_t = \bar{1}$, $\forall t \in \mathbb{N}$. Then we get $\bar{d}(\Delta X_t, \bar{0}) = 0$. Therefore,

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \left(\sum_{\sigma\in P_s, t\in\sigma} \left\{ M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p \right) = 0 < \infty,$$

for some $m > 0$. This implies $(X_t) \in bv(\phi, M, p)$.

Now, consider a subset K of the set of natural numbers $\mathbb{N}$, such that $K = \{t \in \mathbb{N} \colon t = 2n - 1, \forall n \in \mathbb{N}\}$. Define the canonical pre-image $bv_K(\phi, M, p)$ of the K-step set $bv_K(\phi,M,p)$ of $bv(\phi, M, p)$ as follows: let $Y_t \in bv_K(\phi, M, p)$. Then $Y_t = \begin{cases} X_t, & \text{for } t \in K, \\ \bar{0}, & \text{for } t \notin K. \end{cases}$

Therefore,

$$\sum_{s=1}^{\infty}\frac{1}{\phi_s}\Bigg(\sum_{\sigma\in P_s, t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta Y_t,\bar{0})}{m}\right)\right\}^p\Bigg)=\sum_{s=1}^{\infty}\frac{1}{m^4}=\infty,$$

for any fixed $m>0$. This implies, $(Y_t)\notin bv(\phi,M,p)$. Therefore, the class of sequences $bv(\phi,M,p)$ is *not monotone* in general. □

Theorem 4. *The class of sequences $bv(\phi,M,p)$ is not convergence-free in general.*

Proof. The theorem can be proved by the following example.

Consider $M(x)=x^2$, $\forall x\in[0,\infty)$ and $\phi_s=s$, $\forall s\in\mathbb{N}$. Let $p=2$ and $X_t=\bar{4}$, $\forall t\in\mathbb{N}$. Then we get $\bar{d}(\Delta X_t,\bar{0})=0$. Therefore,

$$\sum_{s=1}^{\infty}\frac{1}{\phi_s}\Bigg(\sum_{\sigma\in P_s, t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p\Bigg)=0<\infty,$$

for some $m>0$. This implies $(X_t)\in bv(\phi,M,p)$.

Now, consider another sequence $Y_t=\overline{\left(\frac{1}{t}\right)}$, $\forall t\in\mathbb{N}$. Thus, $\bar{d}(\Delta Y_t,\bar{0})=$ $=\frac{1}{k}-\frac{1}{k+1}$. Then we have for each fixed $m>0$:

$$\sum_{s=1}^{\infty}\frac{1}{\phi_s}\Bigg(\sum_{\sigma\in P_s, t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta Y_t,\bar{0})}{m}\right)\right\}^p\Bigg)=\sum_{s=1}^{\infty}\frac{1}{s}\Bigg(\sum_{\sigma\in P_s, t\in\sigma}\frac{1}{m}\left(\frac{1}{k}-\frac{1}{k+1}\right)^2\Bigg)=\infty.$$

Therefore, the class of sequences $bv(\phi,M,p)$ is *not convergence-free* in general. □

Theorem 5. *The class of sequences $bv(\phi,M,p)$ is not solid in general.*

Proof. The theorem can be proved by the following example.

Consider $M(x)=x$, $\forall x\in[0,\infty)$ and $\phi_s=1$, $\forall s\in\mathbb{N}$. Let $p=1$ and $X_t=\bar{3}$, $\forall t\in\mathbb{N}$. Then we get $\bar{d}(\Delta X_t,\bar{0})=0$. Therefore,

$$\sum_{s=1}^{\infty}\frac{1}{\phi_s}\Bigg(\sum_{\sigma\in P_s, t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}V\right)\right\}^p\Bigg)=0<\infty,$$

for some $m>0$. This implies $(X_t)\in bv(\phi,M,p)$.

Now, consider a sequence (α_t) of scalars, such that

$$\alpha_t = \begin{cases} 1, & \text{for even } t, \\ 0, & \text{otherwise.} \end{cases} \implies \alpha_t X_t = \begin{cases} \bar{t}, & \text{for even } t, \\ \bar{0}, & \text{otherwise.} \end{cases}$$

Therefore,

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \Bigg(\sum_{\sigma \in P_s, t \in \sigma} \left\{ M\left(\frac{\bar{d}(\alpha_t \Delta X_t, \bar{0})}{m} \right) \right\}^p \Bigg) = \infty$$

for any fixed $m > 0$. This implies $(\alpha_t X_t) \notin bv(\phi, M, p)$. Therefore, the class of sequences $bv(\phi, M, p)$ is *not solid* in general. □

Note. The above theorem can be directly proved using Remark 2 and Theorem 3.

Theorem 6. *$bv(M, p) \subseteq bv(\phi, M, p)$ if $\sum_{s=1}^{\infty} \frac{1}{\phi_s} < \infty$.*

Proof. Let $(X_t) \in bv(M, p)$. This implies, for some $m > 0$,

$$\sum_{t=1}^{\infty} \left\{ M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p < \infty.$$

Hence, there exists a positive integer t_0, such that

$$\sum_{t \in \sigma, \sigma \in P_s} \left\{ M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p < 1,$$

for $\forall s \geqslant t_0$ and for some $m > 0$.

Therefore, for some $m > 0$,

$$\sum_{s \geqslant t_0} \frac{1}{\phi_s} \Bigg(\sum_{\sigma \in P_s, t \in \sigma} \left\{ M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p \Bigg) \leqslant \sum_{s \geqslant t_0} \frac{1}{\phi_s} < \infty.$$

This implies

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \Bigg(\sum_{\sigma \in P_s, t \in \sigma} \left\{ M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p \Bigg) < \infty,$$

for some $m > 0$.

Hence, $(X_t) \in bv(\phi, M, p)$, which implies $bv(M, p) \subseteq bv(\phi, M, p)$. □

Theorem 7.

(i) $bv(\phi, M_1, p) \subseteq bv(\phi, M \circ M_1, p)$;

(ii) $bv(\phi, M_1, p) \cap bv(\phi, M_2, p) \subseteq bv(\phi, M_1 + M_2, p)$.

Proof. *(i)* Consider $\alpha > 0$ and $\beta > 0$ that satisfy $M^p(\alpha) = \beta$. Let $(X_t) \in bv(\phi, M_1, p)$. This implies, for some $m > 0$:

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t \in \sigma} \left\{ M_1 \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p = \alpha \text{ (say) } < \infty.$$

As M is a continuous non-decreasing function, then

$$M^p \left(\sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t \in \sigma} \left\{ M_1 \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p \right) = M^p(\alpha) \implies$$

$$\implies \sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t \in \sigma} \left\{ M \circ M_1 \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p = \beta < \infty.$$

This implies $(X_t) \in bv(\phi, M \circ M_1 p)$.

Therefore, $bv(\phi, M_1, p) \subseteq bv(\phi, M \circ M_1, p)$.

(ii) Let $(X_t) \in bv(\phi, M_1, p) \cap bv(\phi, M_2, p)$. This implies, for some $m > 0$,

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t \in \sigma} \left\{ M_1 \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p < \infty$$

and

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t \in \sigma} \left\{ M_2 \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p < \infty.$$

Now, for some $m > 0$

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t \in \sigma} \left\{ (M_1 + M_2) \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p =$$

$$= \sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t \in \sigma} \left\{ M_1 \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) + M_2 \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p \leqslant$$

$$\leqslant \max(1, 2^{(p-1)}) \left[\sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t \in \sigma} \left\{ M_1 \left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m} \right) \right\}^p + \right.$$

$$+\sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{\sigma\in P_s,t\in\sigma}\left\{M_2\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{p}\Bigg] < \infty.$$

This implies $(X_t) \in bv(\phi, M_1 + M_2, p)$.

Therefore, $bv(\phi, M_1, p) \cap bv(\phi, M_2, p) \subseteq bv(\phi, M_1 + M_2, p)$. □

Theorem 8. *Let $0 < q \leqslant p < \infty$. Then $bv(\phi, M, q) \subseteq bv(\phi, M, p)$.*

Proof. Let $(X_t) \in bv(\phi, M, q)$ and $0 < q \leqslant p < \infty$. Therefore, for some $m > 0$,

$$\sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{\sigma\in P_s,t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{q} < \infty.$$

This implies that there exists a natural number t', such that

$$\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{q} < 1, \quad \forall t \geqslant t'.$$

Therefore,

$$\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{p} \leqslant \left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{q}, \quad \forall t \geqslant t'.$$

This implies

$$\sum_{\sigma\in P_s,t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{p} \leqslant \sum_{\sigma\in P_s,t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{q} \implies$$

$$\implies\sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{\sigma\in P_s,t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{p} \leqslant \sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{\sigma\in P_s,t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{q} \implies$$

$$\implies \sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{\sigma\in P_s,t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^{p} < \infty.$$

This implies $(X_t) \in bv(\phi, M, p)$. Therefore, $bv(\phi, M, q) \subseteq bv(\phi, M, p)$. □

Theorem 9. *$bv(\phi, M, p) \subseteq bv(\beta, M, p)$ if and only if $\sup_{s\geqslant 1}\left(\frac{\phi_s}{\beta_s}\right) < \infty$.*

Proof. Let $\sup_{s\geqslant 1}\left(\frac{\phi_s}{\beta_s}\right) = S(< \infty)$. Therefore, $\phi_s \leqslant S\beta_s$, $\forall s \geqslant 1$.

Consider $(X_t) \in bv(\phi, M, p)$. This implies, for some $m > 0$,

$$\sum_{s=1}^{\infty} \frac{1}{S\beta_s} \sum_{\sigma \in P_s, t\in\sigma} \left\{M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m}\right)\right\}^p \leqslant \sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t\in\sigma} \left\{M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m}\right)\right\}^p < \infty.$$

Therefore, $(X_t) \in bv(\beta, M, p)$. Hence, $bv(\phi, M, p) \subseteq bv(\beta, M, p)$ if $\sup_{s\geqslant 1}\left(\frac{\phi_s}{\beta_s}\right) < \infty$.

Conversely, we need to show $\sup_{s\geqslant 1}\left(\frac{\phi_s}{\beta_s}\right) < \infty$ if $bv(\phi, M, p) \subseteq bv(\beta, M, p)$. Let $(X_t) \in bv(\phi, M, p)$. Therefore, for some $m > 0$,

$$\sum_{s=1}^{\infty} \frac{1}{\phi_s} \sum_{\sigma \in P_s, t\in\sigma} \left\{M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m}\right)\right\}^p < \infty.$$

If possible, let $\sup_{s\geqslant 1}\left(\frac{\phi_s}{\beta_s}\right) = \infty$ and $bv(\phi, M, p) \subseteq bv(\beta, M, p)$. Then there exist a sub-sequence $\left(\frac{\phi_{s_i}}{\beta_{s_i}}\right)$ of $\left(\frac{\phi_s}{\beta_s}\right)$ that satisfies $\lim_{i\to\infty}\left(\frac{\phi_{s_i}}{\beta_{s_i}}\right) = \infty$. Then there exist $n \in \mathbb{N}$ for each $T \in \mathbb{R}^+$(set of positive real numbers) that satisfies $\frac{\phi_{s_i}}{\beta_{s_i}} > T, \forall s_i > n$. This implies, for some $m > 0$,

$$\sum_{s=1}^{\infty} \frac{1}{\beta_s} \sum_{\sigma \in P_s, t\in\sigma} \left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p > \sum_{t=1}^{\infty} \frac{T}{\phi_s} \sum_{\sigma \in P_s, t\in\sigma} \left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p.$$

For sufficiently large T, we get:

$$\sum_{s=1}^{\infty} \frac{1}{\beta_s} \sum_{\sigma \in P_s, t\in\sigma} \left\{M\left(\frac{\bar{d}(\Delta X_t, \bar{0})}{m}\right)\right\}^p = \infty,$$

which implies $(X_t) \notin bv(\beta, M, p)$. But this is a contradiction to the fact that $(X_t) \in bv(\phi, M, p)$ and $bv(\phi, M, p) \subseteq bv(\beta, M, p)$. Therefore, our assumption was wrong. So, $\sup_{s\geqslant 1}\left(\frac{\phi_s}{\beta_s}\right) < \infty$.

Hence, $\sup_{s\geqslant 1}\left(\frac{\phi_s}{\beta_s}\right) < \infty$ if $bv(\phi, M, p) \subseteq bv(\beta, M, p)$. □

Corollary 1. *$bv(\phi, M, p) = bv(\beta, M_1, p)$ if and only if $\sup_{s\geqslant 1}\left(\frac{\phi_s}{\beta_s}\right) < \infty$ and $\sup_{s\geqslant 1}\left(\frac{\beta_s}{\phi_s}\right) < \infty$.*

Theorem 10. *$bv(C,M,p)\subseteq bv(\phi,M,p)$ if* $\liminf_s \dfrac{s+1}{s+1-\phi_s}\geqslant 1, \forall s\in\mathbb{N}$.

Proof. Let $X\in bv(C,M,p)$. This implies, for some $m>0$,

$$\sum_{n=1}^{\infty}\frac{1}{n+1}\left(\sum_{t=1}^{n}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p\right)<\infty.$$

Suppose that $\liminf_s \dfrac{s+1}{s+1-\phi_s}\geqslant 1, \forall s\in\mathbb{N}$. Therefore, there exists a positive integer $\gamma>0$, such that $\dfrac{1+\gamma}{\gamma}\geqslant\dfrac{s+1}{\phi_s}$. Now,

$$\sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{\sigma\in P_s,t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p=$$

$$=\sum_{s=1}^{\infty}\frac{s+1}{\phi_s}\cdot\frac{1}{s+1}\sum_{t=1}^{\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p-$$

$$-\sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{t\in\{1,2,..,s\}-\sigma,\sigma\in P_s}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p\leqslant$$

$$\leqslant\sum_{s=1}^{\infty}\frac{1+\gamma}{\gamma}\cdot\frac{1}{s+1}\sum_{t=1}^{\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p-$$

$$-\sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{t_0\in\{1,2,..,s\}-\sigma,\sigma\in P_s}\left\{M\left(\frac{\bar{d}(\Delta X_{t_0},\bar{0})}{m}\right)\right\}^p.$$

Since $X\in bv(C,M,A,p)$, we get, for some $m>0$,

$$\sum_{s=1}^{\infty}\frac{1}{\phi_s}\sum_{\sigma\in P_s,t\in\sigma}\left\{M\left(\frac{\bar{d}(\Delta X_t,\bar{0})}{m}\right)\right\}^p<\infty.$$

Hence, $X\in bv(\phi,M,p)$, which implies $bv(C,M,p)\subseteq bv(\phi,M,p)$. □

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